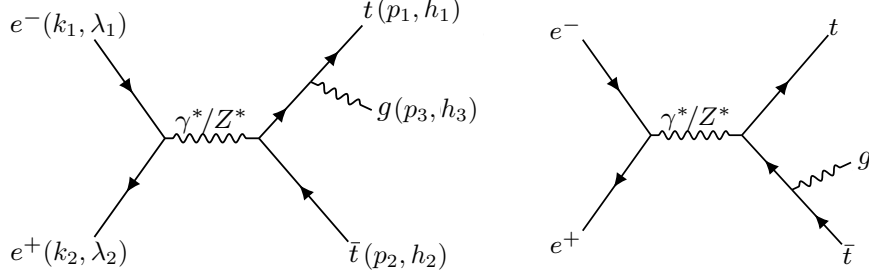


$e^-e^+ \rightarrow t\bar{t}g$ entanglement

We consider the $e^-e^+ \rightarrow t\bar{t}g$ process. The momentum and helicity assignment as well as the tree-level diagrams are shown below.



1 Kinematics

We work in the centre-of-mass frame, which is the same as the lab frame. The total momenta are constrained as

$$(k_1 + k_2)^\mu = (p_1 + p_2 + p_3)^\mu = (\sqrt{s}, \vec{0}), \quad (1)$$

where $\sqrt{s}$ is the centre-of-mass energy. We fix the initial momenta as

$$k_1^\mu = \frac{\sqrt{s}}{2}(1, 0, 0, 1), \quad k_2^\mu = \frac{\sqrt{s}}{2}(1, 0, 0, -1). \quad (2)$$

The gluon and the $t\bar{t}$ subsystem are back-to-back. They are parametrised as

$$p_3^\mu = (E_g, \vec{p}_g), \quad (p_1 + p_2)^\mu = (E_{t\bar{t}}, -\vec{p}_g), \quad \vec{p}_g = E_g(\sin \theta_g, 0, \cos \theta_g), \quad (3)$$

where $E_{t\bar{t}} = \sqrt{m_{t\bar{t}}^2 + E_g^2}$ and $m_{t\bar{t}}$ are the energy and invariant mass of the $t\bar{t}$ subsystem.

At the rest frame of the $t\bar{t}$, the top and antitop momenta are parametrised in the new (x', y', z') coordinate system as

$$p_1' = (\frac{m_{t\bar{t}}}{2}, \vec{q}), \quad p_2' = (\frac{m_{t\bar{t}}}{2}, -\vec{q}), \quad \vec{q} = q(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (4)$$

where $q = \sqrt{\frac{m_{t\bar{t}}^2}{4} - m_t^2}$ is the momentum magnitude. In this coordinate system, the z' axis is taken such that it aligns with the boost direction of the $t\bar{t}$ system, $\propto -\vec{p}_g$. The lab frame momenta p_1 and p_2 in the original (x, y, z) system are obtained by rotating the (x', y', z') system to the original (x, y, z) system and boosting the t and $\bar{t}$ momenta back to the lab frame with the boost vector $\vec{b}$

$$p_1 = \Lambda_{\vec{b}}(R(p_1')), \quad p_2 = \Lambda_{\vec{b}}(R(p_2')), \quad \vec{b} = -\frac{\vec{p}_g}{E_{t\bar{t}}}, \quad (5)$$

where R is the rotation around the y axis (which is the same as the y' axis) by the angle $\theta_g + \pi$.

In this way, all final state momenta p_1 , p_2 and p_3 are determined by the five kinematical parameters

$$\sqrt{s}, \quad E_g, \quad \theta_g, \quad \theta, \quad \phi. \quad (6)$$

The energy conservation $\sqrt{s} = E_g + E_{t\bar{t}}$ relates the gluon energy and the $t\bar{t}$ invariant mass as

$$m_{t\bar{t}}^2 = s - 2\sqrt{s}E_g. \quad (7)$$

The gluon energy ranges

$$0 \leq E_g \leq \frac{s - 4m_t^2}{2\sqrt{s}}. \quad (8)$$

2 Helicity amplitudes

$$\mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} = \sum_I^{\text{diagrams}} \mathcal{M}^{(I)} [\bar{u}(k_1, \lambda_1), v(k_2, \lambda_2), u(p_1, h_1), \bar{v}(p_2, h_2), \epsilon(p_3, h_3)], \quad (9)$$

where $\mathcal{M}^{(I)}$ is the contribution from the diagram I , which is a function of four Dirac spinors corresponding to e^- , e^+ , t and $\bar{t}$ and the polarisation vector ϵ^μ of the gluon. The fermion and anti-fermion spinors $u(p, \lambda)$ and $v(p, \lambda)$ with the helicity $\lambda = \pm$ and the momentum $p^\mu = (E, \vec{p})$ with $E = \sqrt{m^2 + |\vec{p}|^2}$ and $\vec{p} = |\vec{p}|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ are given by [1, 2, 3].

$$u(p, \lambda) = \begin{pmatrix} \omega_{-\lambda}(p) \chi_\lambda(p) \\ \omega_\lambda(p) \chi_\lambda(p) \end{pmatrix}, \quad v(p, \lambda) = \begin{pmatrix} -\lambda \omega_\lambda(p) \chi_{-\lambda}(p) \\ \lambda \omega_\lambda(p) \chi_{-\lambda}(p) \end{pmatrix}, \quad \omega_\lambda(p) = \sqrt{E + \lambda |\vec{p}|},$$

$$\chi_+ = \begin{pmatrix} e^{-i\phi/2} \cos \frac{\theta}{2} \\ e^{i\phi/2} \sin \frac{\theta}{2} \end{pmatrix}, \quad \chi_- = \begin{pmatrix} -e^{-i\phi/2} \sin \frac{\theta}{2} \\ e^{i\phi/2} \cos \frac{\theta}{2} \end{pmatrix}. \quad (10)$$

When $\theta = 0$ and π , the phase ambiguity in $\chi_\pm$ is fixed by setting $\phi = 0$. For the gluon momentum $p_3^\mu = E_g(1, \sin \theta_g, 0, \cos \theta_g)$, the polarisation vector is given by

$$\epsilon^\mu(p_3, \pm) = \pm \frac{1}{\sqrt{2}} (0, -\cos \theta_g, \pm i, \sin \theta_g), \quad (11)$$

We explicitly evaluate the helicity amplitudes (9) with these spinors and polarisation vectors. Since the electron mass is neglected, the amplitudes vanish unless $\lambda_1 = -\lambda_2$

$$\mathcal{M}_{h_1, h_2, h_3}^{++} = \mathcal{M}_{h_1, h_2, h_3}^{--} = 0. \quad (12)$$

2.1 Spin Quantum States

For an initial polarisation (λ_1, λ_2) , given the final state momenta, the unnormalised final state in the spin Hilbert space, $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, is given by

$$|\tilde{\psi}_{\lambda_1, \lambda_2}\rangle = \sum_{h_1, h_2, h_3} \mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} |h_1, h_2, h_3\rangle. \quad (13)$$

The unnormalised 8×8 density matrix is given by

$$R_{(h_1, h_2, h_3), (h'_1, h'_2, h'_3)}^{\lambda_1, \lambda_2} = \mathcal{M}_{h_1, h_2, h_3}^{\lambda_1, \lambda_2} [\mathcal{M}_{h'_1, h'_2, h'_3}^{\lambda_1, \lambda_2}]^*. \quad (14)$$

For unpolarised beams, the corresponding R matrix is

$$R = \frac{1}{4} [R^{+-} + R^{-+}]. \quad (15)$$

The normalised density matrix is immediately obtained as $\rho = R/\text{Tr}[R]$.

The diagonal elements of the R -matrix corresponds to the differential cross-section

$$\frac{d\sigma_{h_1, h_2, h_3}}{d\Phi} = \frac{1}{2s} R_{(h_1, h_2, h_3), (h_1, h_2, h_3)}. \quad (16)$$

In an experimental setup, we must average the density matrix ρ over the events in some selected region Σ , which leads to

$$\rho_\Sigma = \frac{R_\Sigma}{\text{Tr}[R_\Sigma]}, \quad R_\Sigma = \int_\Sigma d\sigma \rho = \int_\Sigma \frac{d\Phi}{2s} R, \quad (17)$$

where the full phase-space integration is

$$\int_{\text{full}} d\Phi = \frac{1}{256 \pi^4} \int_0^{E_g^{\text{max}}} dE_g E_g \sqrt{\lambda(E_g)} \int_{-1}^1 d \cos \theta_g \int_{-1}^1 d \cos \theta \int_0^{2\pi} d\phi \quad (18)$$

with $\lambda(E_g) = 1 - \frac{4m_t^2}{s-2\sqrt{s}E_g}$.

2.2 Projection to fixed gluon helicity

What if the gluon helicity is accurately measured and we prepare the event sample in which all gluons have + helicity? For fixed initial polarisation (λ_1, λ_2) , the corresponding R -matrix is 4×4 and given by

$$R_{(h_1, h_2, +), (h'_1, h'_2, +)}^{\lambda_1, \lambda_2} = \mathcal{M}_{h_1, h_2, +}^{\lambda_1, \lambda_2} [\mathcal{M}_{h'_1, h'_2, +}^{\lambda_1, \lambda_2}]^* \equiv R_{g+}^{\lambda_1, \lambda_2}. \quad (19)$$

Eqs. (15)-(17) for this case will follow in the same way.

3 Entanglement

3.1 Phase-space fully integrated

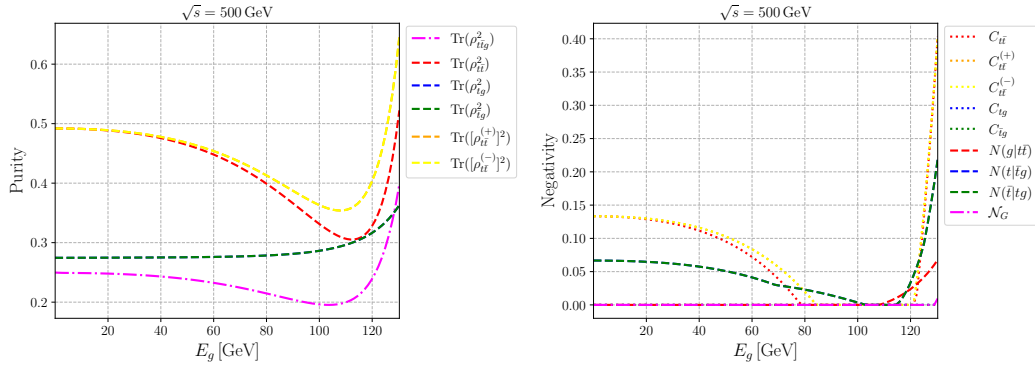


Figure 1: *Left*: Purity, *Right*: Entanglement measures.

3.2 Fixed phase-space

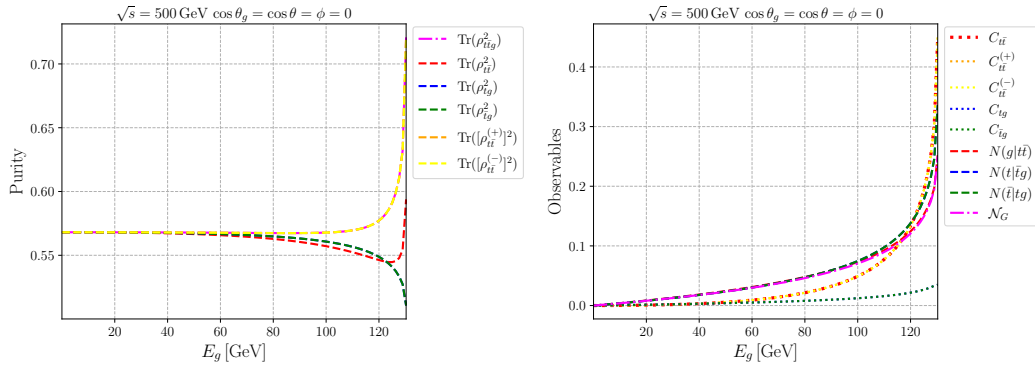


Figure 2: *Left*: Purity, *Right*: Entanglement measures.

3.2.1 $\cos \theta = \phi = 0$

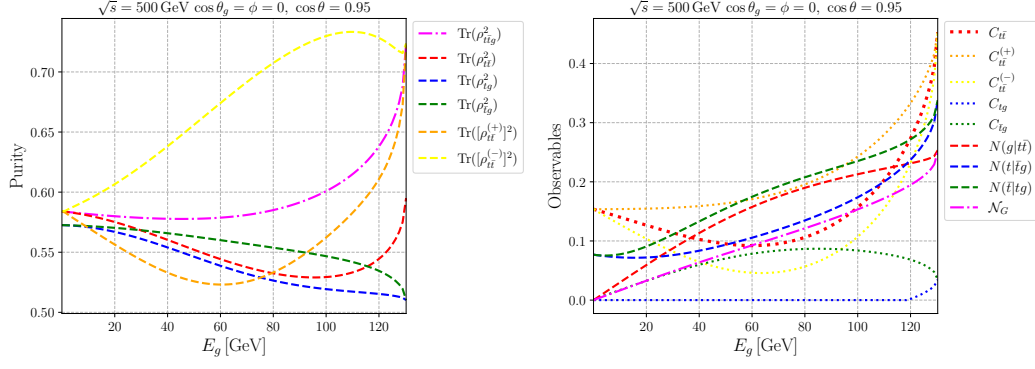


Figure 3: *Left:* Purity, *Right:* Entanglement measures.

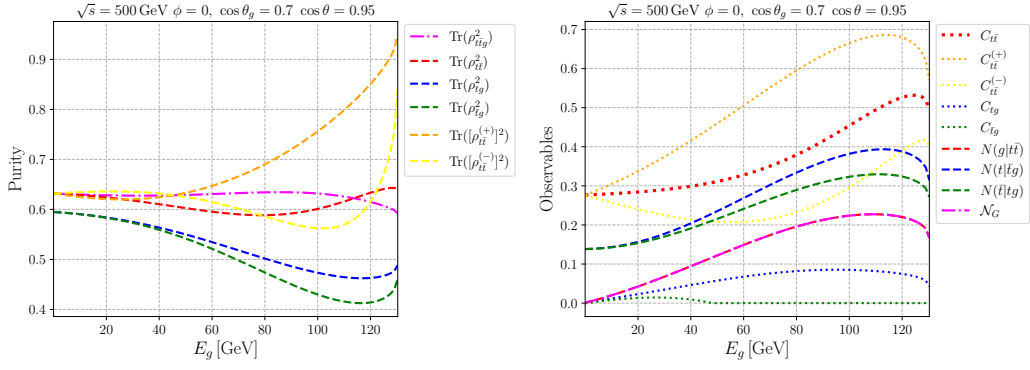


Figure 4: *Left:* Purity, *Right:* Entanglement measures.

3.2.2 $\cos \theta_g = \phi = 0$

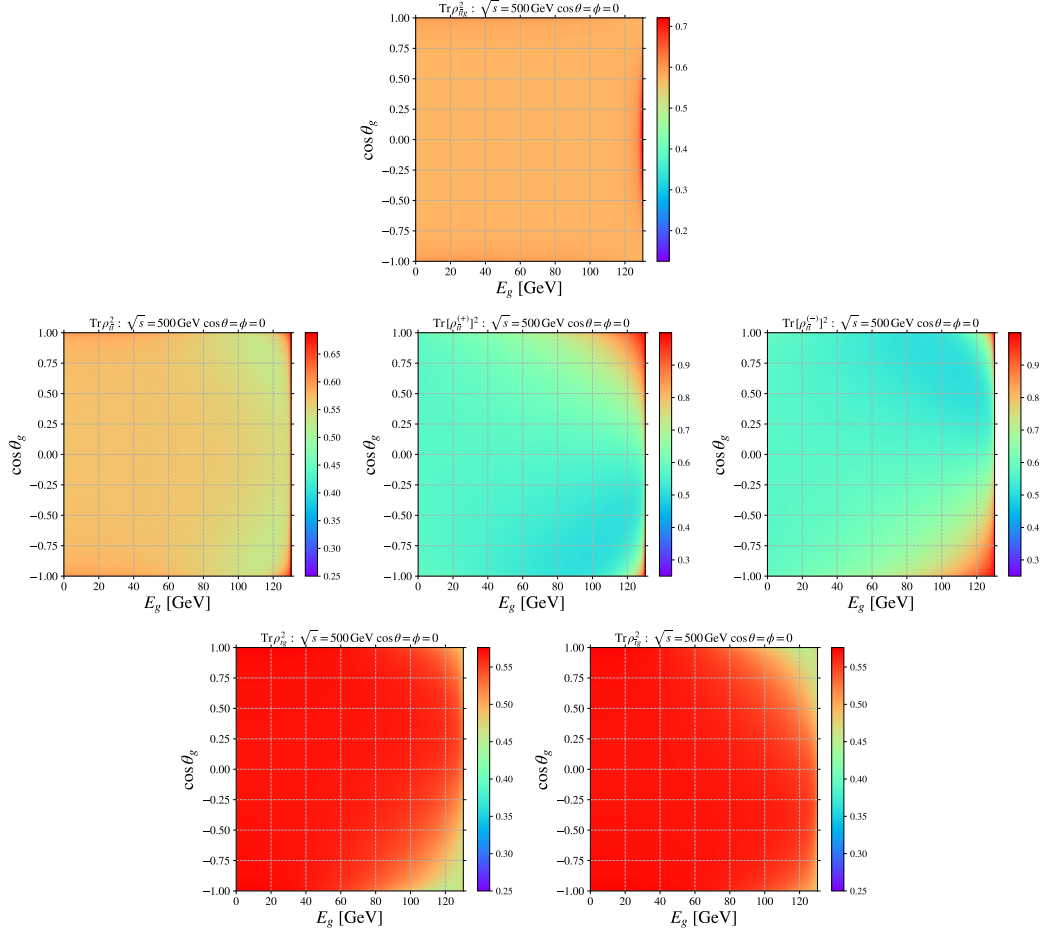


Figure 5: Purity.

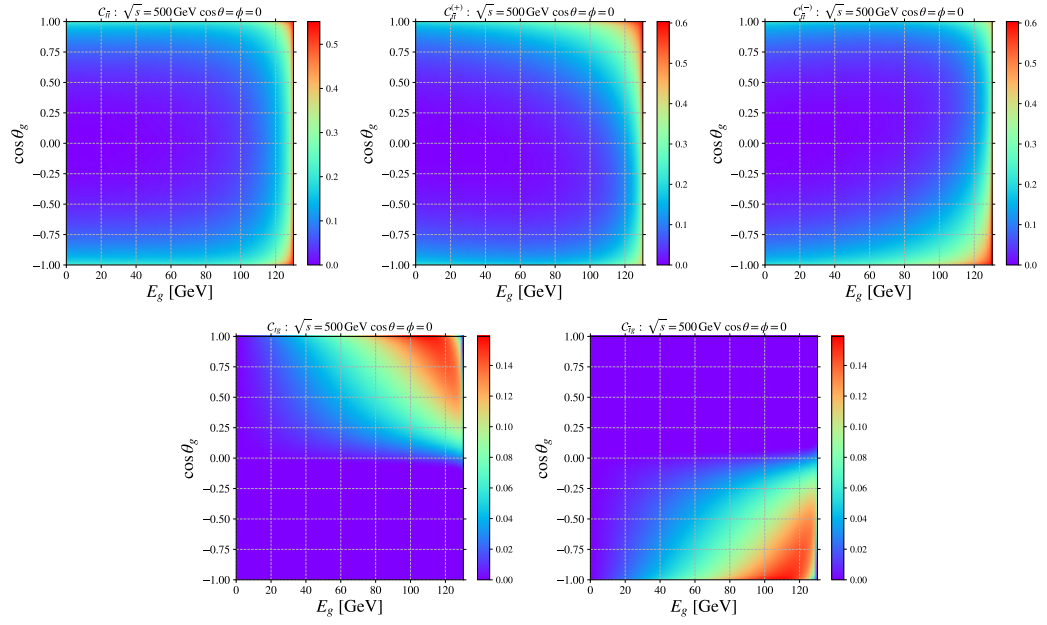


Figure 6: One-to-one concurrence

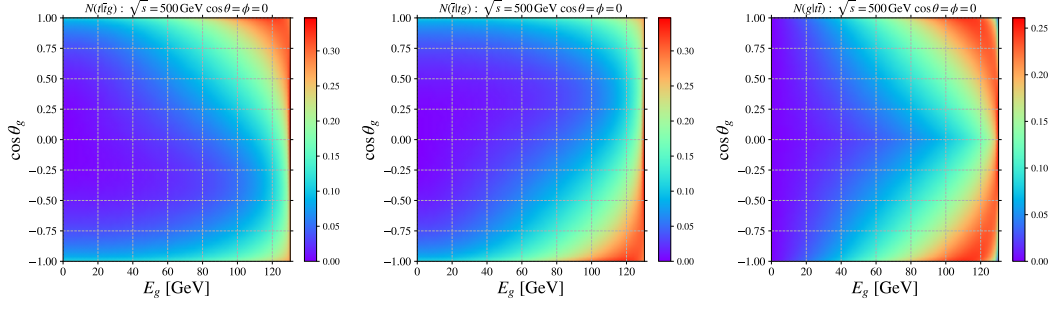


Figure 7: One-to-two negativity

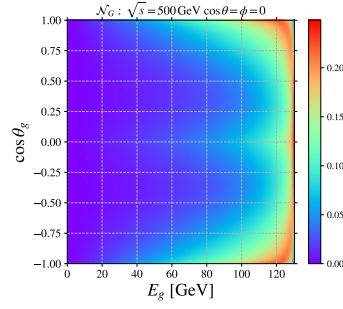


Figure 8: Genuine Multipartite Negativity [4, 5]

3.3 Phase-space partially integrated

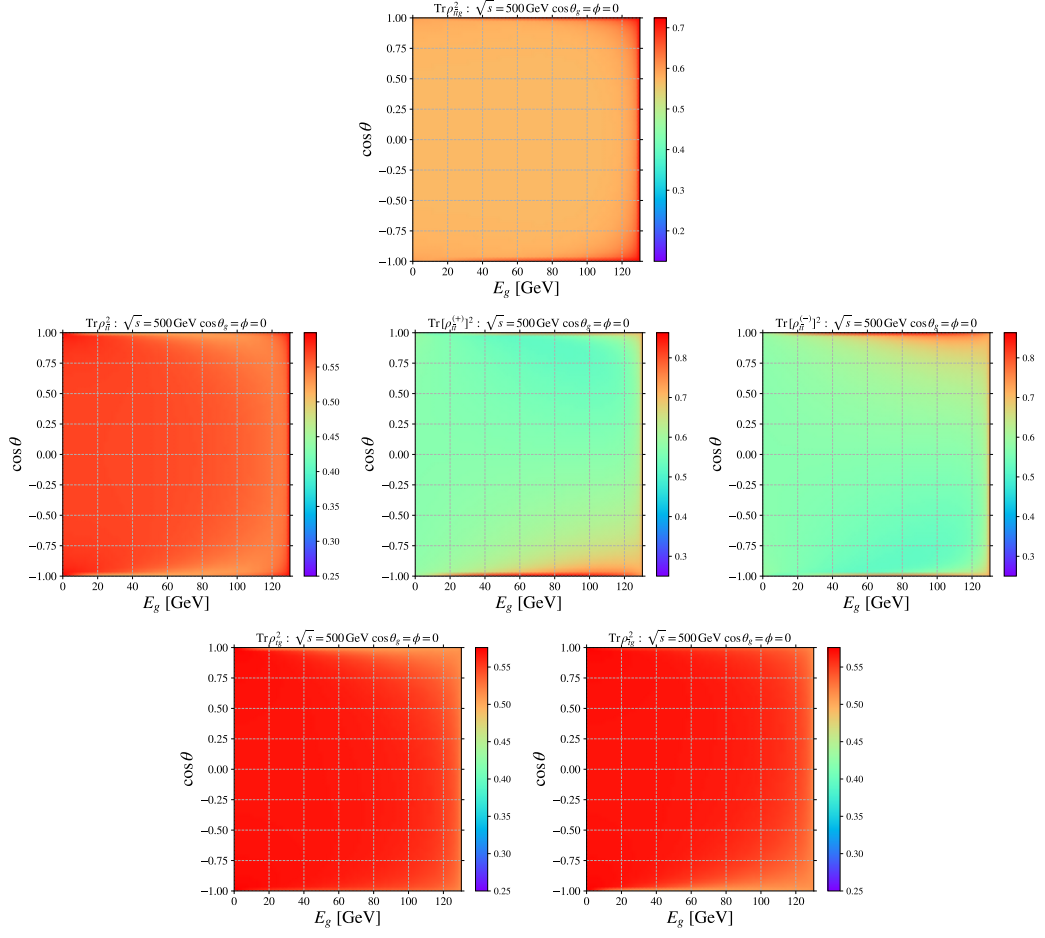


Figure 9: Purity.

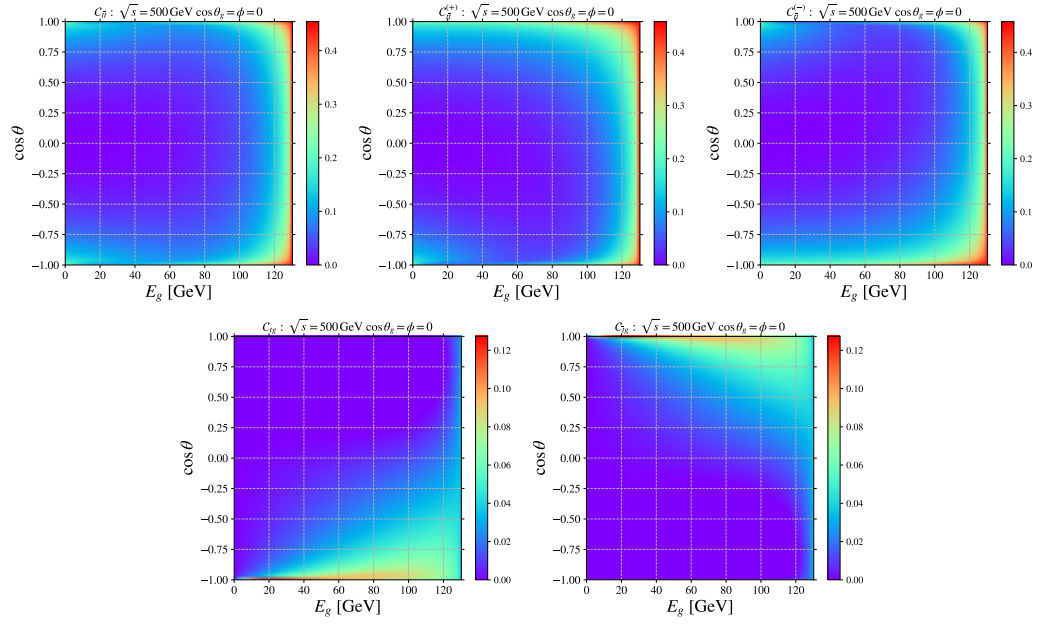


Figure 10: One-to-one concurrence

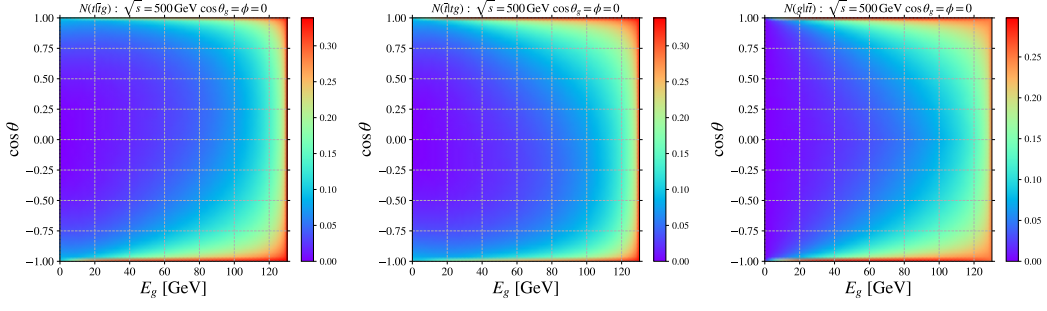


Figure 11: One-to-two negativity

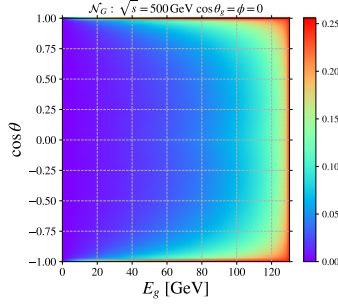


Figure 12: Genuine Multipartite Negativity [4, 5]

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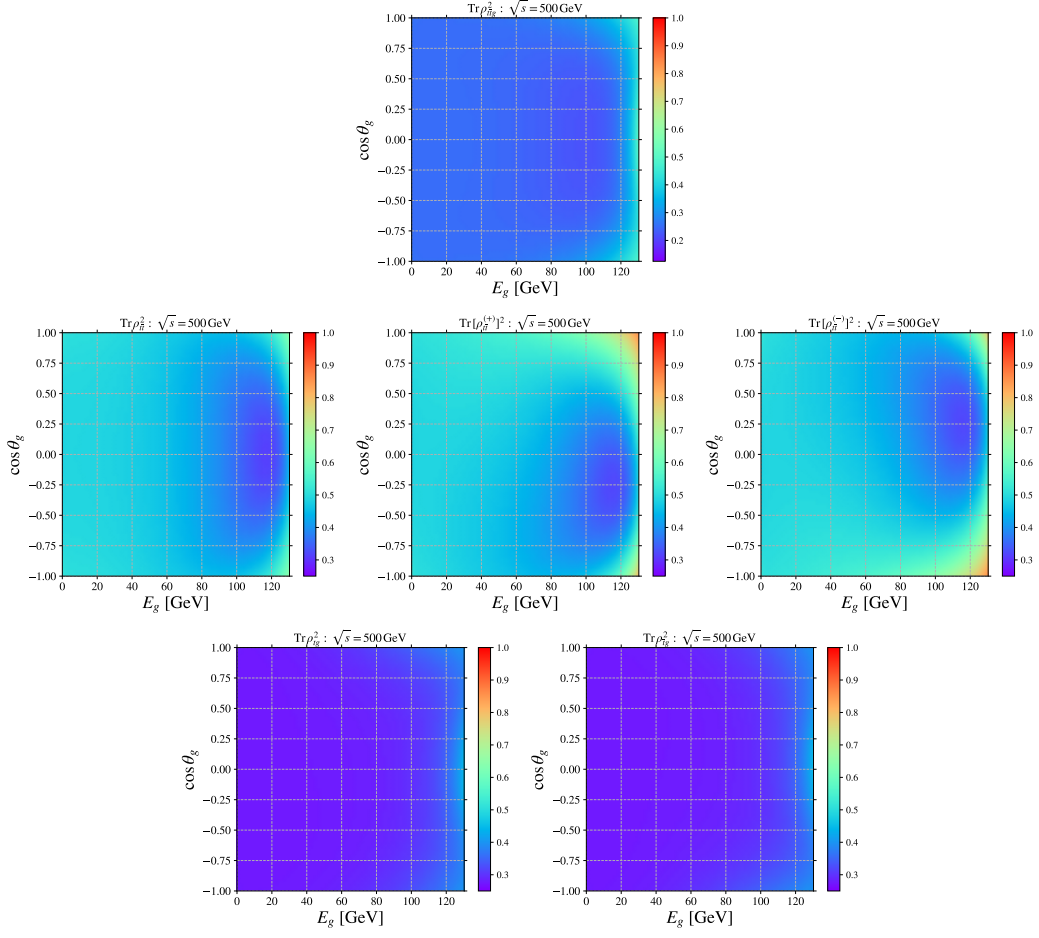


Figure 13: Purity.

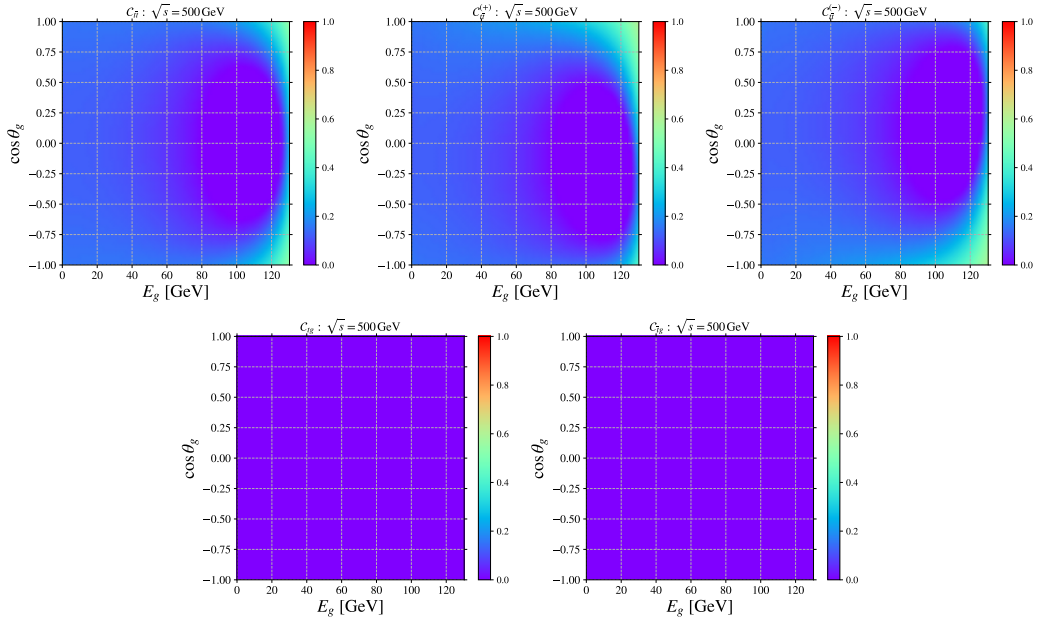


Figure 14: One-to-one concurrence

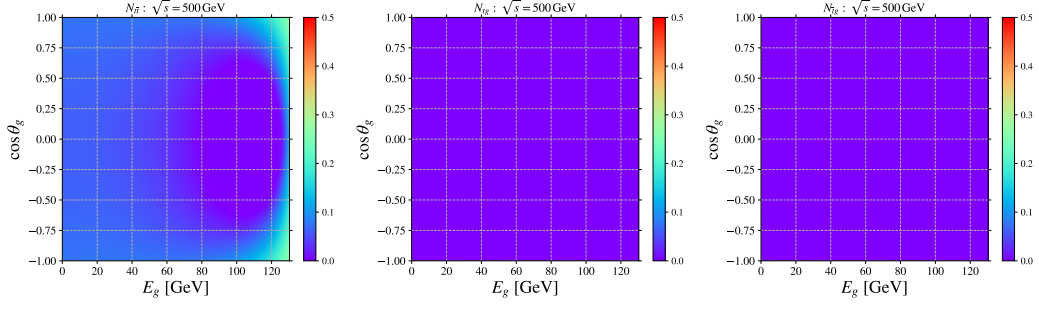


Figure 15: One-to-two negativity

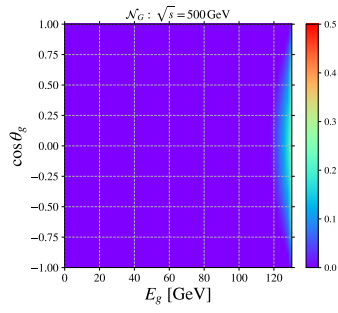


Figure 16: Genuine Multipartite Negativity [4, 5]