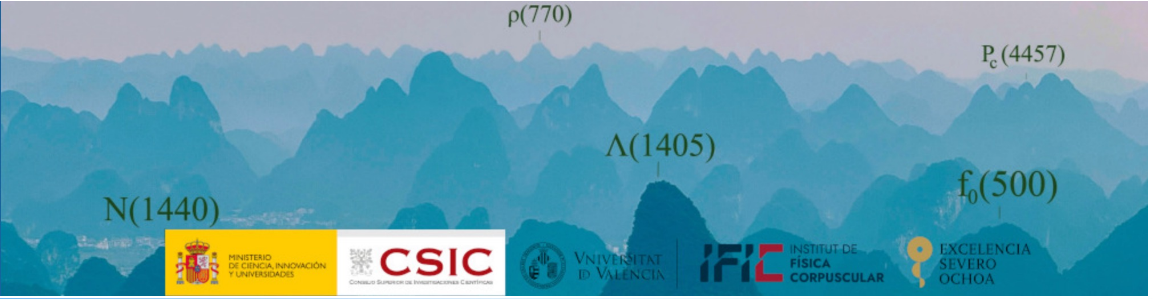


Effective Theories in Hadron and Nuclear Physics
IFIC, 3-5 Sept. 2026

Effective range expansion with the left-hand cut

Feng-Kun Guo

Institute of Theoretical Physics, Chinese Academy of Sciences

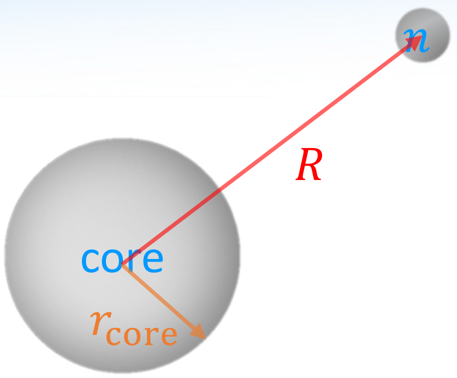
- Neutron-halo scattering — Xu Zhang, Hai-Long Fu, FKG, H.-W. Hammer, PRC 108 (2023) 044304
- $DD^*/D\bar{D}^*$ scattering — Meng-Lin Du et al., PRL 131 (2023) 131903
- Generalized ERE w/ lhc — Meng-Lin Du, FKG, Bing Wu, PRL 135 (2025) 011903;
  W.-J. Wang, B. Wu, M.-L. Du, FKG, PRD 114 (2026) 014003;
  Bo-Yang Liu, B. Wu, J.-W. Fu, M.-L. Du, FKG, U.-G. Meißner, arXiv:2602.03115

Neutron-halo scattering

- One-neutron halo nuclei

  examples: $^{11}\mathrm{Be}$, $^{15}\mathrm{C}$, $^{19}\mathrm{C}$

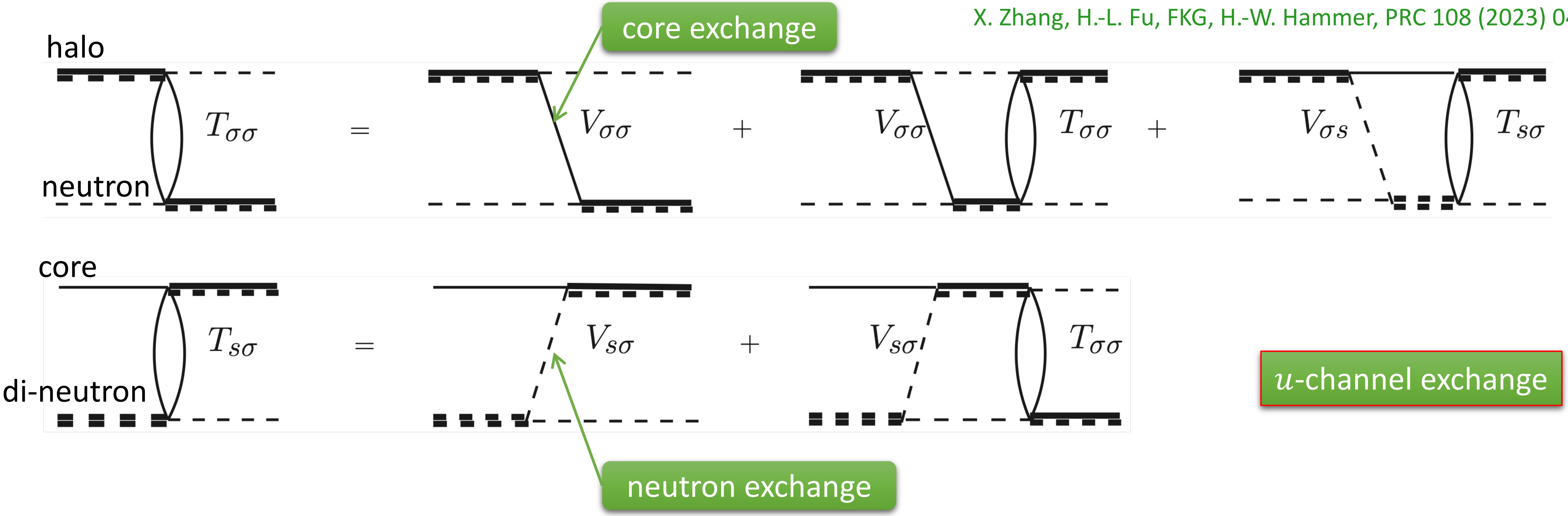
$$R \sim \frac{1}{Q} \sim \frac{1}{\sqrt{2\mu E_n}} \gg r_{\mathrm{core}} \sim \frac{1}{\Lambda} \sim \frac{1}{\sqrt{2\mu E_c^*}}$$



- Halo EFT C. A. Bertulani, H.-W. Hammer, U. van Kolck, NPA 712 (2002) 37; P. F. Bedaque, H.-W. Hammer, U. van Kolck, PLB 569 (2003) 159; …

- Consider scattering between neutron and one-neutron-halo nucleus with total spin $J = 0$

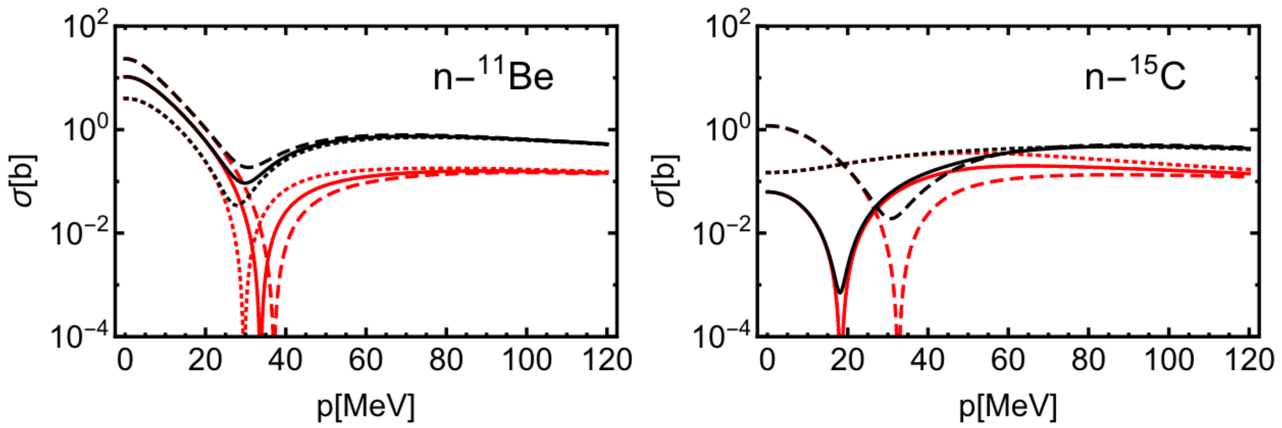
X. Zhang, H.-L. Fu, FKG, H.-W. Hammer, PRC 108 (2023) 044304

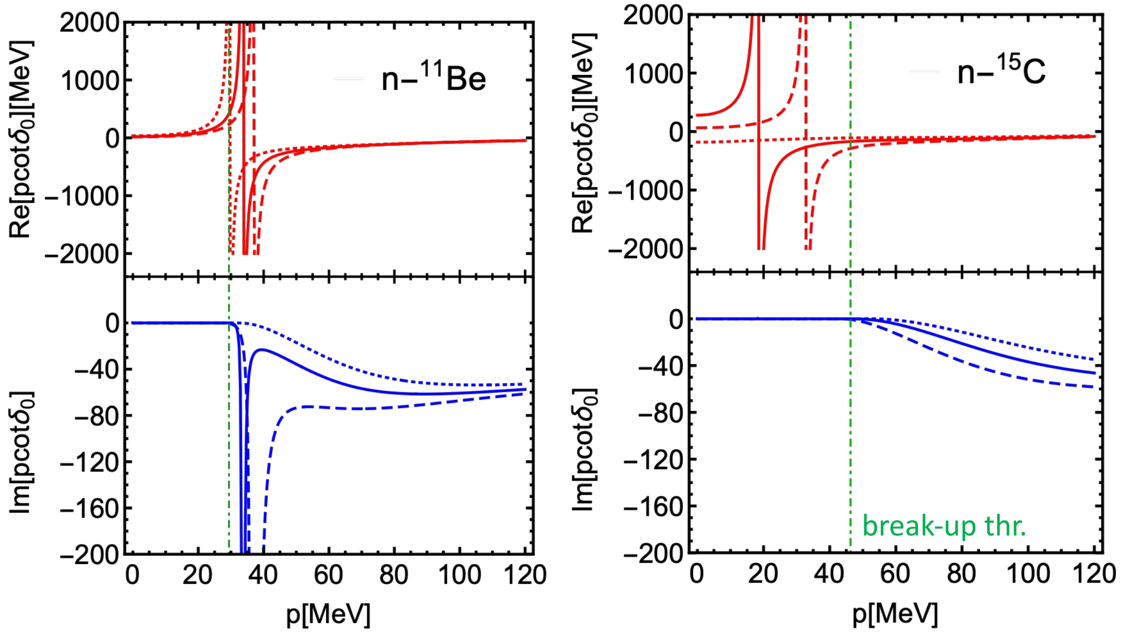


- Inputs: a_s (nn sca. length), B_σ (for neutron halo), B_{2n} (2-neutron separation energy)

Amplitude zeros: neutron-halo scattering

- Inputs: a_s (nn sca. length), B_σ (for neutron halo), B_{2n} (2-neutron separation energy)
- Total cross sections [red: s-wave, black: $l \leq 4$]





➢ Zero in the s-wave cross section!

$$T^l_{\sigma\sigma}(p,p,E) = \frac{2\pi}{\mu_{n\sigma}} \frac{\eta e^{2i\delta^R_l(p)} - 1}{2ip}$$

➢ pole in $p \cot \delta^R_0(p)$

➢ Found before for n-d, n-^{19}C scattering

W.T.H. van Oers, J.D. Seagrave, PLB 24 (1967) 562; …
M. T. Yamashita et al., PLB 670 (2008) 49; …

- ☐ Usual effective range expansion (ERE) fails!

$$k \cot \delta_0 = \frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \cdots$$

- ☐ modified effective range expansion

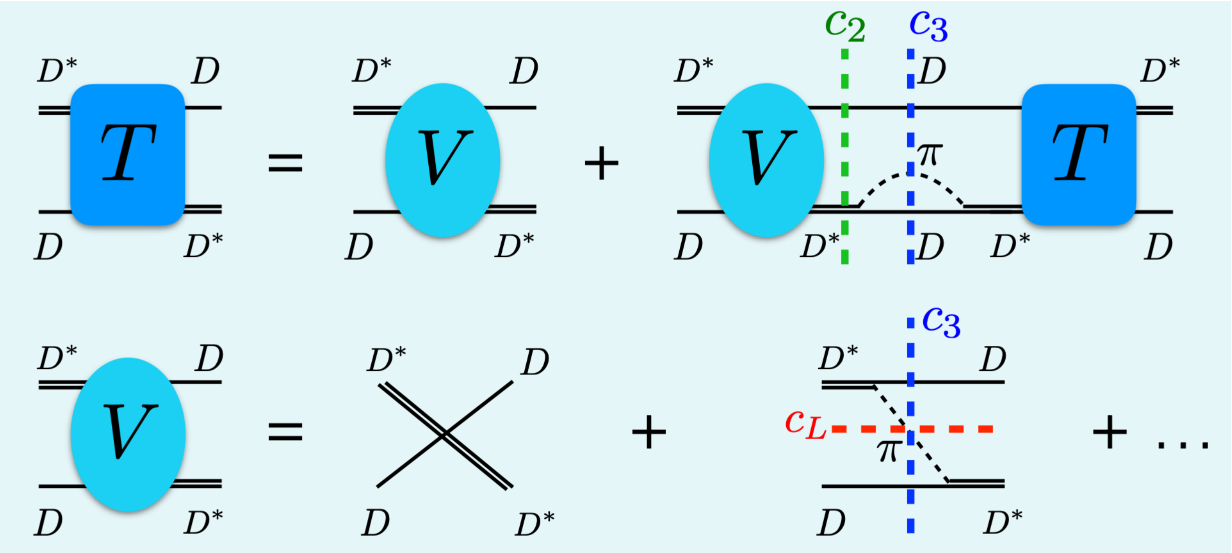
van Oers, Seagrave, PLB 24 (1967) 562

$$k \cot \delta_0 = -A + Bk^2 - \frac{C}{1 + Dk^2}$$

Amplitude zeros: DD^* scattering

M.-L. Du et al., PRL 131 (2023) 131903

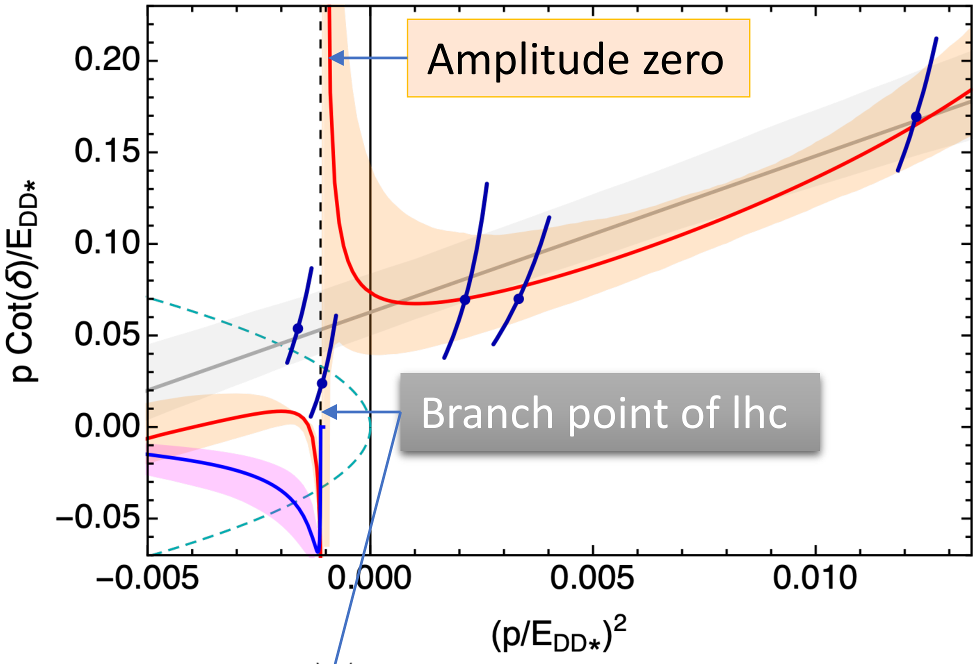
- Similar finding in hadronic physics: DD^* scattering with unphysical pion mass
  - □ $D^* \to D\pi$ in P-wave: D^* as a $D\pi$ dimer

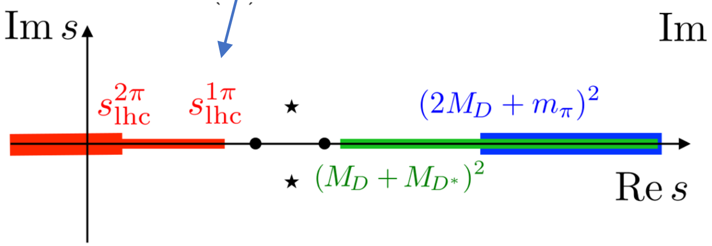


  - □ For unphysically large M_π, $m_{D^*} < m_D + M_\pi$, left-hand-cut
  - □ Usual effective range expansion fails!

Various solutions in finite volume:

L. Meng et al., PRD 109 (2024) L071506; A. Raposo, M. Hansen, JHEP 08 (2024) 075;
R. Bubna et al., JHEP 05 (2024) 168; S. Dawid et al., PLB 864 (2025) 139442; …





Lattice QCD data (M_π = 280 MeV):

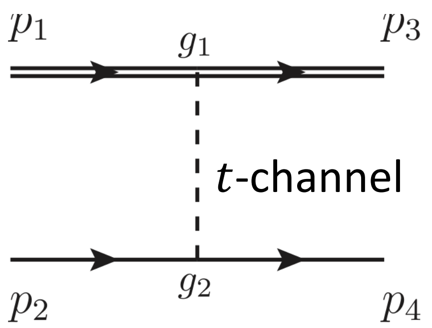
M. Padmanath, S. Prelovsek, PRL 129 (2022) 032002

ERE convergence radius

- Convergence radius for ERE: the nearest singularity
  - **Branch point of the left-hand cut**: Endpoint singularity of projection of t/u-ch. amplitude
    - t-channel

$$L_t\left(k^2\right) \equiv \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{t-m_{\mathrm{ex}}^2} = -\frac{1}{4k^2}\log\frac{m_{\mathrm{ex}}^2/4+k^2}{m_{\mathrm{ex}}^2/4}$$

ERE would fail at $k^2 = -\frac{m_{\mathrm{ex}}^2}{4}$



    - u-channel

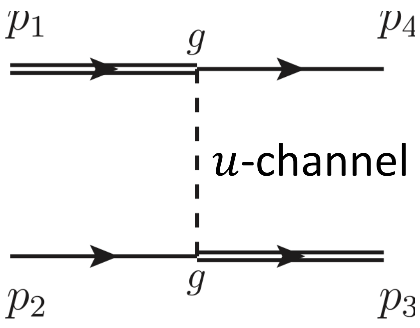
$$L_u\left(k^2\right) \equiv \frac{1}{2}\int_{-1}^{+1}\frac{\mathrm{d}\cos\theta}{u-m_{\mathrm{ex}}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4+k^2}{\mu_+^2/4+\eta^2 k^2}$$

Nonrelativistic approx.

$$\mu_+^2 \equiv \frac{4m_1m_2}{(m_1+m_2)^2}\mu_{\mathrm{ex}}^2$$

$$\mu_{\mathrm{ex}}^2 \equiv m_{\mathrm{ex}}^2 - (m_1-m_2)^2$$

$$\eta^2 \equiv \frac{(m_1-m_2)^2}{(m1+m_2)^2}$$



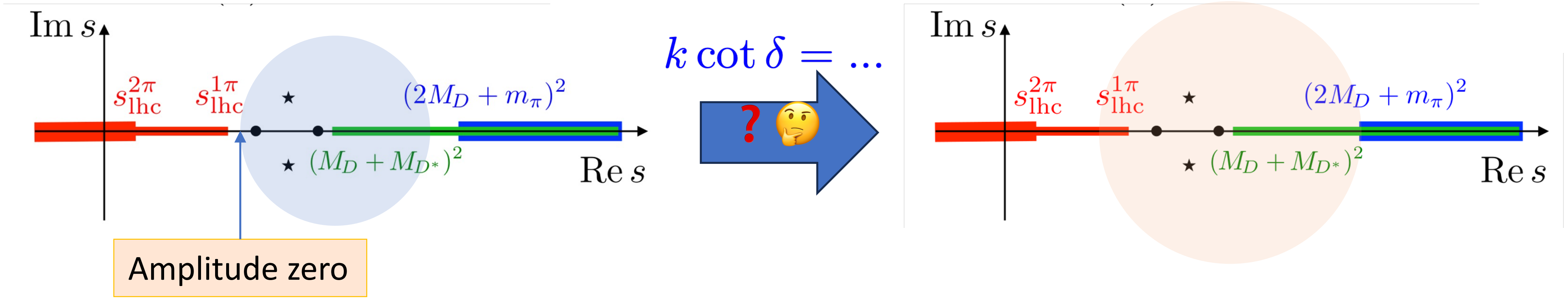
      - Nearby branch point: $k^2 = -\frac{\mu_+^2}{4}$
      - Farther branch point: $k^2 = -\frac{\mu_+^2}{4\eta^2}$
  - **Amplitude zero**: pole of $k\cot\delta$ may be present closer to threshold

Generalization of ERE

- Recent works dealing with the left-hand cut in finite volume

  Meng, Epelbaum, JHEP (2021); Meng et al., PRD (2024); Hansen, Raposo, JHEP (2024); Dawid et al., PRD (2023); Hansen et al., PRD (2024); Bubna et al., JHEP (2024)

- Can we analytically generalize ERE to achieve a larger convergence radius?



- ☐ General issue: hadron physics, nuclear physics, cold atom physics, …
- ☐ Our solution
  - ➢ Generalized ERE from the N/D method

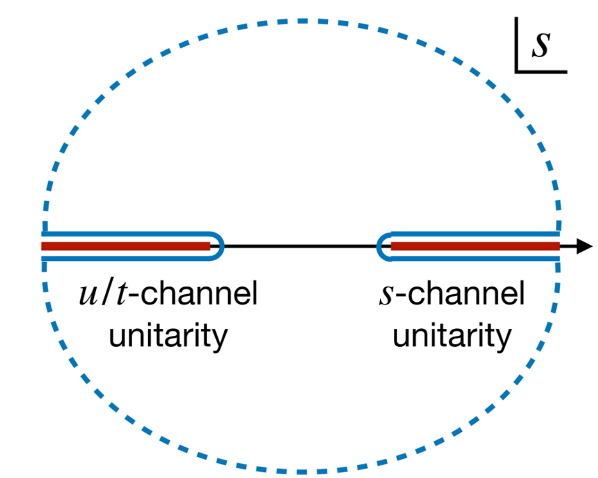
Modified ERE:
H. van Haeringen, L. P. Kok, PRA 26 (1982) 1218;
Manolo' talks

The N/D method

- N/D method of dispersion relation for 2-body scattering:
  - N: left-hand cut (lhc), no right-hand cut
  - D: right-hand cut (rhc), no left-hand cut

$$T(s) = \frac{N(s)}{D(s)}$$

G. Chew, S. Mandelstam (1960);
For pedagogical introduction: J.A. Oller, A Brief Introduction to Dispersion Relations



$$\boxed{\mathrm{Im}D = \mathrm{Im}\frac{N}{T} = N\mathrm{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\mathrm{thr}} \\ 0, & s < s_{\mathrm{thr}} \end{cases}}$$

$$\boxed{\mathrm{Im}N = \begin{cases} \mathrm{Im}TD, & s < s_{\mathrm{lhc}} \\ 0, & s > s_{\mathrm{lhc}} \end{cases}}$$

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s-s_0)^n}{\pi}\int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s-s_0)^{n-\ell}}{\pi}\int_{-\infty}^{s_{\mathrm{left}}} ds' \frac{\mathrm{Im}T(s')D(s')}{(s'-s_0)^{n-\ell}(s'-s)}.$$

If no lhc, N = 1

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + P(s) + G(s)$$

$$T(s) = \frac{1}{D(s)}$$

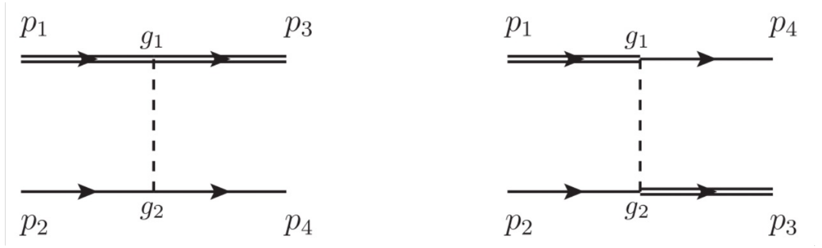
Neglect CDD pole,
low-momentum expansion

$$\boxed{d(k^2) = P(k^2) - ik = \frac{1}{a} + \frac{1}{2}rk^2 + \mathcal{O}(k^4) - ik}$$

For equivalence between the N/D method and the LSE: D.R. Entem, J.A. Oller, Annals Phys. 411 (2019) 167965

Left-hand cut from one-particle exchange (OPE)

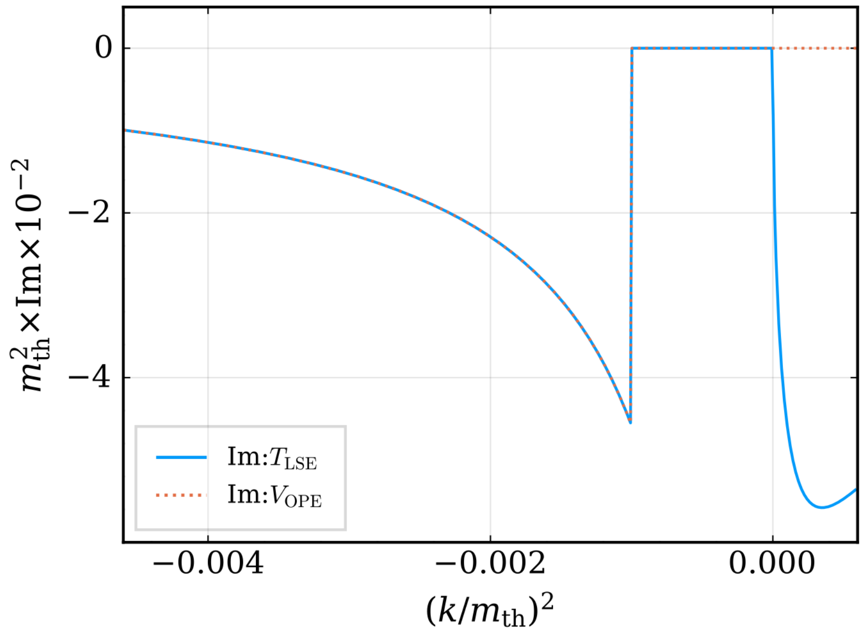
- Starting point: Restrict to OPE, then the lhc of T equals to the lhc of OPE potential
  - ☐ Singularity of integral must come from that of integrand
  - ☐ If the contour does not touch singularity, then integral is regular
    - ➢ Singularities of integrand becomes sing. of integral on other Riemann sheets
    - ➢ lhc of V: $k^2 < -\mu_+^2/4$ (u channel) or $k^2 < -m_{ex}^2/4$ (t channel)



$$T(E;k,k) = V(E;k,k) + \int_0^\infty \frac{l^2 dl}{2\pi^2} V(E;k,l)\, G(E;l)\, T(E;l,k)$$

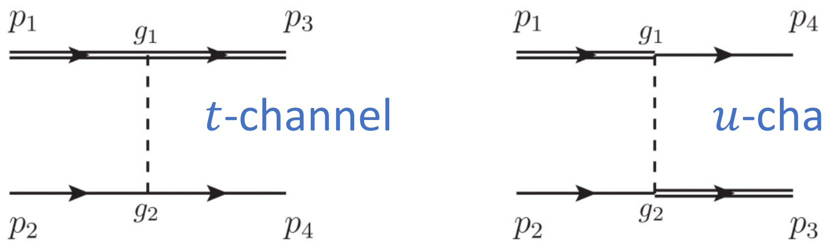
$$\boxed{\mathrm{Im}\, f\left(k^2\right) = c\, \mathrm{Im}\, L\left(k^2\right) = -\frac{c}{4k^2}\pi, \quad \text{for } k^2 < k_{\text{lhc}}^2}$$

c: strength of the lhc, containing product of coupling constants



N/D

- Couplings in different waves do not change the structure



S-wave couplings

$$L_t(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} = -\frac{1}{4k^2}\log\frac{m_{\rm ex}^2/4 + k^2}{m_{\rm ex}^2/4},$$
$$L_u(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{u - m_{\rm ex}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2},$$

P-wave couplings (e.g., $NN\pi$, $D^*D\pi$)

$$\frac{1}{2}\int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_3)^2}{t - m_{\rm ex}^2}\mathrm{d}\cos\theta = -\frac{m_{\rm ex}^2}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} - 1$$
$$\frac{1}{2}\int_{-1}^{+1} \frac{(\mathbf{p}_1 - \mathbf{p}_4)^2}{u - m_{\rm ex}^2}\mathrm{d}\cos\theta \approx -\frac{\mu_{\rm ex}^2}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{u - m_{\rm ex}^2} - 1$$

$\boxed{\mathcal{F}_\ell/2}$

- N/D construction of the amplitude:

$$f(k^2) = \frac{n(k^2)}{d(k^2)} \qquad \begin{array}{ll} \mathrm{Im}\, d(k^2) = -k\, n(k^2), & \text{for } k^2 > 0, \\ \mathrm{Im}\, n(k^2) = d(k^2)\ \mathrm{Im}\, f(k^2), & \text{for } k^2 < k_{\rm lhc}^2. \end{array}$$

N/D

No singularity along lhc, polynomial in k'^2

$= c\,\mathrm{Im}\,L$

$$n\left(k^2\right) = n_m\left(k^2\right) + \frac{\left(k^2\right)^m}{\pi}\int_{-\infty}^{k_{\rm lhc}^2} \frac{d\left(k'^2\right)\,\mathrm{Im}\,f\left(k'^2\right)}{\left(k'^2-k^2\right)\left(k'^2\right)^m}\,\mathrm{d}k'^2$$

$$n\left(k^2\right) = n'_m\left(k^2\right) + \frac{P\left(k^2\right)}{\pi}\int_{-\infty}^{k_{\rm lhc}^2} \frac{\mathrm{Im}\,f\left(k'^2\right)}{k'^2-k^2}\,\mathrm{d}k'^2 = n'_m\left(k^2\right) + P\left(k^2\right) cL\left(k^2\right)$$

$$\begin{aligned} n\left(k^2\right) &= n_0 + n_1 k^2 + \frac{k^2}{\pi}\int_{-\infty}^{k_{\rm lhc}^2} \frac{\left(d_0 + d_1 k'^2\right)\mathrm{Im}\,f\left(k'^2\right)}{\left(k'^2-k^2\right)k'^2}\,\mathrm{d}k'^2 \\ &= n_0 + n_1 k^2 - cL_0 + \left(d_0 + d_1 k^2\right)\frac{c}{\pi}\int_{-\infty}^{k_{\rm lhc}^2} \frac{\mathrm{Im}\,L\left(k'^2\right)}{k'^2-k^2}\,\mathrm{d}k'^2 \\ &= n'_0 + n_1 k^2 + \left(d_0 + d_1 k^2\right) cL\left(k^2\right) \end{aligned}$$

$$\boxed{\propto \tilde{n}\left(k^2\right) + \tilde{g}\left(L\left(k^2\right) - L_0\right)}$$

$$L_0 = L\left(k^2 = 0\right) = -\frac{1}{m_{\rm ex}^2} \text{ or } -\frac{1}{\mu_{\rm ex}^2}$$

divide by a polynomial, so that the normalization is $\tilde{n}(0) = 1$

N/D

$$d(k^2) = d_n(k^2) - \frac{(k^2-k_0^2)^n}{\pi}\int_0^\infty \frac{k' n(k'^2)\,\mathrm{d}k'^2}{(k'^2-k^2)(k'^2-k_0^2)^n}$$

$$d\left(k^2\right) = \tilde{d}\left(k^2\right) - ik\left(\tilde{n}\left(k^2\right) - \tilde{g}L_0\right) - \frac{\tilde{g}}{\pi}\int_0^\infty \frac{k' L\left(k'^2\right)}{k'^2-k^2}\,\mathrm{d}k'^2$$

$$= \tilde{d}\left(k^2\right) - ikn\left(k^2\right) - \tilde{g}d^{\mathrm{R}}\left(k^2\right)$$

- cancelation along lhc, so that $d(k^2)$ is real along lhc
- along rhc, Im $d(k^2) = -k\,n(k^2)$

$$d_t^{\mathrm{R}}(k^2) = \frac{i}{4k}\log\frac{m_{\mathrm{ex}}/2+ik}{m_{\mathrm{ex}}/2-ik},$$

$$d_u^{\mathrm{R}}(k^2) = \frac{i}{4k}\left(\log\frac{\mu_+/2+ik}{\mu_+/2-ik} - \log\frac{\mu_+/2+i\eta k}{\mu_+/2-i\eta k}\right)$$

Generalized ERE with lhc

- Then we obtain the generalized ERE with OPE lhc in an analytic form:

$$\frac{1}{f(k^2)} = \frac{\tilde{d}(k^2) - \tilde{g} d^R(k^2)}{\tilde{n}(k^2) + \tilde{g}\left(L(k^2) - L_0\right)} - ik$$

$\tilde{d}, \tilde{n}$: rational functions

$$L_t(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} = -\frac{1}{4k^2}\log\frac{m_{\rm ex}^2/4 + k^2}{m_{\rm ex}^2/4}$$

$$L_u(k^2) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{u - m_{\rm ex}^2} \approx -\frac{1}{4k^2}\log\frac{\mu_+^2/4 + k^2}{\mu_+^2/4 + \eta^2 k^2}$$

$$d_t^{\rm R}(k^2) = \frac{i}{4k}\log\frac{m_{\rm ex}/2 + ik}{m_{\rm ex}/2 - ik},$$

$$d_u^{\rm R}(k^2) = \frac{i}{4k}\left(\log\frac{\mu_+/2 + ik}{\mu_+/2 - ik} - \log\frac{\mu_+/2 + i\eta k}{\mu_+/2 - i\eta k}\right)$$

$\tilde{d}, \tilde{n}$: if polynomials

$$\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i k^{2i} - \tilde{g} d^{\rm R}(k^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j k^{2j} + \tilde{g}\left(L(k^2) - L_0\right)} - ik$$

Pade-like expansion

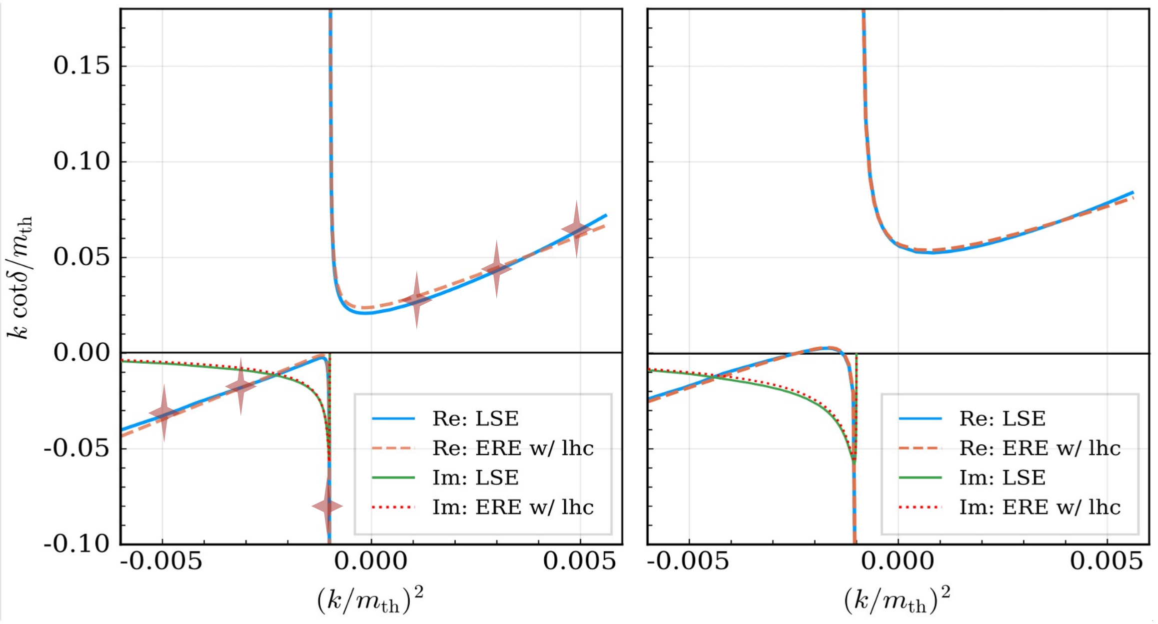
[0,1] approximant

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\rm R}(k^2)}{1 + \tilde{g}\left(L(k^2) - L_0\right)} - ik\right]^{-1} \xrightarrow[\text{no exchange}]{\tilde{g} \to 0} \frac{1}{f(k^2)} = \frac{1}{a} + \frac{1}{2} r k^2 - ik$$

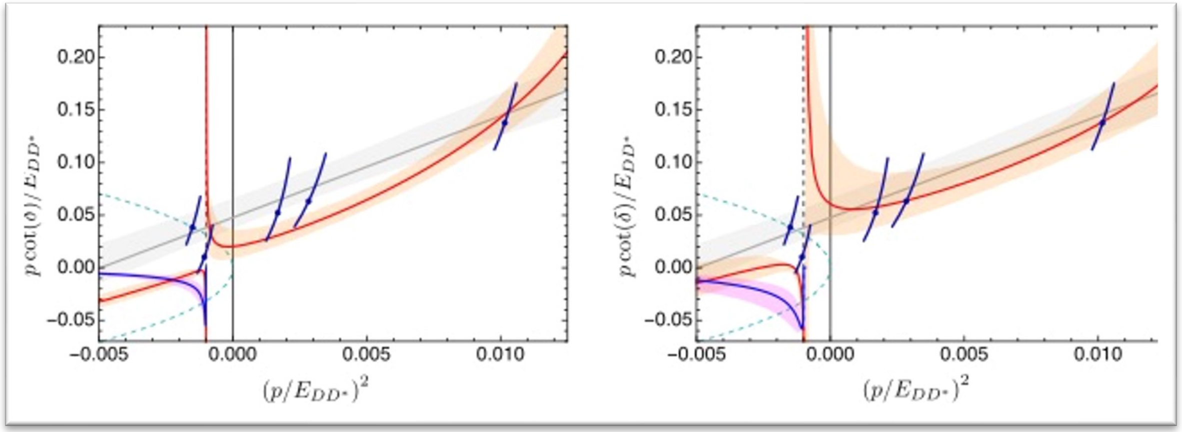
- Scattering length $a = f(k^2 = 0) = \left[\tilde{d}_0 + \frac{\tilde{g}}{\mu_+}(1 - \eta)\right]^{-1}$
- Effective range $r = \left.\frac{\mathrm{d}^2(1/f + ik)}{\mathrm{d}k^2}\right|_{k=0} = 2\tilde{d}_1 - \frac{8\tilde{g}}{3\mu_+^3}(1 - \eta^3) - \frac{4\tilde{g}}{\mu_+^4 a_u}(1 - \eta^4)$

Example

- DD^* scattering with lattice masses ($M_\pi = 280$ MeV, $m_D = 1927$ MeV, $m_{D^*} = 2049$ MeV)
  - Take the $[0,1]$ approximant, 3-parameter fit ($\tilde{d}_0, \tilde{d}_1, \tilde{g}$) to 6 chosen points from solving LSE
  - The amplitude zero is also well captured



$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\rm R}(k^2)}{1 + \tilde{g}\left(L(k^2) - L_0\right)} - ik\right]^{-1}$$



Left, right: corresponding to two fits to lattice QCD data performed in M.-L. Du et al., PRL 131 (2023) 131903

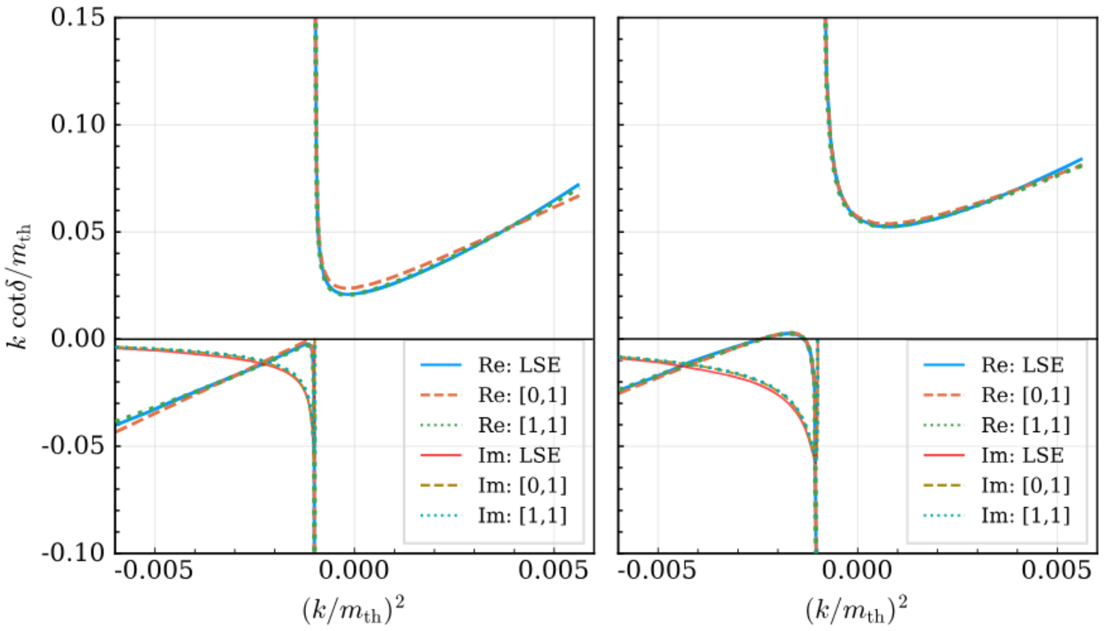
- Coupling extracted from fitting:

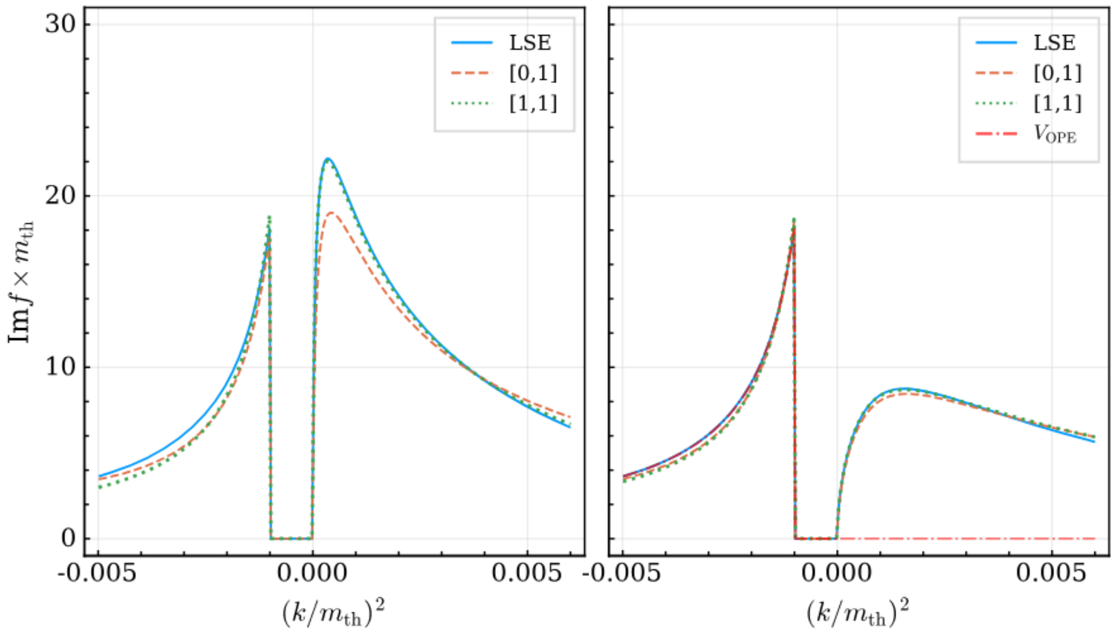
$$g_P = -\frac{2\pi\tilde{g}}{\mu d^{0,{\rm lhc}}\mathcal{F}_\ell} = 9.0\ \text{GeV}^{-2}\ \text{(left)}\ \ 9.4\ \text{GeV}^{-2}\ \text{(right)}$$

Empirical value: $g_P = \frac{g_{D^*D\pi}}{4F_\pi^2} = 9.2\text{GeV}^{-2}$

Example

- Improvable





Amplitude zeros

- The amplitude zeros are also captured in this parametrization
  - Take the leading order, $\tilde{n}(k^2) = 1$, e.g., the [0,1] approximant

$$f_{[0,1]}(k^2) = \left[\frac{\tilde{d}_0 + \tilde{d}_1 k^2 - \tilde{g} d^{\mathrm{R}}(k^2)}{\boxed{1 + \tilde{g}\left(L(k^2) - L_0\right)}} - ik\right]^{-1}$$

  - For a general u-channel exchange, the amplitude has a zero at

$$1 + \tilde{g}\left[L_u\left(k^2_{u,\mathrm{zero}}\right) + \frac{1}{\mu^2_{\mathrm{ex}}}\right] = 0$$

  - For $|m_1 - m_2| \ll m_1 + m_2$, we have $\eta \ll 1$. Neglecting the η terms $\Rightarrow$ analytic solution also for t-channel exchange

$$\eta^2 \equiv \frac{(m_1 - m_2)^2}{(m1 + m_2)^2}$$

$$\boxed{k^2_{t,\ \mathrm{zero}} = -\frac{m^2_{\mathrm{ex}}}{4}\left[1 + \frac{1}{y} W\left(-e^{-y} y\right)\right]}$$

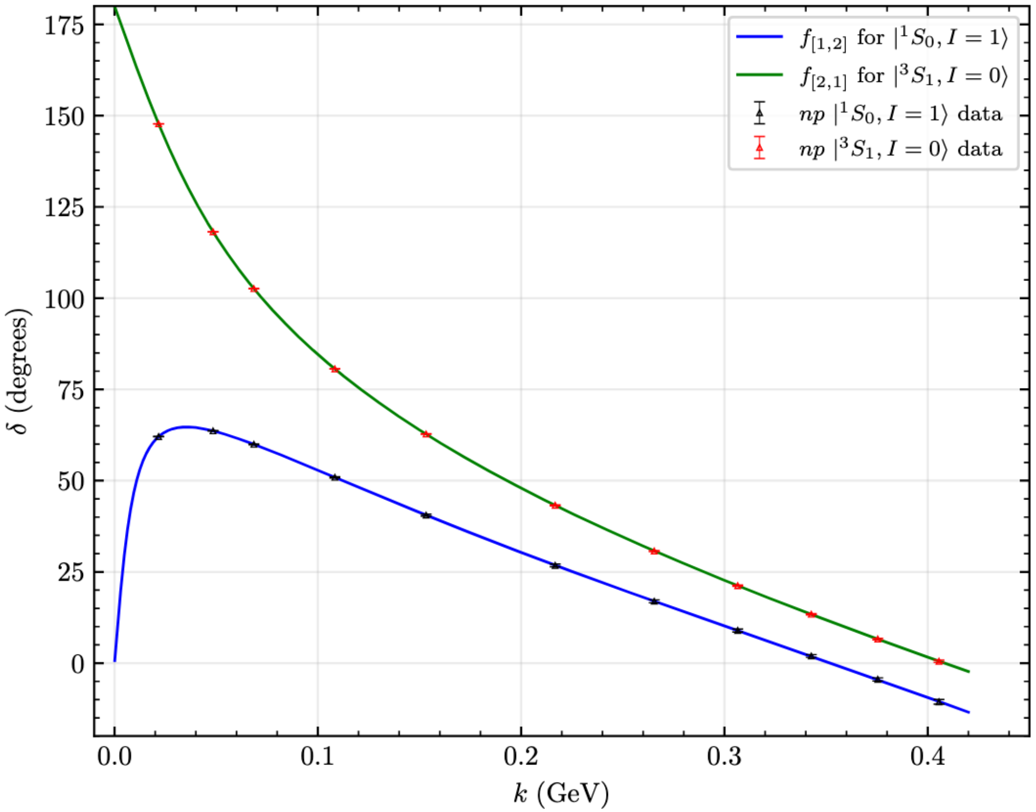
Here, W: the Lambert W function, $y \equiv 1 + m^2_{\mathrm{ex}}/\tilde{g}$

The zero is from interplay between long-range (OPE) and short-range ($\tilde{d}_0, \tilde{d}_1$ in relation between $\tilde{g}$ and couplings g_P; or a_t and r_t) interactions

$$\boxed{y = 1 + \frac{1 + \frac{4}{3} a_t m_{\mathrm{ex}}(1 - \log 4) - \frac{4\pi a_t m^2_{\mathrm{ex}}}{\mu g_P \mathcal{F}_\ell}}{2 + a_t m_{\mathrm{ex}}\left(1 - m_{\mathrm{ex}} r_t / 4\right)}}$$

Applications: Extraction of $NN\pi$ coupling

B.-Y. Liu, B. Wu, J.-W. Fu, M.-L. Du, FKG, U.-G. Meißner, arXiv:2602.03115

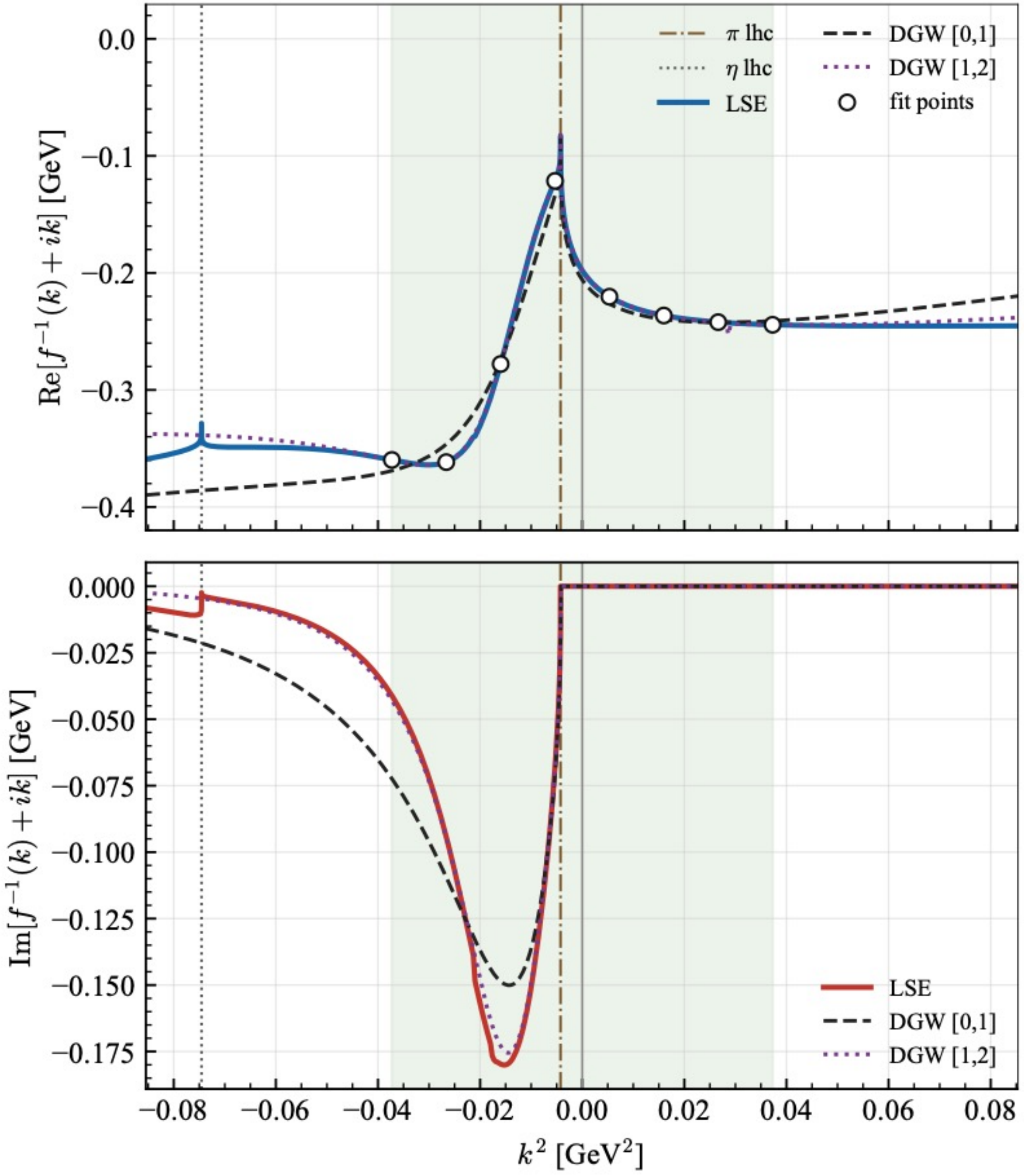


Fit to Nijmegen phase shifts in
V.G.J. Stoks et al., PRC 48 (1993) 792

ERE w/ lhc		E_{pole} (keV)		a (fm)		r (fm)		$\frac{g^2_{\pi N}}{4\pi}$	$\frac{\chi^2}{\text{d.o.f}}$
$\lvert ^1S_0\rangle$	$\lvert ^3S_1\rangle$	$\lvert ^1S_0\rangle$	$\lvert ^3S_1\rangle$	$\lvert ^1S_0\rangle$	$\lvert ^3S_1\rangle$	$\lvert ^1S_0\rangle$	$\lvert ^3S_1\rangle$		
$f_{[2,1]}$	$f_{[1,2]}$	$-66.33(4)$	$-2208(4)$	$23.73(1)$	$-5.418(2)$	$2.68(1)$	$1.741(3)$	$12.2(1.4)$	0.18
$f_{[1,2]}$	$f_{[1,2]}$	$-66.33(4)$	$-2209(4)$	$23.73(1)$	$-5.418(2)$	$2.68(1)$	$1.742(3)$	$12.8(1.2)$	0.13
$f_{[2,1]}$	$f_{[2,1]}$	$-66.34(4)$	$-2211(5)$	$23.73(1)$	$-5.418(2)$	$2.68(1)$	$1.743(3)$	$11.9^{+1.4}_{-1.5}$	0.10
$f_{[1,2]}$	$f_{[2,1]}$	$-66.34(4)$	$-2212(4)$	$23.73(1)$	$-5.418(2)$	$2.68(1)$	$1.744(3)$	$12.7(1.2)$	0.05
$f_{[1,2]}$	-	$-66.4^{+0.6}_{-0.5}$	-	$23.72^{+0.10}_{-0.08}$	-	$2.68^{+0.02}_{-0.03}$	-	$12.4^{+2.2}_{-2.3}$	0.008
-	$f_{[2,1]}$	-	-2222^{+14}_{-12}	-	$-5.416^{+0.002}_{-0.003}$	-	$1.749(7)$	$18.3^{+8.4}_{-8.6}$	0.04
[34]		-66.5	-2237	23.719	-5.403	2.626	1.749	-	-
[35]		$-66.08(13)$	$-2223(6)$	$23.748(10)$	$-5.424(4)$	$2.75(5)$	$1.759(5)$	-	-
[36]		$-66.08(16)$	$-2223(10)$	$23.740(20)$	$-5.419(7)$	$2.77(5)$	$1.753(8)$	-	-
[23]		-	$-2224.7(35)$	-	-	-	-	-	-
[43]		-	$-2224.575(9)$	$23.74(2)$	-	-	-	-	-
[39]		-	-	-	-	-	-	$13.76(8)$	-
[40]		-	-	-	-	-	-	$13.69(27)$	-
[38]		-	-	-	-	-	-	$14.23(11)$	-
[41]		-	-	-	-	-	-	$13.14^{+0.07}_{-0.04}$	-
[42]		-	-	-	-	-	-	$13.6(3)$	-

Applications: BB^* scattering

Y. Zhang, FKG., A. Nefediev, C. Hanhart, P. Pütz, to appear



	a (fm)	r (fm)	g_b
$f_{[0,1]}$	0.96	−2.48	0.50
$f_{[1,2]}$	1.00	−2.66	0.56

Solution from S-wave LSE with one-pion and one-eta exchanges:

$$a = 1.00\ \text{fm}, r = -2.64\ \text{fm}$$

Input value: $g_b = 0.56$

Bonus: Generalizing Weinberg's compositeness relations

Y. Li, FKG, J.-Y. Pang, J.-J. Wu, PRD 105 (2022) L071502

- Assumptions used in Weinberg's derivations
  - Neglecting the non-pole term from the Low equation
  - Approximating the form factor by a constant

S. Weinberg (1965)'s citation record

$$T_{p,k} = V_{p,k} + \frac{g(p)\, g^*(k)}{h_k - E_B} + \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{T_{p,q} T^*_{k,q}}{h_k + i\varepsilon - h_q}$$ w/ $h_k \equiv k^2/(2\mu)$

For ERE up to $\mathcal{O}(p^2)$, is a constant $g(p)$ a consistent approximation?

- Generalization: replacing the constant form factor by a more general separable ansatz $T_{p,k} = t_k\, g(p)\, g^*(k)$

Unitarity: $\mathrm{Im}\, t^{-1}(W) \propto |g(k)|^2 \Rightarrow$ twice-subtracted dispersion relation

$$t^{-1}(W) = (W - E_B) + (W - E_B)^2 \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|g(q)|^2}{(h_q - E_B)^2 (h_q - W)}$$

Then, we get

$$t(W) = \frac{1}{1 - F(W)} \frac{1}{W - E_B}, \qquad F(W) \equiv (W - E_B) \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|g(q)|^2}{(h_q - E_B)^2 (W - h_q)}$$

Compositeness: $F(\infty) = \int_0^\infty \frac{q^2 dq}{(2\pi)^3} \frac{|\langle q|\hat{V}|B\rangle|^2}{(h_q - E_B)^2} = \int_0^\infty \frac{q^2 dq}{(2\pi)^3} |\langle q|B\rangle|^2 = X$

Bonus: Generalizing Weinberg's compositeness relations

- Construct dispersion relation for $F_1(W) \equiv \dfrac{\ln[1-F(W)]}{W-E_B}$, $\mathrm{Im}\, F_1(E+i\varepsilon) = -\dfrac{\delta_B(E)}{E-E_B}\theta(E)$

  Then we get $$F(W) = 1 - \exp\left(\frac{W-E_B}{\pi}\int_0^\infty dE \frac{-\delta_B(E)}{(E-W)(E-E_B)}\right)$$

$$\boxed{X = 1 - \exp\left(\frac{1}{\pi}\int_0^\infty dE \frac{\delta_B(E)}{E-E_B}\right) \in [0,1]}$$

- Using $\mathrm{Im}\, F\left(h_p + i\epsilon\right) = -\frac{\pi p \mu}{(2\pi)^3}\frac{|g(p)|^2}{h_p - E_B}$, we get

$$|g(p)|^2 = -\frac{(2\pi)^3}{\pi\, p\mu}(h_p - E_B)\, \sin\delta_B(E) \exp\left[\frac{h_p - E_B}{\pi} \fint_0^\infty dE \frac{-\delta_B(E)}{(E-h_p)(E-E_B)}\right]$$

- Consider ERE $p\cot\delta_B \approx -\dfrac{8\pi^2}{\mu}\mathrm{Re}T^{-1}(h_p) = \dfrac{1}{a} + \dfrac{r}{2}p^2 + \mathcal{O}(p^4)$, we finally get

$$g^2(p) = \frac{8\pi^2}{\mu^2 R} \times \begin{cases} X_W + \mathcal{O}(p^4) & \text{for } a \in [-R, 0]\ \&\ r \le 0 \\ \frac{a^2}{R^2}\frac{1}{1+(a+R)^2 p^2} + \mathcal{O}(p^4) & \text{for } a < -R\ \&\ r > 0 \end{cases}$$ constant

contains $\mathcal{O}(p^2)$ terms, thus not self-consistent if using a constant g^2 but still work up to $\mathcal{O}(p^2)$ in ERE. Weinberg's relations do not hold in this case

Bonus: Generalizing Weinberg's compositeness relations

- Poles of the T-matrix with ERE up to $\mathcal{O}(p^2)$: $\frac{1}{a}+\frac{r}{2}p^2-i\,p=\frac{r}{2}(p-p_+)(p-p_-)$

$$p_-=\frac{i}{R},\qquad p_+=-\frac{i}{R+a}$$ with $R=\frac{1}{\sqrt{2\mu|E_B|}}$; r is expressed as $r=\frac{2R}{a}(R+a)$

- For $a\in[-R,0]$, then $r<0$, one bound state and one virtual state pole

$$g^2(p)=\frac{8\pi^2}{\mu^2 R}X_W+\mathcal{O}(p^4),\qquad X=X_W\simeq\sqrt{\frac{1}{1+2r/a}}$$

- For $a<-R$, then $r>0$, two bound state poles (the remote one $\sim i/\beta$ is unphysical)

$$g^2(p)=\frac{8\pi^2}{\mu^2 R}\frac{a^2}{R^2}\frac{1}{1+(a+R)^2p^2}+\mathcal{O}(p^4),\qquad X\simeq 1-e^{-\infty}=1$$

For the deuteron, $R=4.31$ fm, $a=-5.42$ fm, $|a+R|\sim\beta^{-1}\sim m_\pi^{-1}$

Summary and outlook

- We have derived a generalized ERE, taking into account explicitly the left-hand-cut due to one-particle exchange (either t- or u-channel)

$$\boxed{\frac{1}{f_{[m,n]}(k^2)} = \frac{\sum_{i=0}^{n} \tilde{d}_i k^{2i} - \tilde{g} d^{\mathrm{R}}(k^2)}{1 + \sum_{j=1}^{m} \tilde{n}_j k^{2j} + \tilde{g}\left(L(k^2) - L_0\right)} - ik}$$

- Should be applicable in a wide range of systems
  - ☐ Hadron physics: $D^{(*)}D^*, D^{(*)}\bar{D}^*, B^{(*)}B^*, B^{(*)}\bar{B}^*, ND^*, \Sigma_{(c)}^{(*)}D^*$, …
  - ☐ Nuclear physics: OPE for NN; nucleon exchange for halo nucleus scattering; …
  - ☐ Cold atom physics: atom-dimer scattering with atom exchange

Thank you for your attention!

One-neutron halo nuclei

H.-W. Hammer, C. Ji, D. R. Phillips, JPG 44 (2017) 103002

S-wave 1-neutron halos	$^{11}\mathrm{Be}$	$^{15}\mathrm{C}$	$^{19}\mathrm{C}$
Experimental values			
J^P	$1/2^+$	$1/2^+$	$1/2^+$
One-neutron separation energy: $S_{1n}[\mathrm{MeV}]$	$0.50164(25)$	$1.2181(8)$	$0.58(9)$
Core excitation energy: $E_c^*[\mathrm{MeV}]$	$3.36803(3)$	$6.0938(2)$	$1.62(2)$

Neutron-halo scattering: amplitude zeros

- Found before for n-d, n-^{19}C scattering

W.T.H. van Oers, J.D. Seagrave, PLB 24 (1967) 562; …
M. T. Yamashita, T. Frederico, L. Tomio, PLB 670 (2008) 49; …

  - ☐ Leads to a modified effective range expansion (ERE)

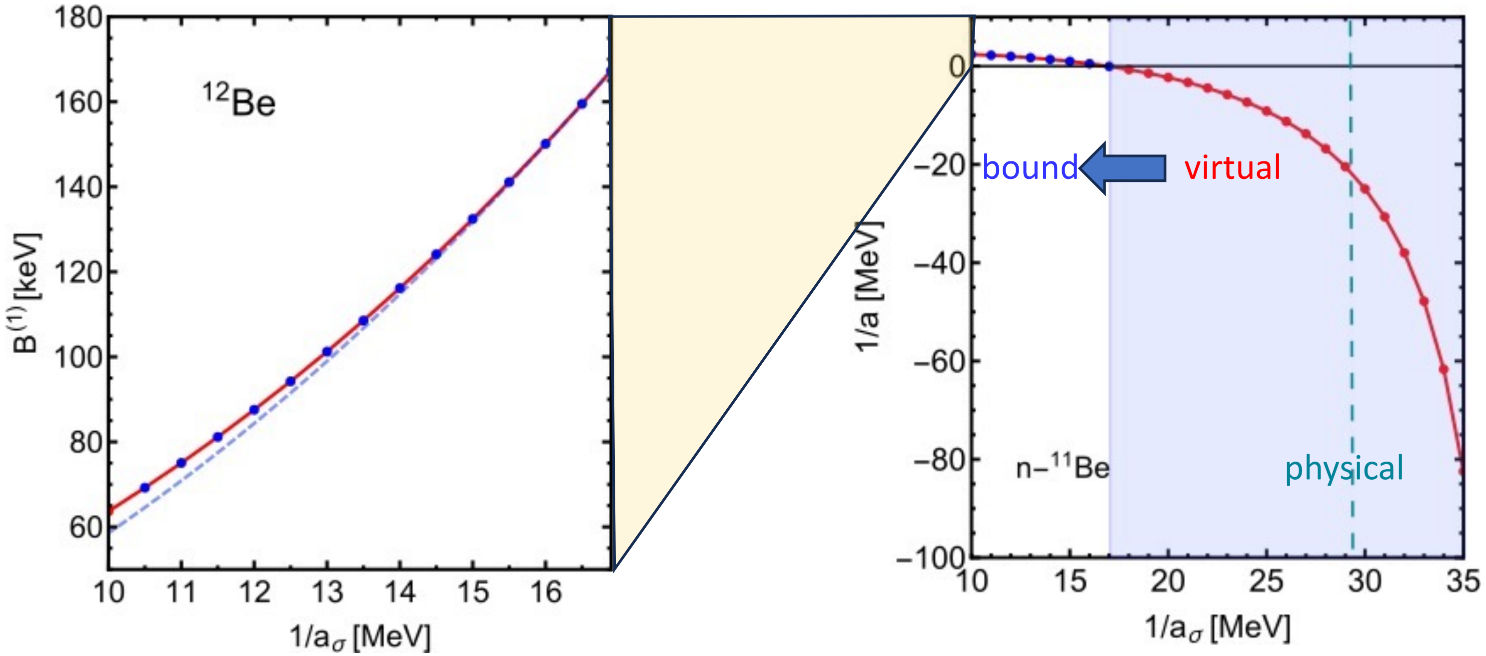
$$k \cot \delta_0 = -A + Bk^2 - \frac{C}{1 + Dk^2}$$

  - ☐ Manifestation of a 3-body virtual state

B. A. Girard, M. G. Fuda, PRC 19 (1979) 579; …

    - ➢ Increasing the nh scattering length a_σ => excited Efimov state

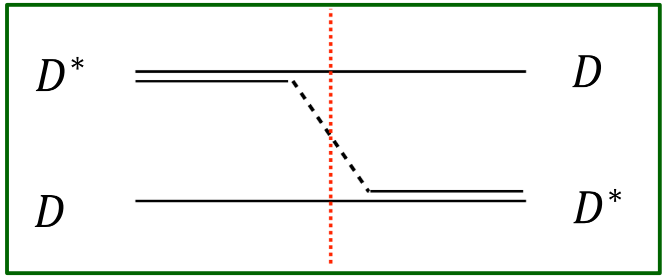
nh scattering length changes sign

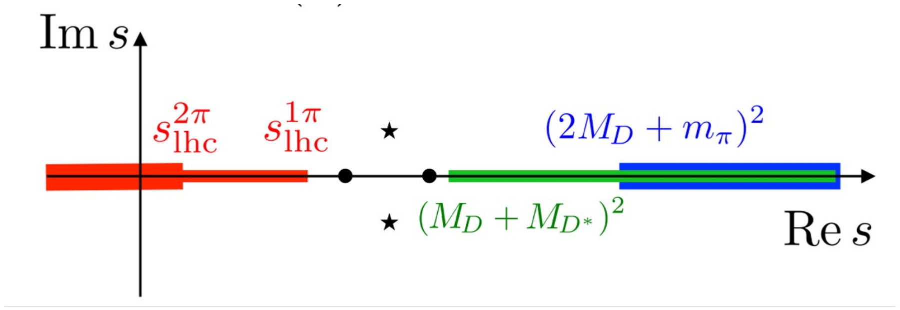


Left-hand cut for DD^* scattering

M.-L. Du et al., PRL 131 (2023) 131903

- u-channel one-pion exchange





☞ two-body branch point:

$$E = m_D + m_{D^*}$$

$$\Longrightarrow p_{\rm rhc_2}^2 = 0$$

☞ three-body branch point:

$$E = m_D + m_{D^*} + M_\pi$$

$$\Longrightarrow \left(\frac{p_{\rm rhc_3}}{E_{DD^*}}\right)^2 = +0.019$$

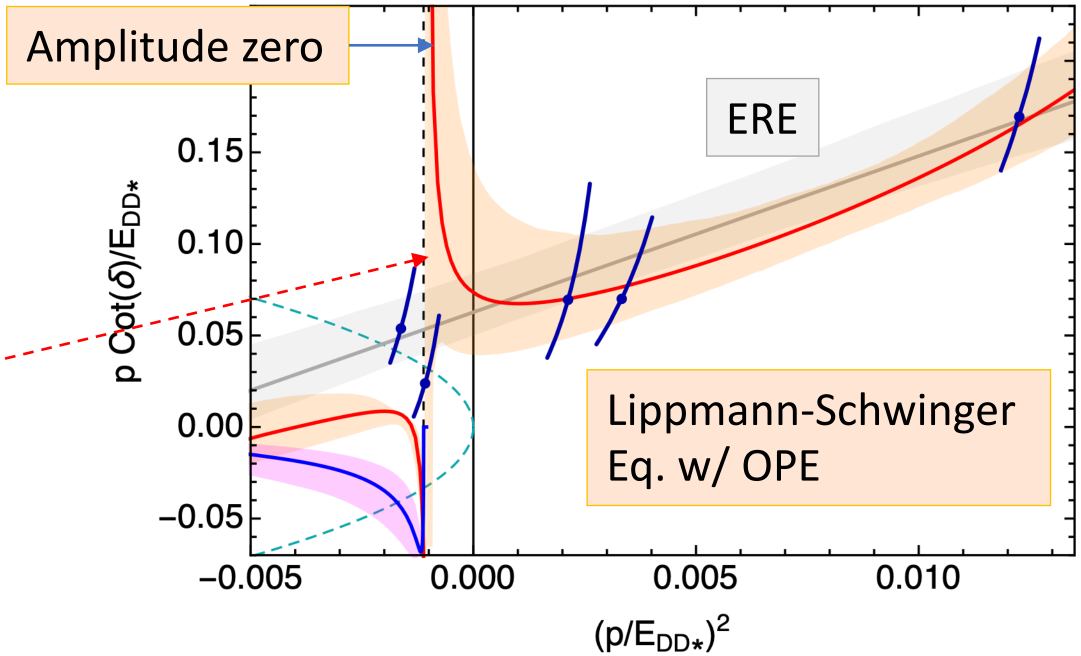
☞ left-hand cut branch point:

$$\Longrightarrow \left(\frac{p_{\rm lhc}^{1\pi}}{E_{DD^*}}\right)^2 = -0.001 \qquad -\frac{\mu_+^2}{4}$$

$$\left(\frac{\tilde{p}_{\rm lhc}^{1\pi}}{E_{DD^*}}\right)^2 = -0.190 \qquad -\frac{\mu_+^2}{4\eta^2}$$

Lattice QCD data ($M_\pi = 280$ MeV, $m_D = 1927$ MeV, $m_{D^*} = 2049$ MeV):

M. Padmanath, S. Prelovsek, PRL 129 (2022) 032002

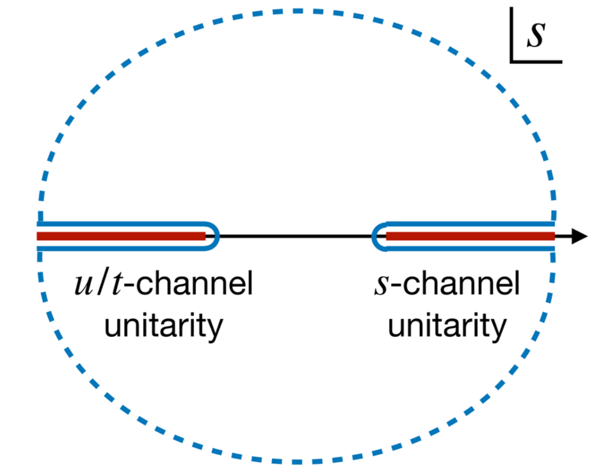


The N/D method

- N/D method of dispersion relation for 2-body scattering:
  - N: left-hand cut (lhc), no right-hand cut
  - D: right-hand cut (rhc), no left-hand cut

$$T(s) = \frac{N(s)}{D(s)}$$

G. Chew, S. Mandelstam (1960);
For pedagogical introduction: J.A. Oller, A Brief Introduction to Dispersion Relations

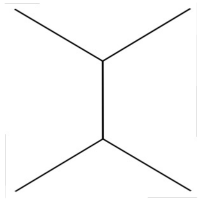


$$\boxed{\mathrm{Im}D = \mathrm{Im}\frac{N}{T} = N\mathrm{Im}\frac{1}{T} = \begin{cases} -N\rho, & s > s_{\rm thr} \\ 0, & s < s_{\rm thr} \end{cases}}$$

$$\boxed{\mathrm{Im}N = \begin{cases} \mathrm{Im}TD, & s < s_{\rm lhc} \\ 0, & s > s_{\rm lhc} \end{cases}}$$

$$D(s) = \sum_i \frac{\gamma_i}{s - s_i} + \sum_{m=0}^{n-1} a_m s^m - \frac{(s-s_0)^n}{\pi} \int_{s_{\rm thr}}^{\infty} ds' \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)^n},$$

$$N(s) = \sum_{m=0}^{n-\ell-1} b_m s^m + \frac{(s-s_0)^{n-\ell}}{\pi} \int_{-\infty}^{s_{\rm left}} ds' \frac{\mathrm{Im}T(s')D(s')}{(s'-s_0)^{n-\ell}(s'-s)}.$$

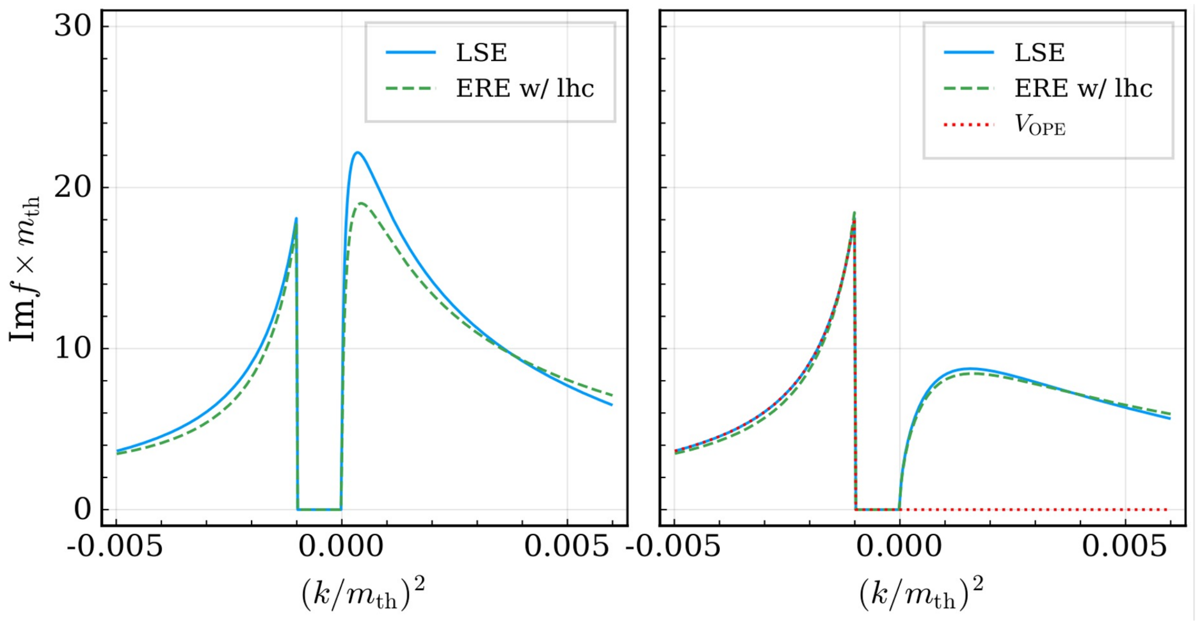


$$L_t\left(k^2\right) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{\mathrm{d}\cos\theta}{t - m_{\rm ex}^2} = -\frac{1}{4k^2}\log\frac{m_{\rm ex}^2/4 + k^2}{m_{\rm ex}^2/4}$$

e.g., along the lhc, $k^2 < -\frac{m_{\rm ex}^2}{4} \Rightarrow \mathrm{Im}\,T \neq 0$

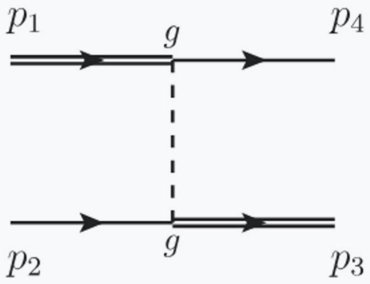
Example

- [0,1] approximant



$9.0\ \mathrm{GeV}^{-2}$ $9.4\ \mathrm{GeV}^{-2}$ fit result $$g_P = -\frac{2\pi\tilde{g}}{\mu d^{0,\mathrm{lhc}}\mathcal{F}_\ell}$$

$$g^2_{D^* D\pi}/(4F^2) = 9.2\ \mathrm{GeV}^{-2}$$



$$-\frac{2\pi}{\mu}\mathrm{Im}f = \mathrm{Im}T = \mathrm{Im}V_{\rm OPE}(k^2) = g_P\frac{-\pi}{4k^2}\mathcal{F}_\ell, \quad \text{for } k^2 < k^2_{\rm lhc}$$

$$\mathrm{Im}\, n(k^2) = -\tilde{g}\frac{\pi}{4k^2}, \quad \text{for } k^2 < k^2_{\rm lhc}$$

$$d_u^{0,\mathrm{lhc}} = \tilde{d}_0 - \frac{\tilde{d}_1\mu_+^2}{4} + \frac{\mu_+}{2}\left(1+\frac{\tilde{g}}{\mu_{\rm ex}^2}\right) + \frac{\tilde{g}\log[2/(1+\eta)]}{\mu_+}$$

The generalized ERE is improvable

- Better description keeping more terms. Take the [1,1] approximant for DD^* scattering at $M_\pi = 280$ MeV