$S \to t\bar{t}g$ entanglement

Let's consider the process $\phi \to t\bar{t}g$, in which a massive scalar boson of momentum q, with $q^2 = m_\phi^2 = s$, decays into a $t\bar{t}$ pair plus a gluon. The momenta are assigned as in the following diagram.

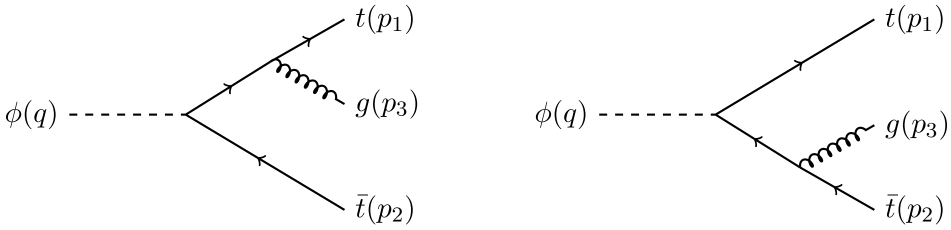

Figure 1: $S \to t\bar{t}g$ diagrams.

Up to the proportionality constant, the amplitudes for the fixed helicities, h_1, h_2, h_3, are given by

$$\mathcal{M}_{h_1,h_2,h_3} \;\propto\; \sqrt{2} \cdot \bar{u}(p_1, h_1) \left[\gamma^\mu \frac{\not{p}_1 + \not{p}_3 + m_t}{2\,(p_1 \cdot p_3)} + \frac{-\not{p}_2 - \not{p}_3 + m_t}{2\,(p_2 \cdot p_3)} \gamma^\mu \right] v(p_2, h_2)\, \epsilon^*_\mu(p_3, h_3)\,. \quad (1)$$

The final state in the helicity space, $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, is given by

$$|\psi\rangle \;=\; \frac{1}{\mathcal{N}} \sum_{h_1,h_2,h_3} \mathcal{M}_{h_1,h_2,h_3} |h_1, h_2, h_3\rangle \quad (2)$$

with

$$\mathcal{N}^2 \;=\; \sum_{h_1,h_2,h_3} |\mathcal{M}_{h_1,h_2,h_3}|^2\,. \quad (3)$$

1 Kimematics

We work in the $t\bar{t}$ center-of-mass frame, defined by

$$p_1 + p_2 = \left(m_{t\bar{t}},\, \vec{0}\right). \quad (4)$$

In this frame, we choose to align the z-axis with the spatial momentum of the top quark, so that

$$p_1 = \left(\frac{m_{t\bar{t}}}{2},\, 0,\, 0,\, P\right), \qquad p_2 = \left(\frac{m_{t\bar{t}}}{2},\, 0,\, 0,\, -P\right), \qquad P = \frac{1}{2}\sqrt{m_{t\bar{t}}^2 - 4m_t^2}\,. \quad (5)$$

Similarly, we choose the x-axis so that the gluon momentum is fully contained in the xz-plane, namely

$$p_3 = E_g\,(1, \sin\theta, 0, \cos\theta), \quad (6)$$

for a polar angle θ (in the $t\bar{t}$ rest frame) between top-quark and gluon. Using momentum conservation, $q = p_1 + p_2 + p_3$, the scalar boson momentum can be written as

$$q = (m_{t\bar{t}} + E_g,\; E_g \sin\theta,\; 0,\; E_g \cos\theta), \quad (7)$$

which implies $\boldsymbol{q} = \boldsymbol{p}_3$. The invariant mass of the scalar boson is fixed by $q^2 = s$, from which we obtain

$$m_{t\bar{t}} = \sqrt{E_g^2 + s} - E_g\,. \quad (8)$$

The kinematic range of E_g, in the $t\bar{t}$ rest frame, follows from the bounds on $m_{t\bar{t}}$. The maximum gluon energy is reached when the $t\bar{t}$ system is produced at threshold, $m_{t\bar{t}}^{\min} = 2m_t$, which yields

$$E_g^{\max} = \frac{s - 4m_t^2}{4m_t}\,. \tag{9}$$

Conversely, the lower bound corresponding to the soft-gluon limit $E_g \to 0$, is found maximizing the quarks invariant mass $m_{t\bar{t}}^{\max} = \sqrt{s}$. Therefore, we can equivalently parametrize the kinematics in terms of E_g or $m_{t\bar{t}}$,

$$E_g \in \left[0,\ \frac{s - 4m_t^2}{4m_t}\right] \quad \Longleftrightarrow \quad m_{t\bar{t}} \in \left[2m_t,\ \sqrt{s}\,\right]. \tag{10}$$

However, in our framework we prefer the use of E_g, since it directly quantifies the influence of the emitted gluon (which we interpret as the environment) on the $t\bar{t}$ system. The gluon polarization is

$$\epsilon(p_3, h_3) = \frac{1}{\sqrt{2}}\,(0,\, \cos\theta,\, h_3 i,\, -\sin\theta), \tag{11}$$

which is derived from $\frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$ by rotating about y-axis of an angle θ.

2 Helicity amplitudes

To compute the helicity amplitudes, we use

$$\bar{u}(h_1) = \begin{pmatrix} \omega_{h_1}\chi_{h_1}^\dagger, & \omega_{-h_1}\chi_{h_1}^\dagger \end{pmatrix}, \qquad v(h_2) = \begin{pmatrix} -h_2\omega_{h_2}\chi_{h_2}, \\ h_2\omega_{-h_2}\chi_{h_2} \end{pmatrix}, \tag{12}$$

where

$$\omega_h \equiv \sqrt{\frac{m_{t\bar{t}}}{2} + hP}, \qquad \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{13}$$

and

$$\not{\epsilon}^*(h_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} & -[\epsilon_{h_3}] \\ [\epsilon_{h_3}] & \end{pmatrix}, \qquad [\epsilon_{h_3}] = \begin{pmatrix} -s_\theta & c_\theta - h_3 \\ c_\theta + h_3 & s_\theta \end{pmatrix}, \tag{14}$$

$$(\not{p}_1 + \not{p}_3) = \begin{pmatrix} & \Omega_-^{(1)} \\ \Omega_+^{(1)} & \end{pmatrix}, \qquad \Omega_\pm^{(1)} = \begin{pmatrix} \frac{m_{t\bar{t}}}{2} + E_g \pm (E_g c_\theta + P) & \pm E_g s_\theta \\ \pm E_g s_\theta & \frac{m_{t\bar{t}}}{2} + E_g \mp (E_g c_\theta + P) \end{pmatrix}, \tag{15}$$

$$(\not{p}_2 + \not{p}_3) = \begin{pmatrix} & \Omega_-^{(2)} \\ \Omega_+^{(2)} & \end{pmatrix}, \qquad \Omega_\pm^{(2)} = \begin{pmatrix} \frac{m_{t\bar{t}}}{2} + E_g \pm (E_g c_\theta - P) & \pm E_g s_\theta \\ \pm E_g s_\theta & \frac{m_{t\bar{t}}}{2} + E_g \mp (E_g c_\theta - P) \end{pmatrix}, \tag{16}$$

$$\begin{aligned}
\Omega_p &\equiv \frac{1}{2}(\Omega_+^{(1)} + \Omega_-^{(1)}) = \frac{1}{2}(\Omega_+^{(2)} + \Omega_-^{(2)}) = (\frac{m_{t\bar{t}}}{2} + E_g) \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \\
\Omega_m^{(1)} &\equiv \frac{1}{2}(\Omega_+^{(1)} - \Omega_-^{(1)}) = \begin{pmatrix} E_g c_\theta + P & E_g s_\theta \\ E_g s_\theta & -(E_g c_\theta + P) \end{pmatrix}, \\
\Omega_m^{(2)} &\equiv \frac{1}{2}(\Omega_+^{(2)} - \Omega_-^{(2)}) = \begin{pmatrix} E_g c_\theta - P & E_g s_\theta \\ E_g s_\theta & -(E_g c_\theta - P) \end{pmatrix}.
\end{aligned} \tag{17}$$

The part of amplitude that is proportional to m_t is given by

$$\sqrt{2}\cdot \bar{u}(h_1)\not{\epsilon}^*(h_3)v(h_2) = -h_2(\omega_{h_1}\omega_{-h_2} + \omega_{-h_1}\omega_{h_2})\chi_{h_1}^\dagger[\epsilon_{h_3}]\chi_{h_2} \tag{18}$$

$$-h_2(\omega_{h_1}\omega_{-h_2}+\omega_{-h_1}\omega_{h_2}) \equiv W_{h_1h_2} = \begin{cases} -2m_t & :++ \\ +m_{t\bar{t}} & :+- \\ -m_{t\bar{t}} & :-+ \\ +2m_t & :-- \end{cases} \tag{19}$$

$$\chi^\dagger_{h_1}[\epsilon_{h_3}]\chi_{h_2} \equiv X_{h_1h_2h_3} = \begin{cases} -s_\theta & :+++ \\ -s_\theta & :++- \\ c_\theta - 1 & :+-+ \\ c_\theta + 1 & :+-- \\ c_\theta + 1 & :-++ \\ c_\theta - 1 & :-+- \\ s_\theta & :--+ \\ s_\theta & :--- \end{cases} \tag{20}$$

The part of amplitude that is not proportional to m_t is given by

$$\sqrt{2}\cdot\bar{u}(h_1)\not{\epsilon}^*(h_3)(\not{p}_1+\not{p}_3)v(h_2) \; = \; h_2\left[\omega_{h_1}\omega_{h_2}\chi^\dagger_{h_1}[\epsilon_{h_3}\Omega^{(1)}_+]\chi_{h_2} + \omega_{-h_1}\omega_{-h_2}\chi^\dagger_{h_1}[\epsilon_{h_3}\Omega^{(1)}_-]\chi_{h_2}\right] \equiv Y_{h_1h_2h_3}$$

$$Y_{h_1h_2h_3} = \begin{cases} m_{t\bar{t}}[\epsilon_{h_3}\Omega_p]_{11} + 2P[\epsilon_{h_3}\Omega^{(1)}_m]_{11} & :++h_3 \\ -2m_t[\epsilon_{h_3}\Omega_p]_{12} & :+-h_3 \\ 2m_t[\epsilon_{h_3}\Omega_p]_{21} & :-+h_3 \\ -m_{t\bar{t}}[\epsilon_{h_3}\Omega_p]_{22} + 2P[\epsilon_{h_3}\Omega^{(1)}_m]_{22} & :--h_3 \end{cases} \tag{21}$$

$$\sqrt{2}\cdot\bar{u}(h_1)(\not{p}_2+\not{p}_3)\not{\epsilon}^*(h_3)v(h_2) \; = \; -h_2\left[\omega_{h_1}\omega_{h_2}\chi^\dagger_{h_1}[\Omega^{(2)}_-\epsilon_{h_3}]\chi_{h_2} + \omega_{-h_1}\omega_{-h_2}\chi^\dagger_{h_1}[\Omega^{(2)}_+\epsilon_{h_3}]\chi_{h_2}\right] \equiv Z_{h_1h_2h_3}$$

$$Z_{h_1h_2h_3} = \begin{cases} -m_{t\bar{t}}[\Omega_p\epsilon_{h_3}]_{11} + 2P[\Omega^{(2)}_m\epsilon_{h_3}]_{11} & :++h_3 \\ 2m_t[\Omega_p\epsilon_{h_3}]_{12} & :+-h_3 \\ -2m_t[\Omega_p\epsilon_{h_3}]_{21} & :-+h_3 \\ m_{t\bar{t}}[\Omega_p\epsilon_{h_3}]_{22} + 2P[\Omega^{(2)}_m\epsilon_{h_3}]_{22} & :--h_3 \end{cases} \tag{22}$$

Writing $2(p_{1(2)}\cdot p_3) = E_g(m_{t\bar{t}} \pm 2P\cos\theta)$, the total helicity amplitudes are given by

$$\mathcal{M}_{h_1h_2h_3} \propto \frac{M^{(t)}_{h_1h_2h_3}}{E_g(m_{t\bar{t}} - 2P\cos\theta)} + \frac{M^{(\bar{t})}_{h_1h_2h_3}}{E_g(m_{t\bar{t}} + 2P\cos\theta)} \tag{23}$$

with

$$\begin{aligned} M^{(t)}_{h_1h_2h_3} &= m_t W_{h_1h_2} X_{h_1h_2h_3} + Y_{h_1h_2h_3}\,, \\ M^{(\bar{t})}_{h_1h_2h_3} &= m_t W_{h_1h_2} X_{h_1h_2h_3} - Z_{h_1h_2h_3}\,. \end{aligned} \tag{24}$$

It happens that these numerators are equal:

$$M_{h_1h_2h_3} \;\equiv\; M^{(t)}_{h_1h_2h_3} \;=\; M^{(\bar{t})}_{h_1h_2h_3} \;=\; \begin{cases} A_+\cdot\sin\theta, & :+++ \\ A_-\cdot\sin\theta, & :++- \\ -B\,(-1+\cos\theta)\,, & :+-+ \\ -B\,(1+\cos\theta)\,, & :+-- \\ B\,(1+\cos\theta)\,, & :-++ \\ B\,(-1+\cos\theta)\,, & :-+- \\ A_-\cdot\sin\theta, & :--+ \\ A_+\cdot\sin\theta. & :--- \end{cases} \tag{25}$$

with

$$A_{\pm} \;=\; \left[4m_t^2 - m_{t\bar{t}}^2 - E_g\left(m_{t\bar{t}} \pm 2P\right)\right], \qquad B \;=\; 2E_g m_t \tag{26}$$

Therefore, the helicity amplitudes are be written as

$$\mathcal{M}_{h_1h_2h_3} \;\propto\; \frac{2m_{t\bar{t}}}{E_g(m_{t\bar{t}}^2 \sin^2\theta + 4m_t^2\cos^2\theta)} \cdot M_{h_1h_2h_3}\,. \tag{27}$$

3 Entanglement at a fixed phase-space point

In this section, we fix E_g and θ and analytically compute various entanglement quantities of the $t\bar{t}g$ system.

3.1 Entanglement between t and $\bar{t}$

3.1.1 Projecting the gluon polarisation

If the gluon polarisation is detected as $h_3 = +$ or $h_3 = -$, the total state $|\psi\rangle$ is collapses to the $t\bar{t}$ state

$$\begin{aligned}
|\psi_{g+}\rangle \;&=\; \frac{1}{N}\left[A_+ \cdot \sin\theta|++\rangle \,+\, B(1-\cos\theta)|+-\rangle \,+\, B(1+\cos\theta)|-+\rangle \,+\, A_- \cdot \sin\theta|--\rangle\right],\\
|\psi_{g-}\rangle \;&=\; \frac{1}{N}\left[A_- \cdot \sin\theta|++\rangle \,-\, B(1+\cos\theta)|+-\rangle \,-\, B(1-\cos\theta)|-+\rangle \,+\, A_+ \cdot \sin\theta|--\rangle\right],\\
|N|^2 \;&=\; 4B^2 + (A_+^2 + A_-^2 - 2B^2)\sin^2\theta\,.
\end{aligned} \tag{28}$$

respectively. We can see that $|\psi_{g+}\rangle$ and $|\psi_{g-}\rangle$ are spin-flipped partners:

$$|\widetilde{\psi}_{g\pm}\rangle \equiv (\sigma_y \otimes \sigma_y)|\psi_{g\pm}^*\rangle = |\psi_{g\mp}\rangle \tag{29}$$

For pure states, the concurrence is defined by $\mathcal{C}(|\psi\rangle) = |\langle\psi|\widetilde{\psi}\rangle|$. This immediately gives

$$\begin{aligned}
\mathcal{C}(|\psi_{g+}\rangle) \;=\; \mathcal{C}(|\psi_{g-}\rangle) \;=\; |\langle\psi_{g+}|\psi_{g-}\rangle| \;&=\; \frac{2(A_+A_- - B^2)\sin^2\theta}{4B^2 + (A_+^2 + A_-^2 - 2B^2)\sin^2\theta}\\
&=\; \frac{\left(m_{t\bar{t}}^2 - 4m_t^2\right)\left(m_{t\bar{t}}\left(m_{t\bar{t}} + 2E_g\right) - 4m_t^2\right)\sin^2\theta}{\left(m_{t\bar{t}}^2 - 4m_t^2\right)\left(m_{t\bar{t}}(m_{t\bar{t}} + 2E_g) - 4m_t^2\right)\sin^2\theta + E_g^2(8m_t^2\cos^2\theta + 2m_{t\bar{t}}^2\sin^2\theta)}\,.
\end{aligned} \tag{30}$$

Clearly, the $t\bar{t}$ entanglement is the same between $|\psi_{g+}\rangle$ and $|\psi_{g-}\rangle$. For $E_g \to 0$, we can easily see that $\mathcal{C}(|\psi_{g+}\rangle) = \mathcal{C}(|\psi_{g-}\rangle) \to 1$ for all θ, as $A_+ = A_-$ and $B = 0$ in this limit. This result can be also checked by using the well-known formula, $\mathcal{C}(|\psi\rangle) = 2|ps - qr|$ for $|\psi\rangle = p|++\rangle + q|+-\rangle + r|-+\rangle + s|--\rangle$.

3.1.2 Tracing out the gluon polarisation

If we do not measure the gluon polarisation and focus only on the $t\bar{t}$ spin space, the $t\bar{t}$ state is represented by the density matrix

$$\rho_{t\bar{t}} \;=\; \sum_{h_3=\pm}\langle h_3|\psi\rangle\langle\psi|h_3\rangle \;=\; \frac{1}{2}\left[|\psi_{g+}\rangle\langle\psi_{g+}| \,+\, |\psi_{g-}\rangle\langle\psi_{g-}|\right] \tag{31}$$

The concurrence $\mathcal{C}_{t\bar{t}} = \mathcal{C}(\rho_{t\bar{t}})$ is given by

$$\mathcal{C}_{t\bar{t}} \;=\; \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4})\,. \tag{32}$$

where λ_i $(i = 1, \dots, 4)$ are the eigenvalues of $\rho_{t\bar{t}} \cdot \tilde{\rho}_{t\bar{t}}$ in the descendent order, where $\tilde{\rho}_{t\bar{t}} \equiv (\sigma_y \otimes \sigma_y)\rho^*_{t\bar{t}}(\sigma_y \otimes \sigma_y)$. Due to the relation Eq. (29), we have $\tilde{\rho}_{t\bar{t}} = \rho_{t\bar{t}}$ in our case. To compute the eigenvalues, we go to the orthogonal basis

$$|\pm\rangle \equiv \frac{|\psi_{g+}\rangle \pm |\psi_{g-}\rangle}{\sqrt{2(1 \pm \beta)}}, \qquad \beta \equiv \langle\psi_{g+}|\psi_{g-}\rangle = \mathcal{C}(|\psi_{g\pm}\rangle), \tag{33}$$

where we used the fact that $\langle\psi_{g+}|\psi_{g-}\rangle \in \mathbb{R}$. In this basis, $\rho_{t\bar{t}}$ is written by

$$\rho_{t\bar{t}} = \frac{1}{2}\left[\,(1+\beta)|+\rangle\langle+| + (1-\beta)|-\rangle\langle-|\,\right] \tag{34}$$

and the two nonzero eigenvalues of $\rho^2_{t\bar{t}}$ are

$$\lambda_1 = \frac{(1+\beta)^2}{4}, \qquad \lambda_2 = \frac{(1-\beta)^2}{4}. \tag{35}$$

We therefore find

$$\mathcal{C}_{t\bar{t}} = \sqrt{\lambda_1} - \sqrt{\lambda_2} = \beta = \mathcal{C}(|\psi_{g\pm}\rangle). \tag{36}$$

Namely, the $t\bar{t}$ entanglement after tracing out the gluon polarisation is the same as those after projecting the gluon polarisation to $+$ and $-$. The expression of $\mathcal{C}(|\psi_{g\pm}\rangle)$ is given in (30).

We can easily compute the purity $\gamma_{t\bar{t}}$ of $\rho_{t\bar{t}}$ as:

$$\gamma_{t\bar{t}} \equiv \mathrm{Tr}[\rho^2_{t\bar{t}}] = \lambda_1 + \lambda_2 = \frac{1}{2}(1+\beta^2) = \frac{1}{2}(1+\mathcal{C}^2_{t\bar{t}}). \tag{37}$$

One can also write

$$\mathcal{C}_{t\bar{t}} = \sqrt{2\gamma_{t\bar{t}} - 1}. \tag{38}$$

In this expression we can see that the $t\bar{t}$ pair is maximally entangled when $\rho_{t\bar{t}}$ is pure and disentangled when $\rho_{t\bar{t}}$ is maximally mixed ($\gamma_{t\bar{t}} = \frac{1}{2}$).

3.2 Entanglement between $t\bar{t}$ and g

For a fixed phase-space point (E_g and θ), the final state $|\psi\rangle$ of the $t\bar{t}g$ system is pure. In this case, the concurrence between $t\bar{t}$ and g can be computed as

$$\mathcal{C}_{g(t\bar{t})} = \sqrt{2(1 - \mathrm{Tr}[\rho^2_{t\bar{t}}])}, \qquad \rho_{t\bar{t}} \equiv \mathrm{Tr}_g|\psi\rangle\langle\psi|. \tag{39}$$

From λ_1 and λ_2 in Eq. (35), we can immediately find

$$\mathcal{C}_{g(t\bar{t})} = \sqrt{1 - \mathcal{C}^2_{t\bar{t}}}. \tag{40}$$

We see that as $\mathcal{C}_{t\bar{t}}$ decreases, $\mathcal{C}_{g(t\bar{t})}$ monotonically increases.

3.3 Entanglement for other partitions

Let us consider the projected state where the top-quark helicity is $h_1 = \pm$:

$$\begin{aligned}
|\psi_{t+}\rangle &= \frac{1}{N}\left[A_+ \sin\theta|++\rangle + A_- \sin\theta|+-\rangle + B(1-\cos\theta)|-+\rangle - B(1+\cos\theta)|--\rangle\right], \\
|\psi_{t-}\rangle &= \frac{1}{N}\left[B(1+\cos\theta)|++\rangle - B(1-\cos\theta)|+-\rangle + A_- \sin\theta|-+\rangle + A_+ \sin\theta|--\rangle\right],
\end{aligned} \tag{41}$$

where N is given in Eq. (28). Interestingly, these states are orthogonal

$$\langle\psi_{t+}|\psi_{t-}\rangle = 0. \tag{42}$$

By tracing out the top helicities, the density operator for the $\bar{t}g$ subsystem is

$$\rho_{\bar{t}g} \,=\, \frac{1}{2}\left(|\psi_{t+}\rangle\langle\psi_{t+}| \,+\, |\psi_{t-}\rangle\langle\psi_{t-}| \right) \tag{43}$$

Since $|\psi_{t+}\rangle$ and $|\psi_{t-}\rangle$ are orthogonal, this is maximally mixed state:

$$\gamma_{\bar{t}g} \,=\, \mathrm{Tr}[\rho^2_{\bar{t}g}] \,=\, \frac{1}{2}\,. \tag{44}$$

We conclude $\bar{t}$ and g are not entangled:

$$\mathcal{C}_{\bar{t}g} \,=\, 0 \tag{45}$$

The entanglement between t vs $\bar{t}g$ can also be calculated straightforwardly. Eq. (43) implies $\mathrm{Tr}[\rho^2_{\bar{t}g}] = \frac{1}{2}$, which leads to

$$\mathcal{C}_{t(\bar{t}g)} \,=\, \sqrt{2(1 - \mathrm{Tr}[\rho^2_{\bar{t}g}])} \,=\, 1\,. \tag{46}$$

Namely, t is maximally entangled with the subsystem $\bar{t}g$.

Due to the symmetry between t and $\bar{t}$, we also have

$$\mathcal{C}_{tg} \,=\, 0, \qquad \mathcal{C}_{\bar{t}(tg)} \,=\, 1\,. \tag{47}$$

3.4 The monogamy relation

The one-to-two concurrence and one-to-one concurrence are connected by the monogamy relation:

$$\begin{aligned}
\mathcal{C}^2_{g(t\bar{t})} &= \mathcal{C}^2_{gt} + \mathcal{C}^2_{g\bar{t}} + \tau, \\
\mathcal{C}^2_{t(\bar{t}g)} &= \mathcal{C}^2_{t\bar{t}} + \mathcal{C}^2_{tg} + \tau, \\
\mathcal{C}^2_{\bar{t}(tg)} &= \mathcal{C}^2_{t\bar{t}} + \mathcal{C}^2_{\bar{t}g} + \tau,
\end{aligned} \tag{48}$$

where τ is called the 3-tangle. Since $\mathcal{C}_{tg} \,=\, \mathcal{C}_{\bar{t}g} \,=\, 0$, from the first equation we conclude

$$\tau \,=\, \mathcal{C}^2_{g(t\bar{t})} \,=\, 1 - \mathcal{C}^2_{t\bar{t}}\,. \tag{49}$$

This is consistent with the second and third equations by noting $\mathcal{C}_{t(\bar{t}g)} = \mathcal{C}_{\bar{t}(tg)} = 1$.

3.5 The genuine tripartite entanglement

The genuine tripartite entanglement F_3 is quantified by the area of the concurrence triangle, whose three sides are given by $\mathcal{C}_{g(t\bar{t})}$, $\mathcal{C}_{t(\bar{t}g)}$ and $\mathcal{C}_{\bar{t}(tg)}$. The explicit formula is

$$F_3 \,=\, \left[\tfrac{16}{3}Q(Q - \mathcal{C}_{g(t\bar{t})})(Q - \mathcal{C}_{t(\bar{t}g)})(Q - \mathcal{C}_{\bar{t}(tg)})\right]^{\frac{1}{2}}\,, \tag{50}$$

with $Q = \frac{1}{2}[\mathcal{C}_{g(t\bar{t})} + \mathcal{C}_{t(\bar{t}g)} + \mathcal{C}_{\bar{t}(tg)}]$. Substituting the above results, we obtain

$$F_3 \,=\, \mathcal{C}_{g(t\bar{t})}\sqrt{\frac{4 - \mathcal{C}^2_{g(t\bar{t})}}{3}} \,=\, \sqrt{(1 - \mathcal{C}^2_{t\bar{t}})\left(1 + \frac{\mathcal{C}^2_{t\bar{t}}}{3}\right)}\,. \tag{51}$$

The F_3 is a monotonically increasing function of $\mathcal{C}_{g(t\bar{t})}$ $(0 < \mathcal{C}_{g(t\bar{t})} < 1)$, which takes $F_3 = 0$ at $\mathcal{C}_{g(t\bar{t})} = 0$ and $F_3 = 1$ at $\mathcal{C}_{g(t\bar{t})} = 1$. On the other hand, if expressed as a function of $\mathcal{C}_{t\bar{t}}$, the F_3 is a monotonically *decreasing* function of $\mathcal{C}_{t\bar{t}}$ $(0 < \mathcal{C}_{t\bar{t}} < 1)$. The three qubits are maximally (minimally) entangled, $F_3 = 1$ (0), when $t\bar{t}$ are disentangled $\mathcal{C}_{t\bar{t}} = 0$ (maximally entangled $\mathcal{C}_{t\bar{t}} = 1$)

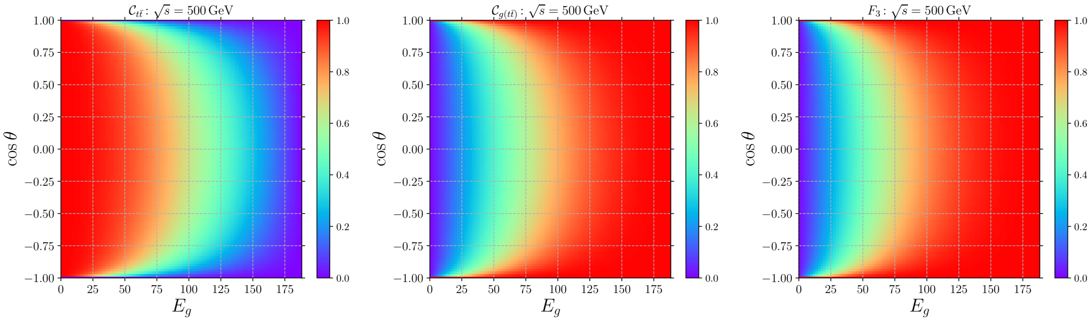

Figure 2: $\mathcal{C}_{t\bar{t}}$, $\mathcal{C}_{g(t\bar{t})}$ and F_3 on the $(E_g, \cos\theta)$ planes.

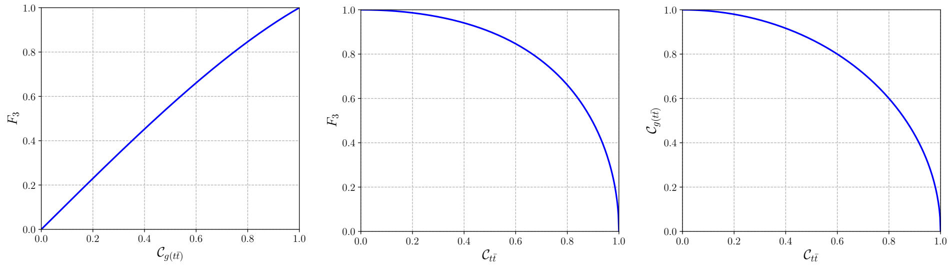

Figure 3: F_3 as functions of $\mathcal{C}_{g(t\bar{t})}$ (left) and $\mathcal{C}_{t\bar{t}}$ (middle). $\mathcal{C}_{g(t\bar{t})}$ as a function of $\mathcal{C}_{t\bar{t}}$ (right).

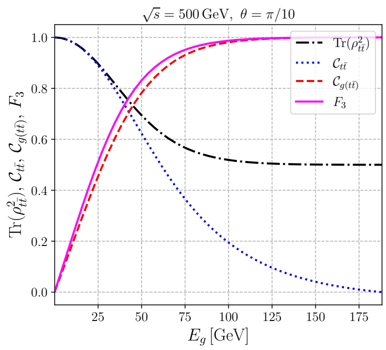

Figure 4: $\mathrm{Tr}(\rho_{t\bar{t}}^2)$, $\mathcal{C}_{t\bar{t}}$, $\mathcal{C}_{g(t\bar{t})}$ and F_3 as a function of E_g for a fixed angle, $\theta = \frac{\pi}{10}$.

3.6 Summary

For a fixed phase-space point (E_g and θ), the following analytical results are obtained:

$$\mathcal{C}(|\psi_{g+}\rangle) = \mathcal{C}(|\psi_{g-}\rangle) = \mathcal{C}_{t\bar{t}} = \frac{\left(m_{t\bar{t}}^2 - 4m_t^2\right)\left(m_{t\bar{t}}\left(m_{t\bar{t}} + 2E_g\right) - 4m_t^2\right)\sin^2\theta}{\left(m_{t\bar{t}}^2 - 4m_t^2\right)\left(m_{t\bar{t}}(m_{t\bar{t}} + 2E_g) - 4m_t^2\right)\sin^2\theta + E_g^2(8m_t^2\cos^2\theta + 2m_{t\bar{t}}^2\sin^2\theta)},$$

$$\mathcal{C}_{tg} = \mathcal{C}_{\bar{t}g} = 0,$$

$$\gamma_{t\bar{t}} = \frac{1 + \mathcal{C}_{t\bar{t}}^2}{2}, \quad \gamma_{tg} = \gamma_{\bar{t}g} = \frac{1}{2}, \tag{52}$$

$$\mathcal{C}_{t(\bar{t}g)} = \mathcal{C}_{\bar{t}(tg)} = 1, \quad \mathcal{C}_{g(t\bar{t})} = \sqrt{1 - \mathcal{C}_{t\bar{t}}^2},$$

$$\tau = \mathcal{C}_{g(t\bar{t})}^2 = 1 - \mathcal{C}_{t\bar{t}}^2, \qquad F_3 = \mathcal{C}_{g(t\bar{t})}\sqrt{\frac{4 - \mathcal{C}_{g(t\bar{t})}^2}{3}} = \sqrt{(1 - \mathcal{C}_{t\bar{t}}^2)\left(1 + \frac{\mathcal{C}_{t\bar{t}}^2}{3}\right)}. \tag{53}$$

4 Entanglement for averaged angle

We consider the entanglement quantities after gluon's angle θ is averaged over.

4.1 Purity

The quantum state of the full $t\bar{t}g$ system is no longer pure after integrating over θ. In Fig. 5, the purity of the full system $t\bar{t}g$ as well as that of three subsystems $t\bar{t}$, tg and $\bar{t}g$ are shown on the left plot. On the right plot, the purity of the $t\bar{t}$ subsystem when the gluon polarisation is projected to $+$ and $-$ are shown. Comparing them with the purity of the full system, we can see that the purity is not lost by projecting the gluon polarisation.

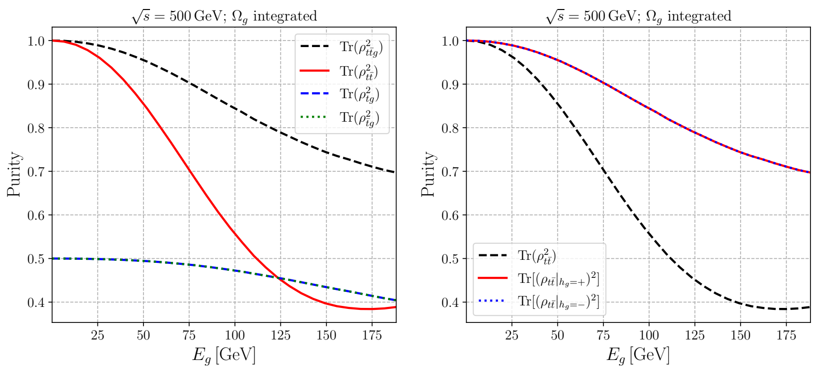

Figure 5: Purity.

4.2 One-to-one entanglement (concurrence)

One-to-one concurrences for various combinations are shown in Fig. 6. The concurrences between t and $\bar{t}$ are plotted on the left plot for three cases: (1) gluon polarisation is averaged over, (2) gluon polarisation is projected to $h_3 = +$, and (3) $h_3 = -$. These three cases give the same concurrence value, as we confirmed in the analysis for a fixed angle. On the right plot, we see that t-g and $\bar{t}$-g are not entangled.

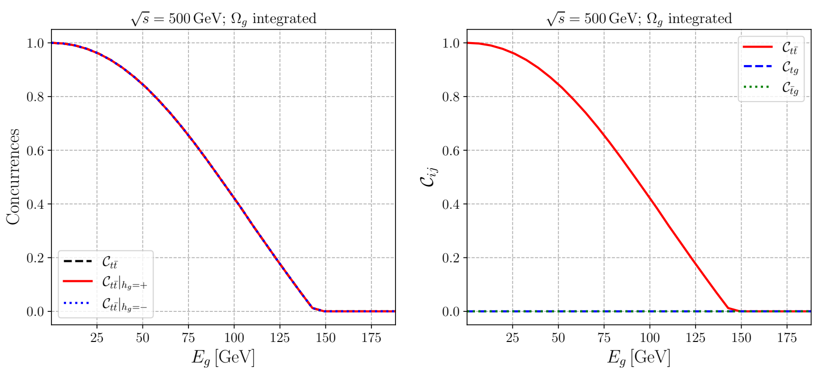

Figure 6: One-to-one concurrence.

4.3 One-to-two entanglement (logarithmic negativity)

For mixed three-qubit (A, B, C) states, one-to-two concurrence cannot be computed in practice. Popular variables to quantify the one-to-other entanglement in arbitrary systems are the negativity and the logarithmic negativity. Both quantities are based on the partial transpose of the density matrix, ρ^{T_i} $(i = A, B, C)$, and the fact that if ρ^{T_i} has at least one negative eigenvalue, the state ρ cannot be written as $\rho = \sum_k p_k \rho_k^{(i)} \otimes \rho_k^{(\text{rest})}$ with $p_k \geq 0$ and $\sum_k p_k = 1$, namely entangled.

The negativity between A and BC is defined by

$$\mathcal{N}_{A|BC} = \frac{||\rho^{T_A}||_1 - 1}{2}, \tag{54}$$

where $||\mathcal{O}||_1 = \mathrm{Tr}\sqrt{\mathcal{O}^\dagger \mathcal{O}}$ is the trace norm or the sum of the singular values of $\mathcal{O}$. Since $\mathrm{Tr}\rho^{T_A} = 1$, $\mathcal{N}_{A|BC}$ can also be computed in terms of the eigenvalues λ_i of ρ^{T_A} as

$$\mathcal{N}_{A|BC} = \sum_i \frac{|\lambda_i| - \lambda_i}{2} = \sum_{\lambda_i < 0} |\lambda_i| \,. \tag{55}$$

The logarithmic negativity is defined by

$$E_N^{A|BC} = \log_2 ||\rho^{T_A}||_1 = \log_2(1 + 2\mathcal{N}_{A|BC}) \,. \tag{56}$$

The logarithmic negativity has particularly nice properties.

- It is an entanglement monotone (non-increasing under LOCC).
- It is additive for a tensor product $E_N^{A|BC}(\rho_1^{ABC} \otimes \rho_2^{ABC}) = E_N^{A|BC}(\rho_1^{ABC}) + E_N^{A|BC}(\rho_2^{ABC})$.
- It gives an upper bound on distillable entanglement.
- It is symmetric under the partial transpose: $E_N^{A|BC} = E_N^{BC|A}$.

However, there exist PPT entangled states, which are entangled (not separable) but have vanishing $\mathcal{N}_{A|BC}$ and $E_N^{A|BC}$.

Fig. 7 show the logarithmic negativities for $g|t\bar{t}$, $t|\bar{t}g$ and $\bar{t}|tg$ partitions as functions of the gluon energy E_g.

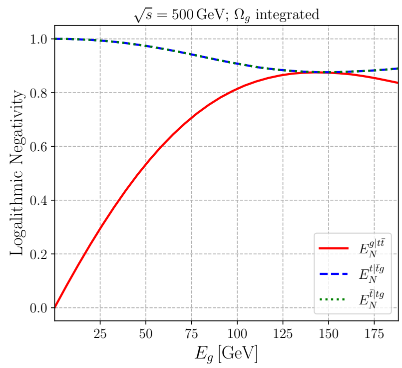

Figure 7: One-to-two logarithmic negativity.