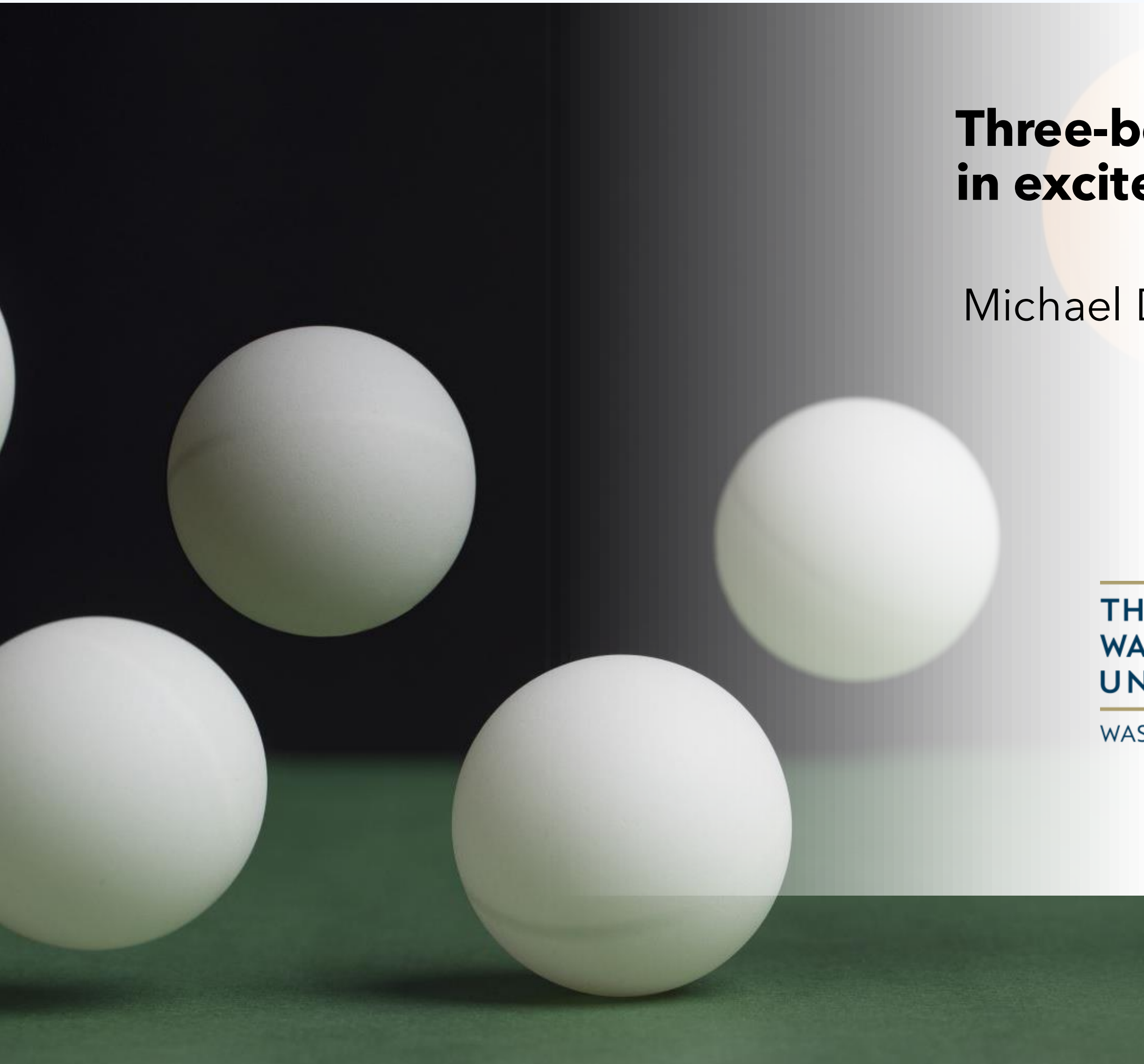
ETHNP (Effective Theories in Hadron and Nuclear Physics)

3-5 September 2026
Universe
Europe/Madrid timezone



Three-body effects in excited hadrons

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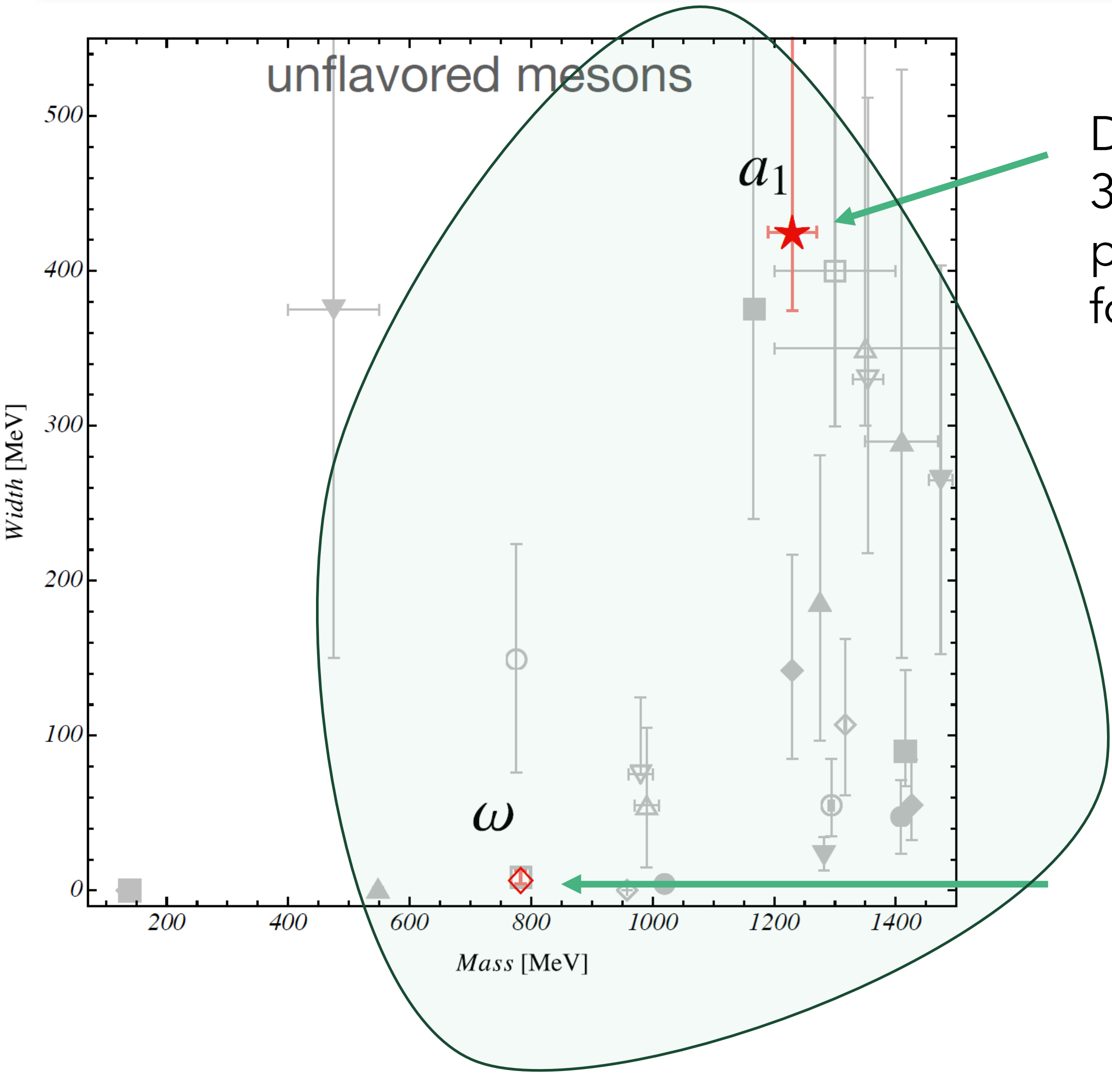
Outline

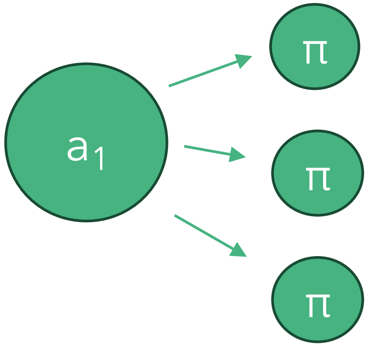
- Constructing a manifestly unitary amplitude
- Finite-volume applications for lattice QCD:
  - $3\pi^+$ allows to see exchange kinematics in data [1911.09047]
  - Three pions in isospin I=2 [2601.16916]
  - The $\pi(1300)$ [2510.09467]
- Infinite-volume: The curious case of the K(1460) [2511.02543]
- Two (πN)- and three$(\pi\pi N)$-body effects in baryons → Jinzi Wu's talk

Work supported by: National Science Foundation Grant No. PHY 2310036

>2-body meson decays



Decays almost 100% to 3 pions (decay to 2 pseudoscalar mesons forbidden)



Lattice data & analysis of 3B resonance [2107.03973]

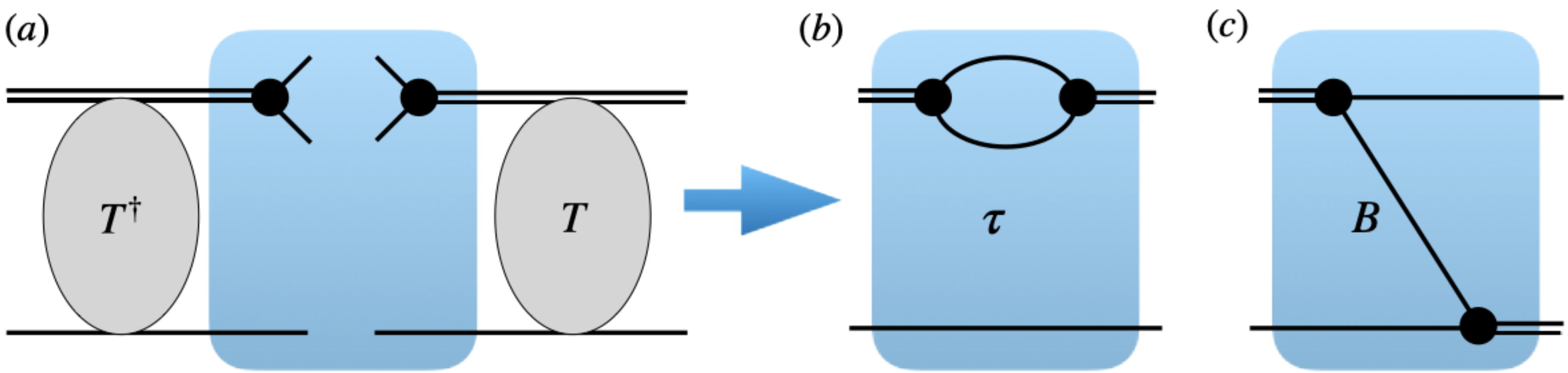
Three-body unitarity with isobars * [1706.06118]

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

$$\times \prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2-m^2)}\right](2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)$$

Idea: To construct a 3B amplitude, start directly from unitarity (based on ideas of 60's); match a general amplitude to it

delta function sets all intermediate particles on-shell

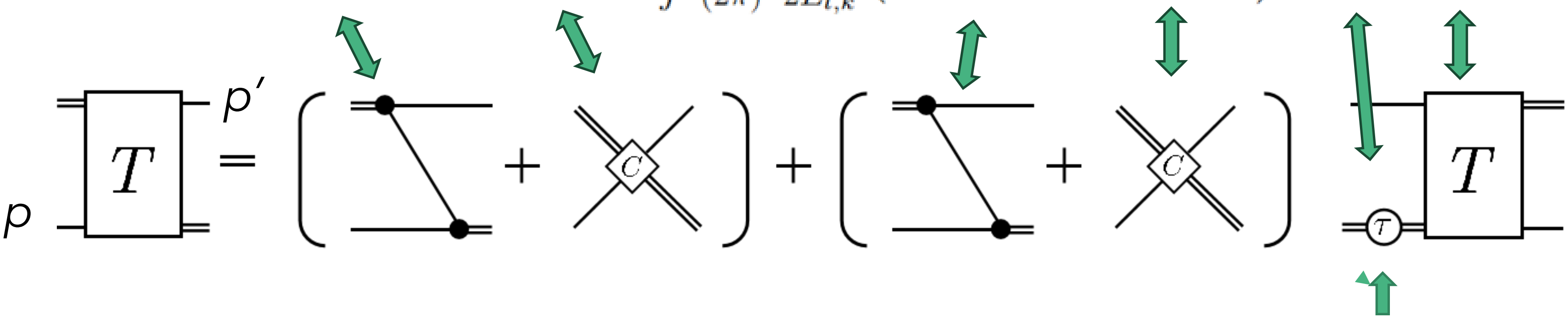


* "Isobar" stands for two-body sub-amplitude which can be resonant or not; can be matched to CHPT expansion to one loop if desired. Isobars are re-parametrizations of full 2-body amplitudes [Bedaque] [Hammer]

Resulting scattering equation [Mai et al./1706.06118]

- 4-dim BSE becomes 3-dim by putting spectator on-shell (choice)

$$\tilde{T}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) = \tilde{B}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) + \tilde{C}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) + \int \frac{\mathrm{d}^3 l}{(2\pi)^3\, 2E_{l,k}} \left(\tilde{B}_{jk}(s,\boldsymbol{p}',\boldsymbol{l}) + \tilde{C}_{jk}(s,\boldsymbol{p}',\boldsymbol{l})\right) \tilde{\tau}_k(\sigma_l)\, \tilde{T}_{ki}(s,\boldsymbol{l},\boldsymbol{p})$$



Exchange B:
- Complex
- Required by unitarity

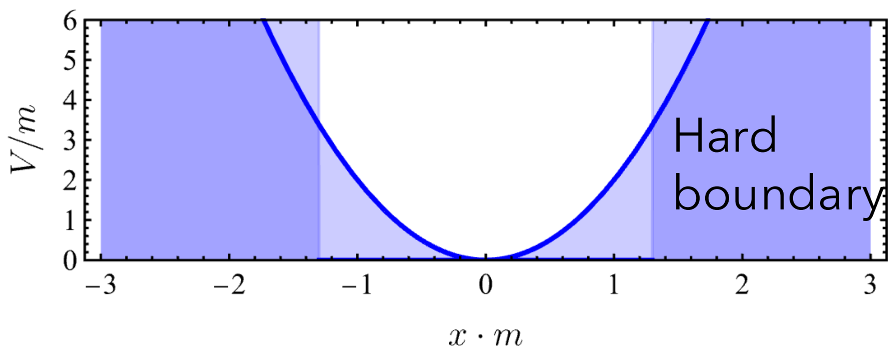
Contact term C:
- Does not destroy unitarity
- <u>Free parametrization: fit to data</u>

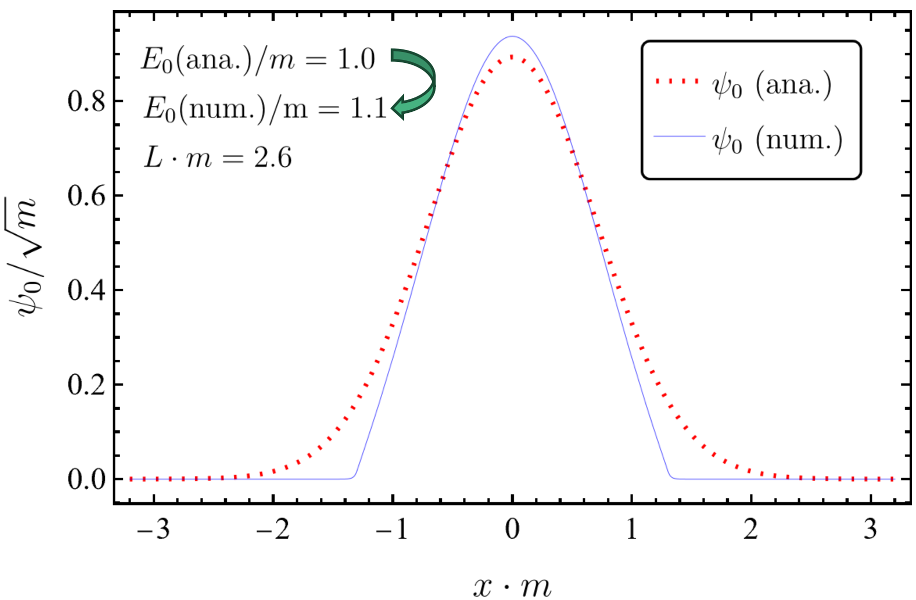
Isobar-spectator Green's functions

Finite-Volume Effects

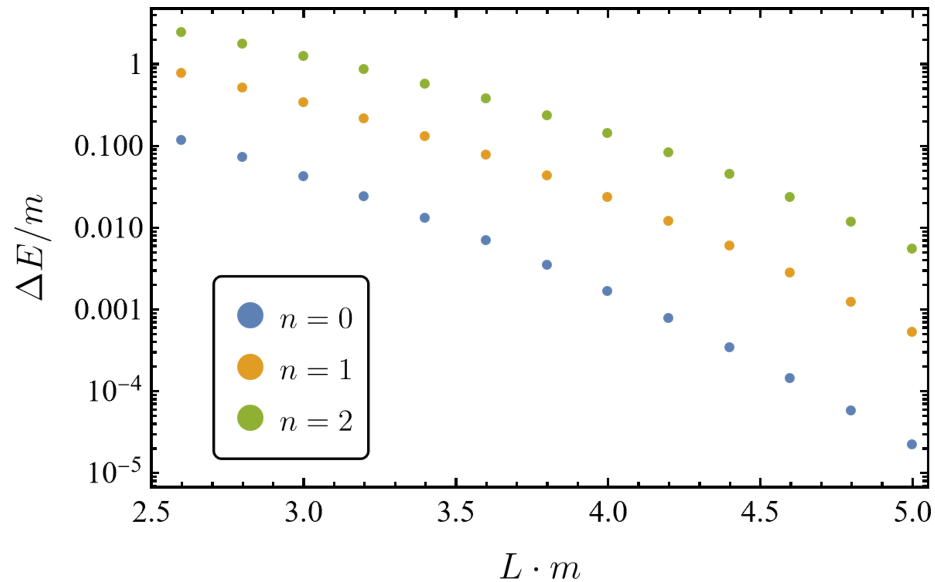
[https://blogs.gwu.edu/doring/]

A wave function is squeezed into a finite volume





Distortion of the energy spectrum as function of box size



Periodic Boundary Conditions

$$\Psi(\vec{x}) \stackrel{!}{=} \Psi(\vec{x} + \hat{\mathbf{e}}_i\, L) = \exp\left(i\, L\, q_i\right) \Psi(\vec{x})$$

$$q_i = \frac{2\pi}{L}\, n_i, \qquad n_i \in \mathbb{Z}$$

only discrete momenta allowed

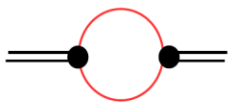
Lattice QCD: Finite-volume unitarity (FVU)

[1709.08222/Mai, MD]

Two-body unitarity

↓

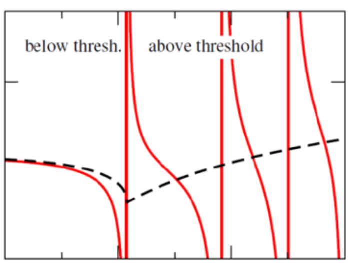
On-shell condition



$\int \to \sum$

↓

Imaginary parts



Infinite → Fin. Vol

↓

Power-law fin-vol. effects

↓

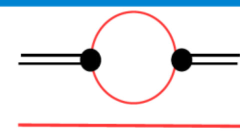
Lüscher

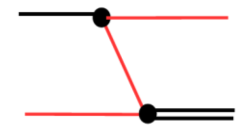
$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\,G(E)\right)$$

Three-body unitarity

↓

On-shell condition





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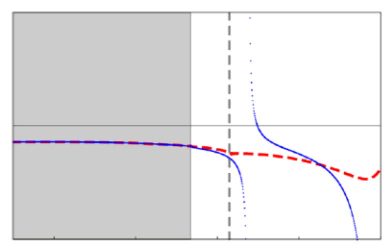
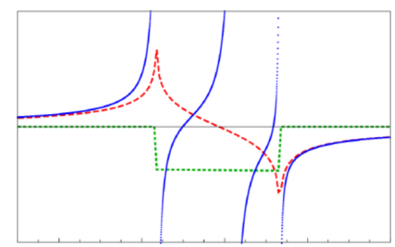
Imaginary parts





↓

Power-law fin-vol. effects

↓

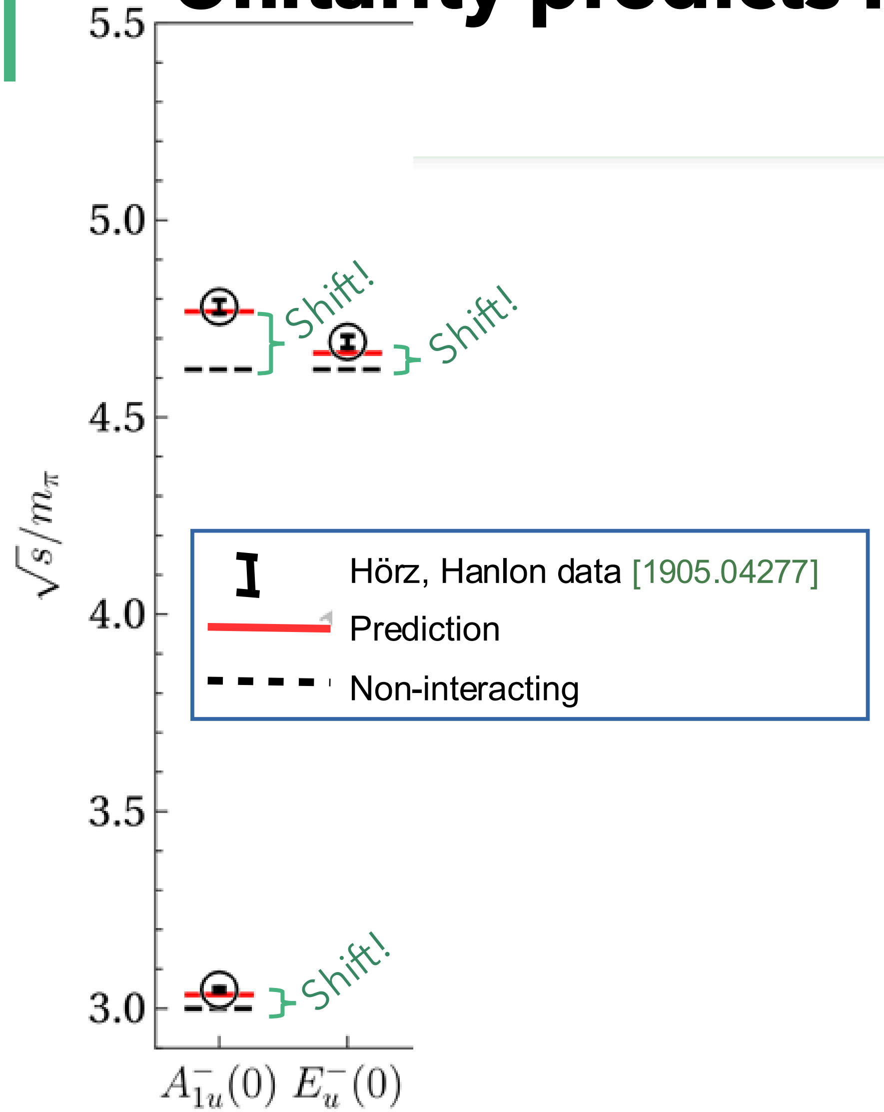
Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{uu'}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_\mathbf{s}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

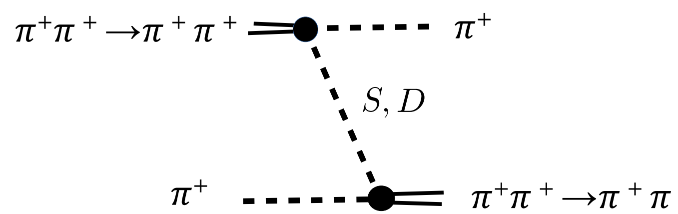
[Similar: RFT, (NR)EFT, …]

Unitarity predicts lattice: $3\pi^+$

[1911.09047, GWQCD/Culver et al.]



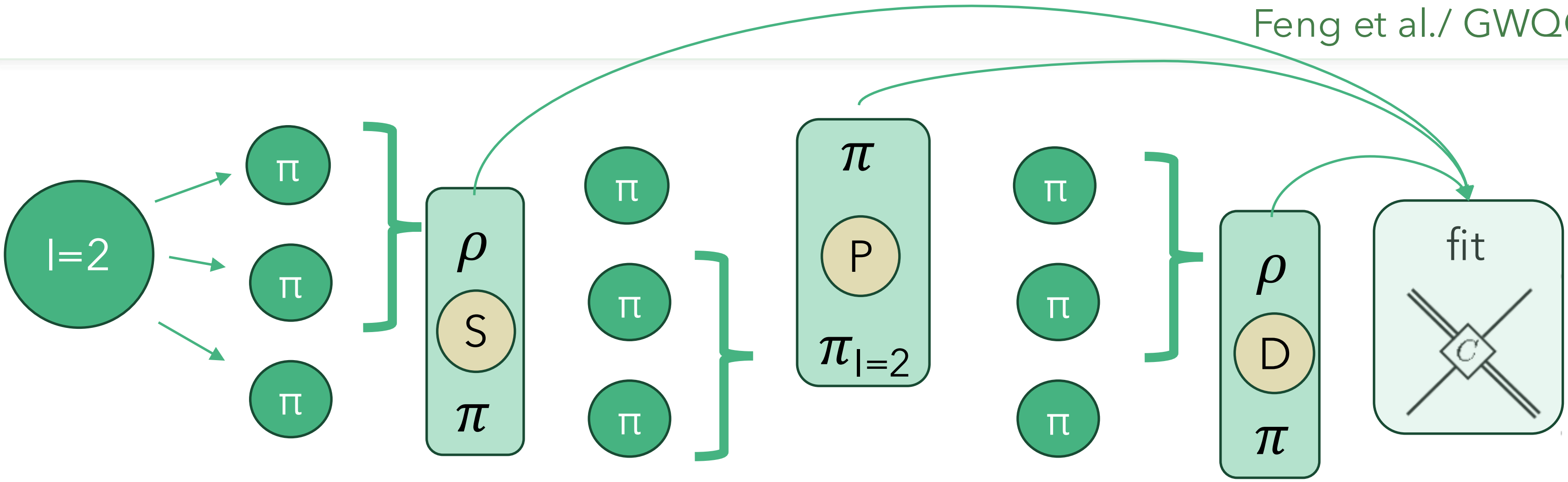
S **D** (lowest participating wave)



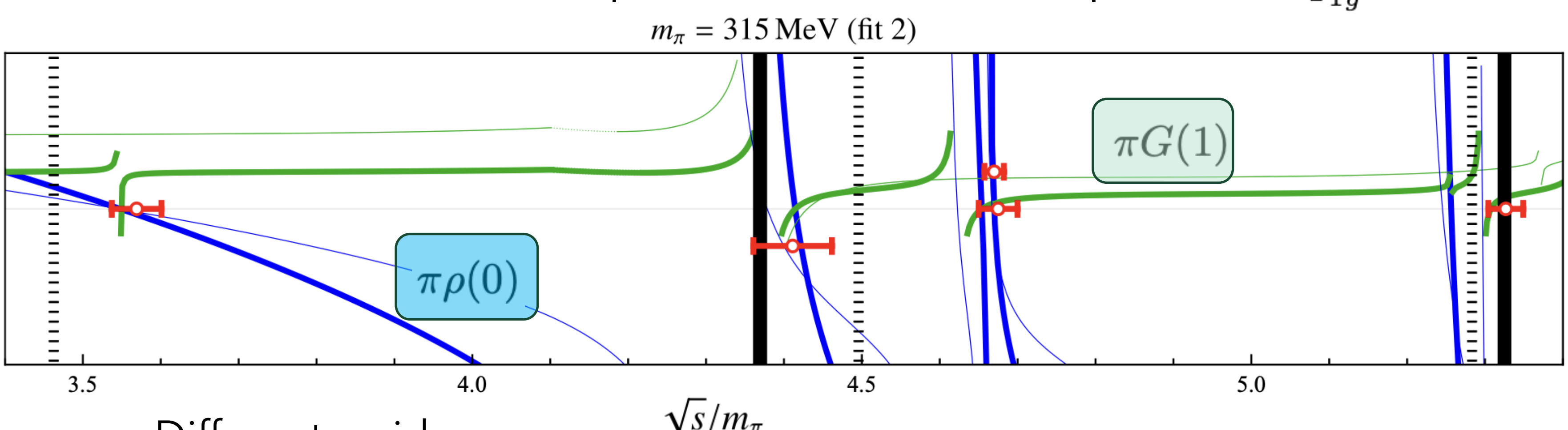
Particle exchange quantitatively predicts energy eigenvalues

From I=3 to I=2: isobar with resonance

[2601.16916, Feng et al./ GWQCD]



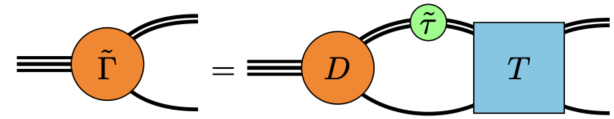
- Finite-volume lattice spectrum and FVU amplitude $T^{-1}_{T_{1g}}$



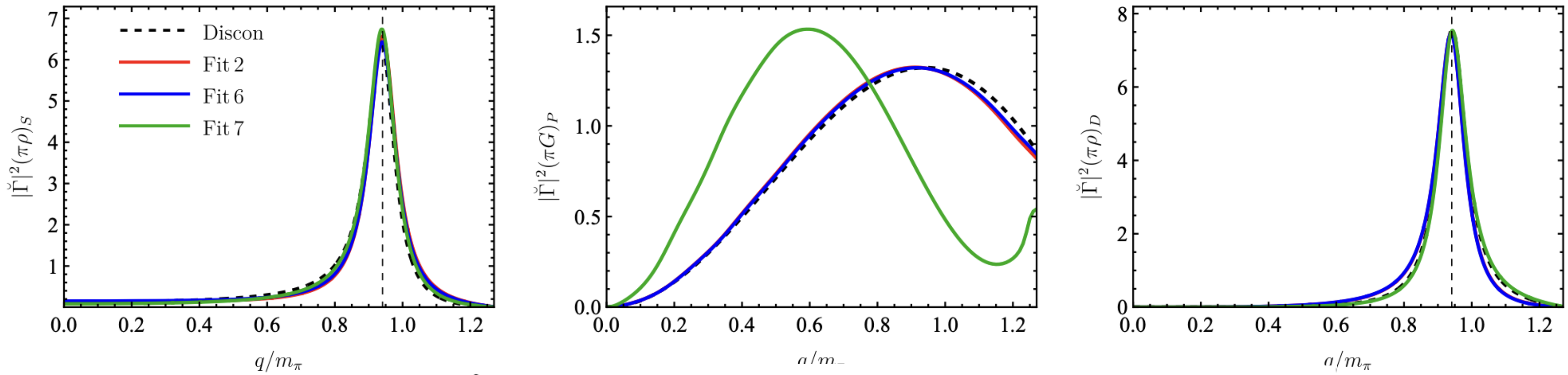
Different residua

See also [2510.24894/RFT]

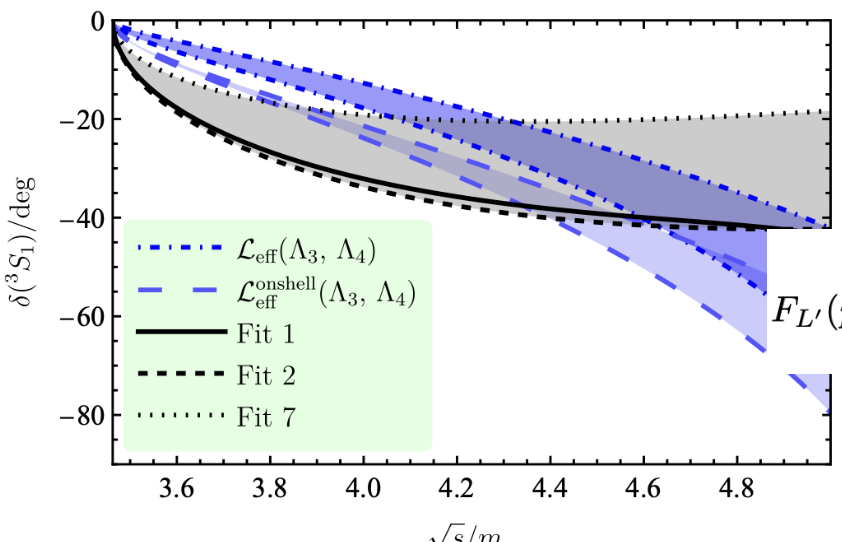
Infinite-volume amplitudes

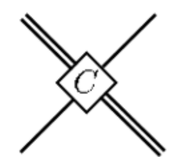


- 3-body production amplitude for fixed energy $\sqrt{s} = 4m_\pi$



- Narrow ρ limit





fit

$$F_{L'}(p')\left(\frac{p'}{m_\pi}\right)^{L'}\breve{c}_{L'L}(s)\left(\frac{p}{m_\pi}\right)^{L}F_L(p)$$

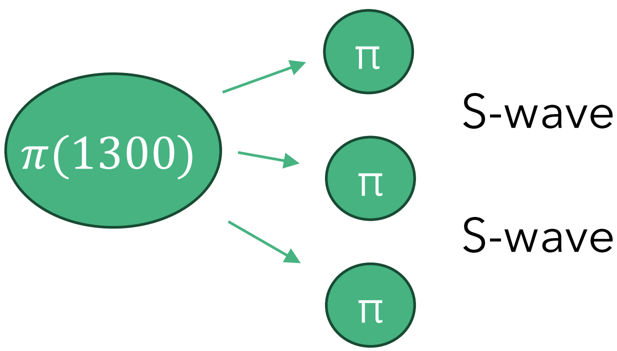
… or predict

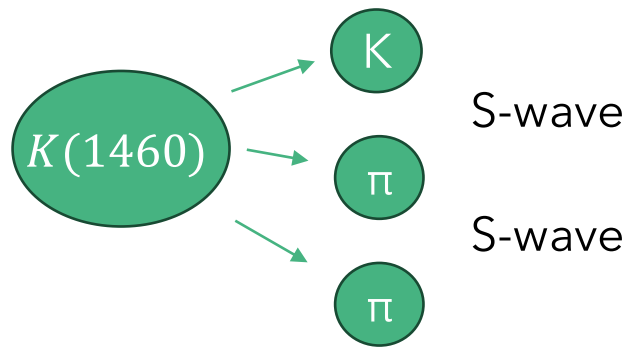
$$\mathcal{L} = -\frac{1}{4f_\pi^2}\,\langle [V^\mu, \partial^\nu V_\mu][P, \partial_\nu P]\,|\,\rangle$$

Excited pseudoscalar mesons

- Adding $J^P=0^+$ quantum numbers to pseudoscalar mesons (π, η, K) in S-wave

  → Excited pseudoscalar mesons





- Found in CLEO [JHEP 05 (2017)]
- Not in Compass
- State found in lattice QCD [Dudek et al., PRD 82, 2010]
- Dynamically generated from πf_0? [Martinez et al. PRD 84, 2011]
- New evidence from lattice QCD [Yan, Mai, Y. Feng, MD et al.2510.09476 [hep-lat]

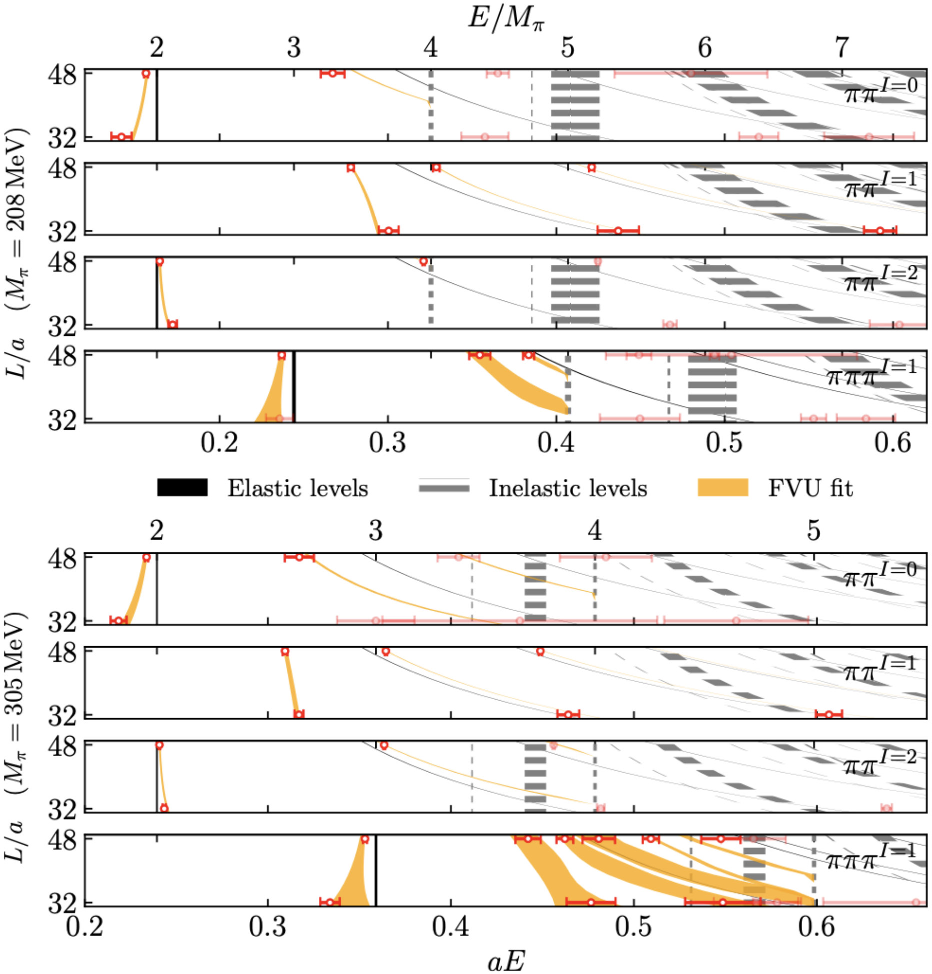
- Experimentally well established
- Hadronic molecule Kf_0-Ka_0? [Albaladejo PRD82 (2010) Martinez PRC 83 (2011); Filikhin PRD 102 (2020), Zhang, Hanhart EPJA 58 (2022)]

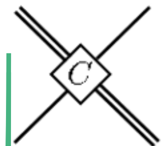
From I=2 to I=1: $\pi(1300)$ on the lattice

- Exploratory spectroscopy on the lattice:

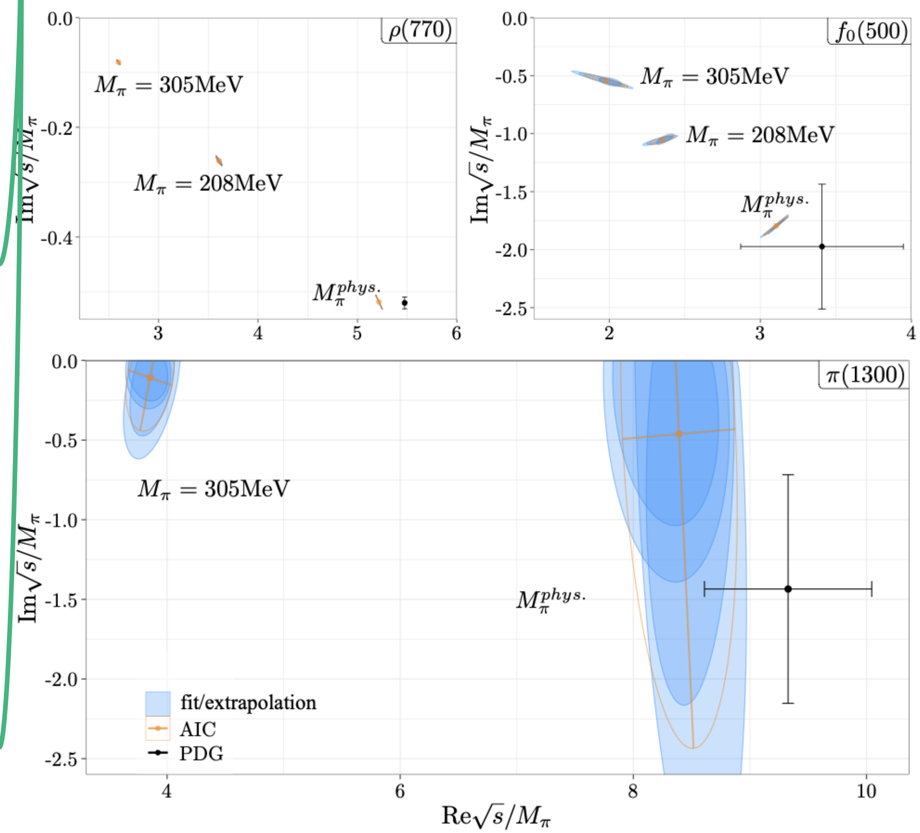
[2510.09467/Yan et al.]

Does the $\pi(1300)$ exist?



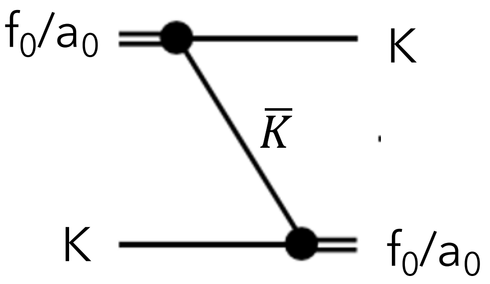


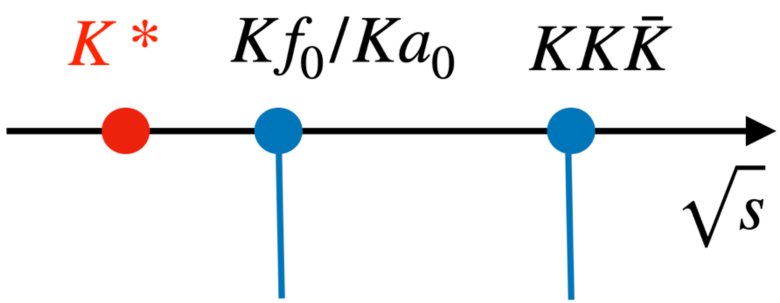
$$c^{\alpha\beta} = c_c^{\alpha\beta} + \frac{c_p^{\alpha\beta}}{s - m_0^2}\,, \qquad \alpha = \beta \in \{\sigma\pi, \rho\pi\}$$



Extension to strangeness: the K(1460)

[2511.02543 ,
MD, K. Khemchandani, A. Martinez

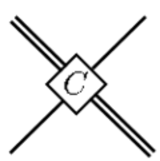




- Hadronic molecule hypothesis (stable, mass-degenerate f_0 and a_0):

Zhang, Hanhart et al. [Eur. Phys. J A 58,22 (2022)] predict binding energy in MeV:

α	Λ [GeV]	$t(a)$	$+t(b)$	$+s(a)$	$+s(b)$	$+$*streched boxes*	$+$*crossed boxes*
0	0.5	0.51 (1)	0.66 (1.29)	0.83 (1.63)	0.52 (1.02)	0.58 (1.14)	0.68 (1.33)
1	0.5	1.63 (1)	2.51 (1.54)	3.93 (2.41)	**1.63** (1)	1.86 (1.14)	2.21 (1.36)
$1^\dagger$	0.5	1.58 (1)	2.44 (1.54)	3.81 (2.41)	1.59 (1.01)	1.81 (1.15)	**2.16** (1.37)



=0

A new model for strange mesons

- Two-body input from experiment when available, chiral models if not: Full coupled-channel amplitudes
- Channels: $K f_0^{K\bar{K}}$ $K f_0^{\pi\pi}$ $K a_0^{K\bar{K}}$ $K a_0^{\pi\eta}$ $\bar{K} K_2^{KK}$ $\pi\kappa^{K\pi}$ $\eta\kappa^{K\pi}$
- Sub-channels have $f_0(\pi\pi, K\bar{K}), a_0(\pi\eta, K\bar{K})$

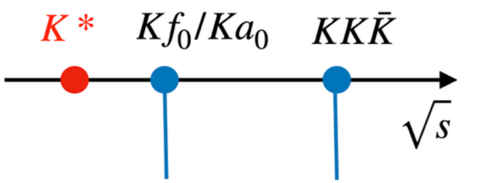
Strangess-2 kaon cluster

- Hanhart result reproduced: With stable isobars, one gets very shallow 3-body bound state in $K f_0^{K\bar{K}}$ $K a_0^{K\bar{K}}$
- Slowly switch on all channels and full cc two-body amplitudes

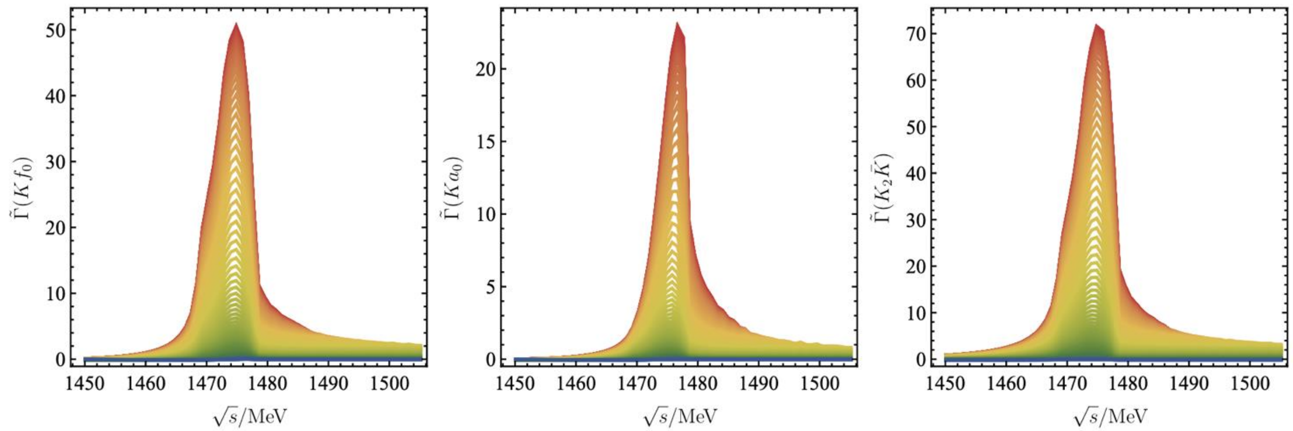
State	$M_R - i\Gamma_R/2$
$f_0(500)$	$471.97 - i189.07$
$f_0(980)$	$988.08 - i17.36$
$a_0(980)$	$982.32 - i53.63$
$K_0^*(700)$	$832.57 - i230.70$

Two-body resonances

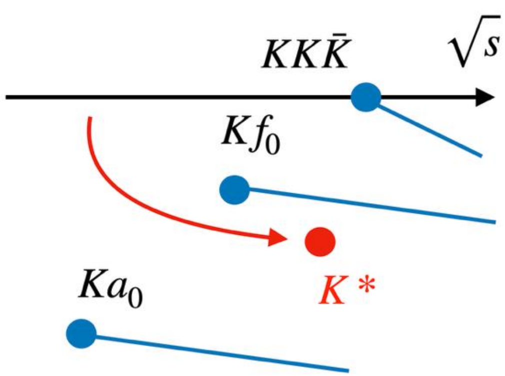
Pole trajectory

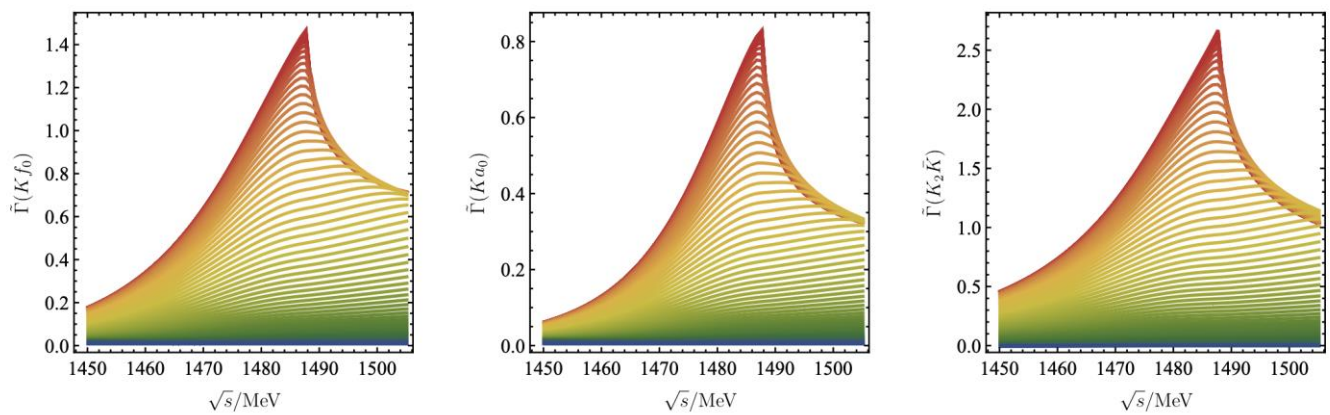


Stable isobars (slightly different model)



Realistic, coupled-channels isobars

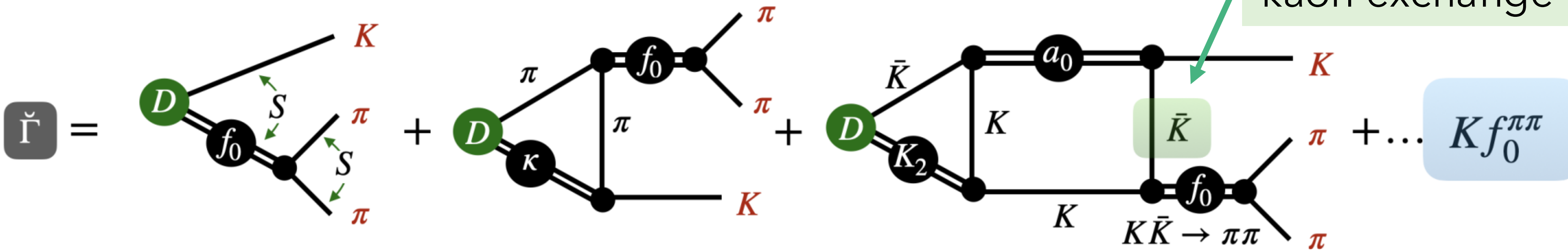


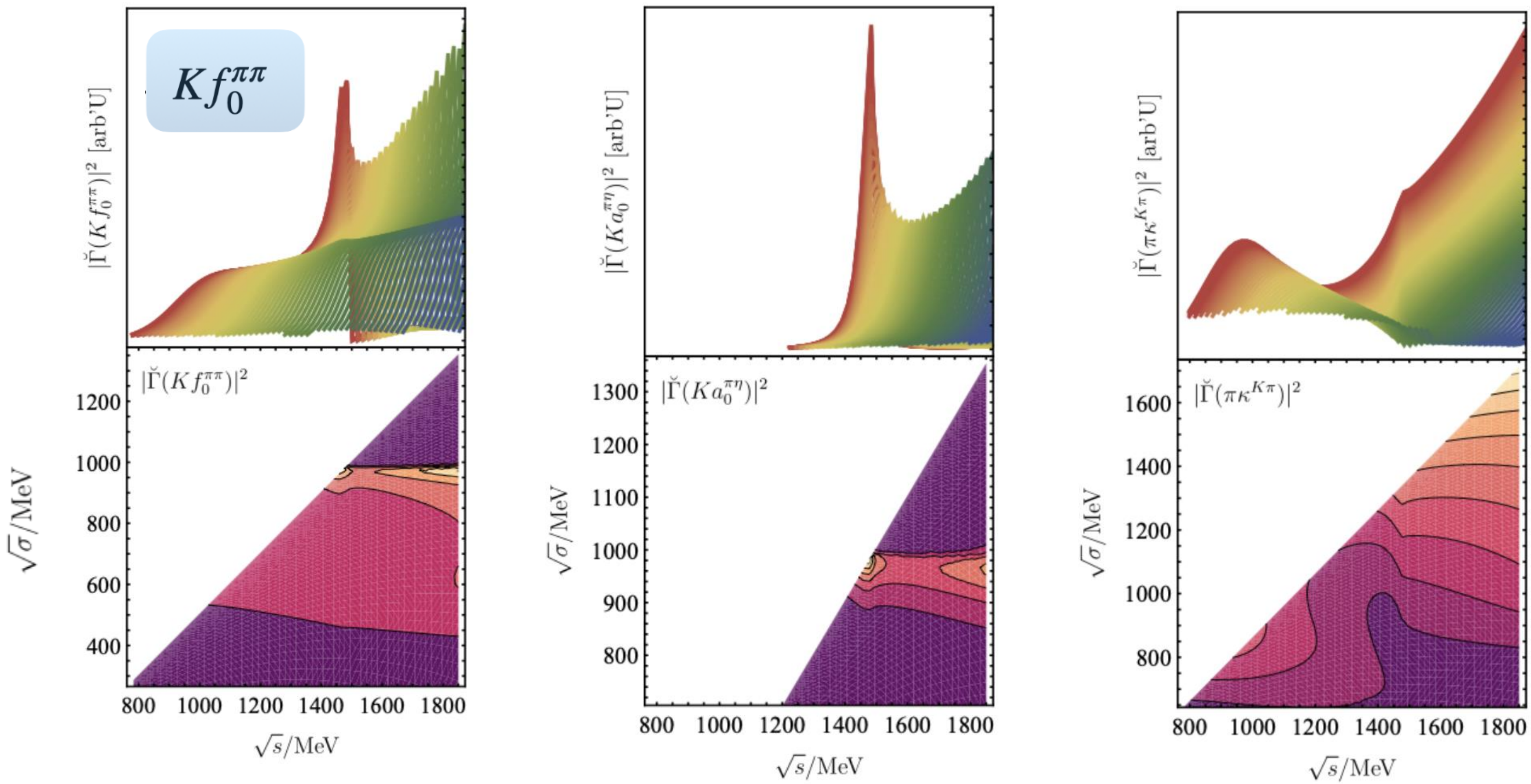


All that remains is an enhanced cusp

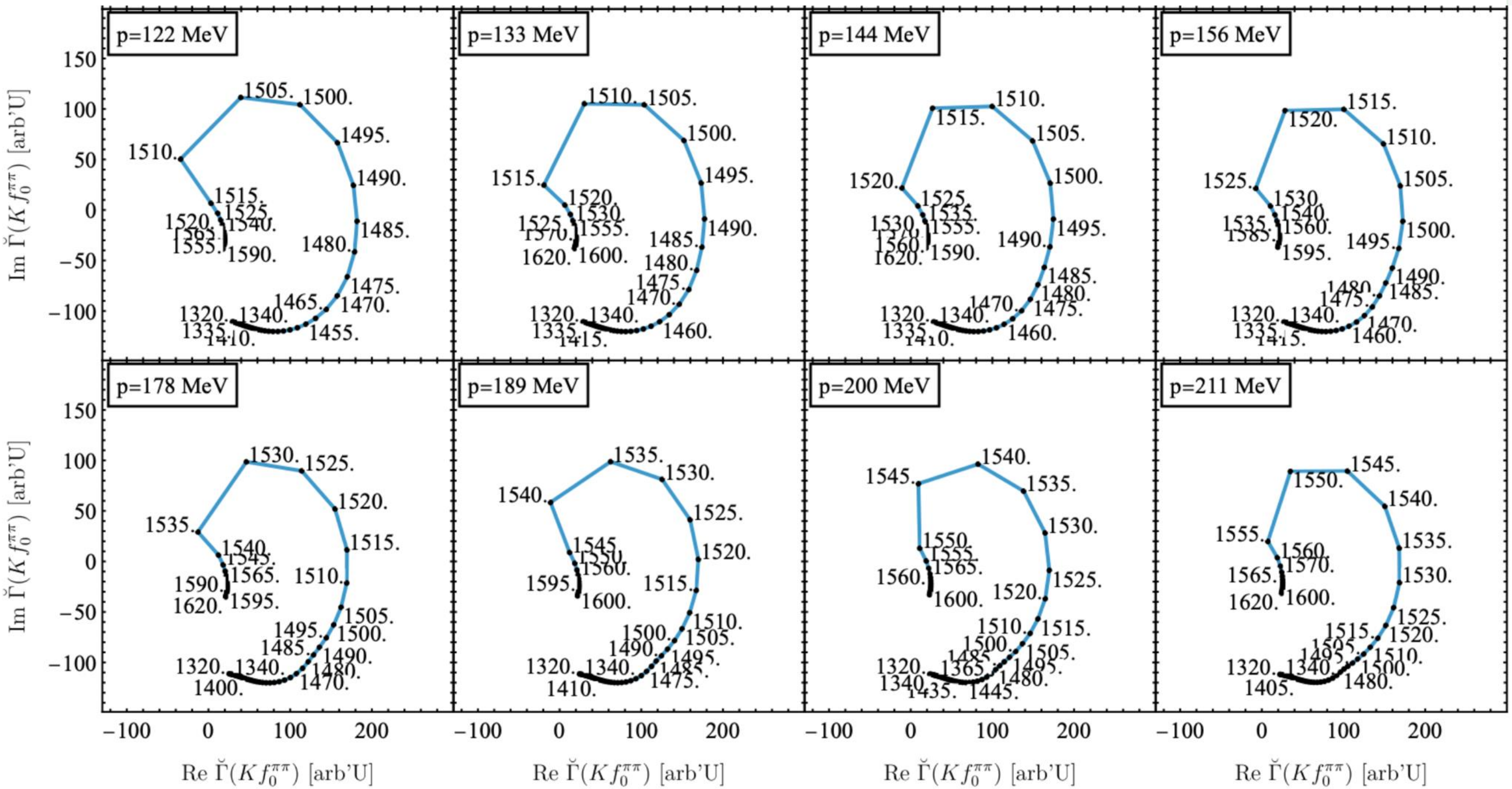
Production reactions

Almost on-shell kaon exchange





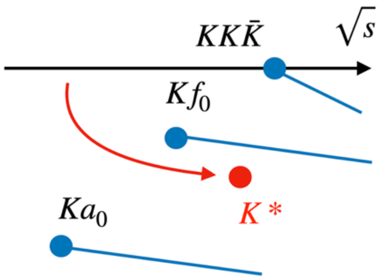
Phase motion



Resonance-like phase motion in the Argand plot (fix spectator momenta)

Summary

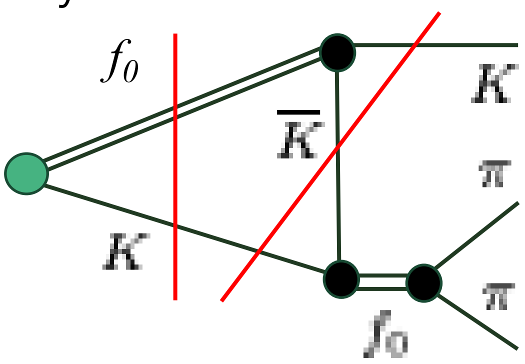
- Strong three-body effects in excited hadrons from the lattice
  - Pions on the lattice in isospin **I=**…
  - …**3**: All-repulsive channel shows the relevance of particle exchange
  - …**2**: Sub-channels/isobars with resonances; allows comparison with EFT
  - …**1**: Exploratory spectroscopy on the lattice allows to determine $\pi(1300)$
- Intricate 3-body dynamics of the the K(1460)



$$Kf_0^{K\bar{K}}\ \ Kf_0^{\pi\pi}\ \ Ka_0^{K\bar{K}}\ \ Ka_0^{\pi\eta}\ \ \bar{K}K_2^{KK}\ \ \pi\kappa^{K\pi}\ \ \eta\kappa^{K\pi}$$

  - Generation from on-shell kaon exchange
  - But realistic isobars & channels necessary by unitarity make the resonance disappear
  - Still, a triangle singularity at 3K threshold

  \+ complex threshold Kf_0 still generate resonance-like strong enhancement

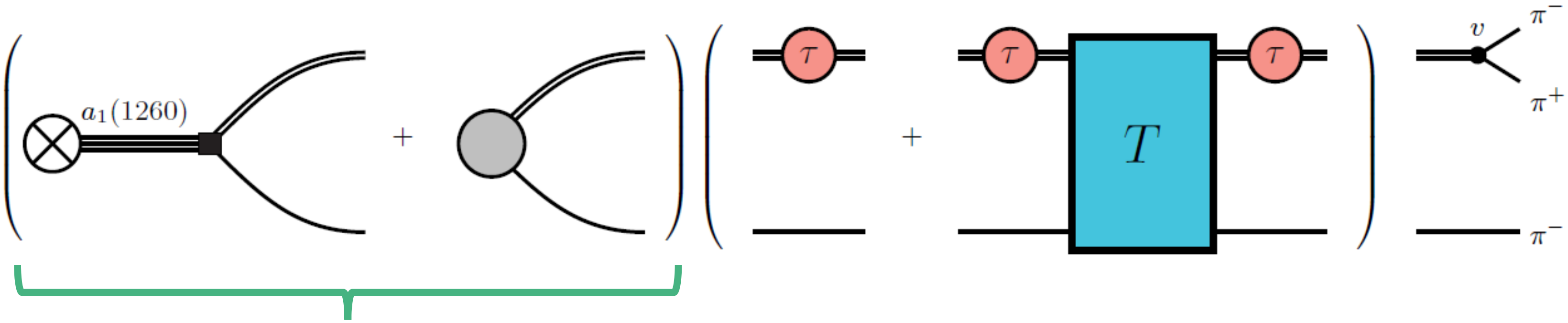


Spare slides

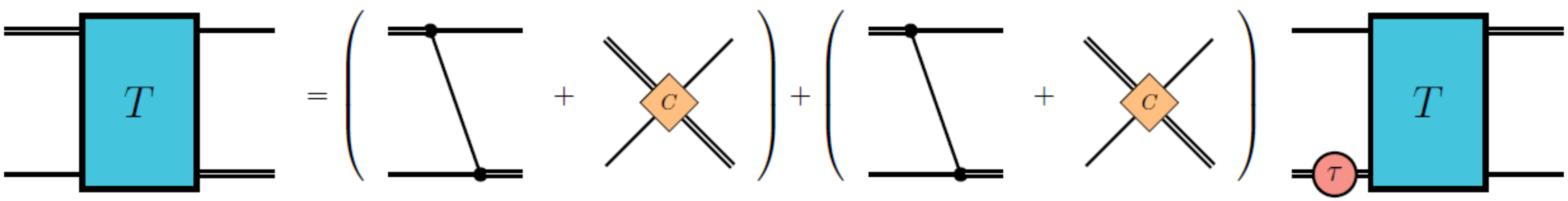
The $a_1(1260)$ and its Dalitz plots

[Sadasivan 2020]

- Disconnected and connected decays for three-body untarity



Additional fit parameters

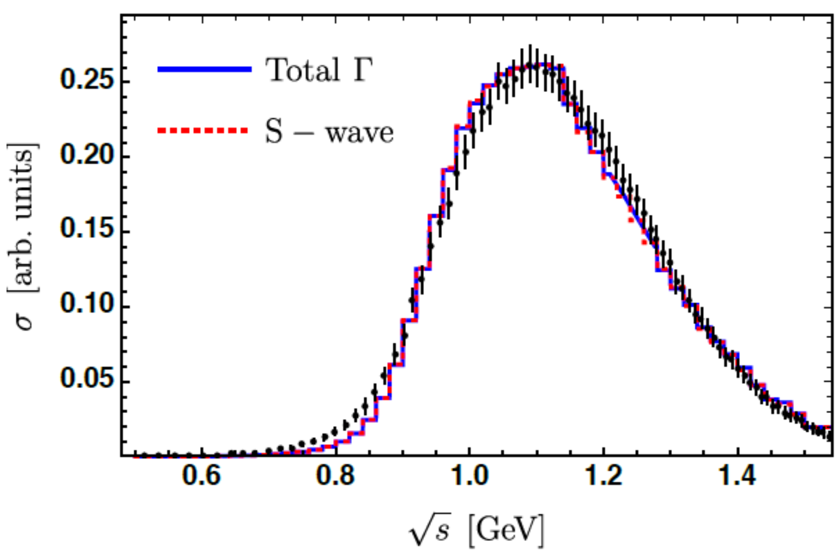


Fitting the lineshape & predicting Dalitz plots

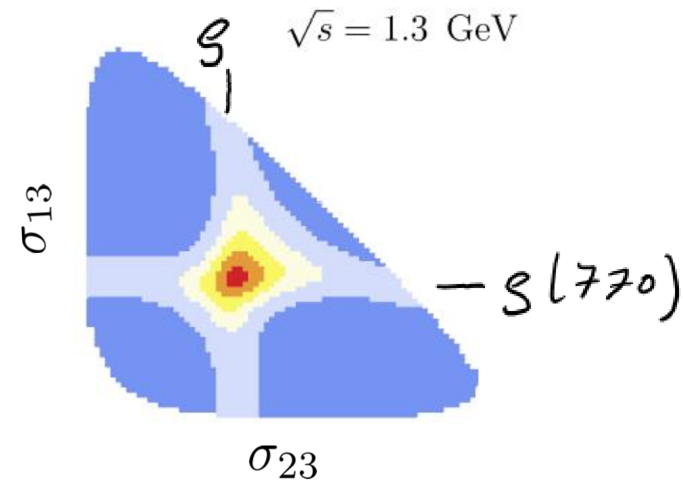
[Sadasivan 2020]

- One can have $\pi\rho$ in S- and D-wave coupled channels
- Fit contact terms to the lineshape from Experiment (ALEPH)

$a_1 \to \pi^-\pi^-\pi^+$ (symmetrize π^-'s!)



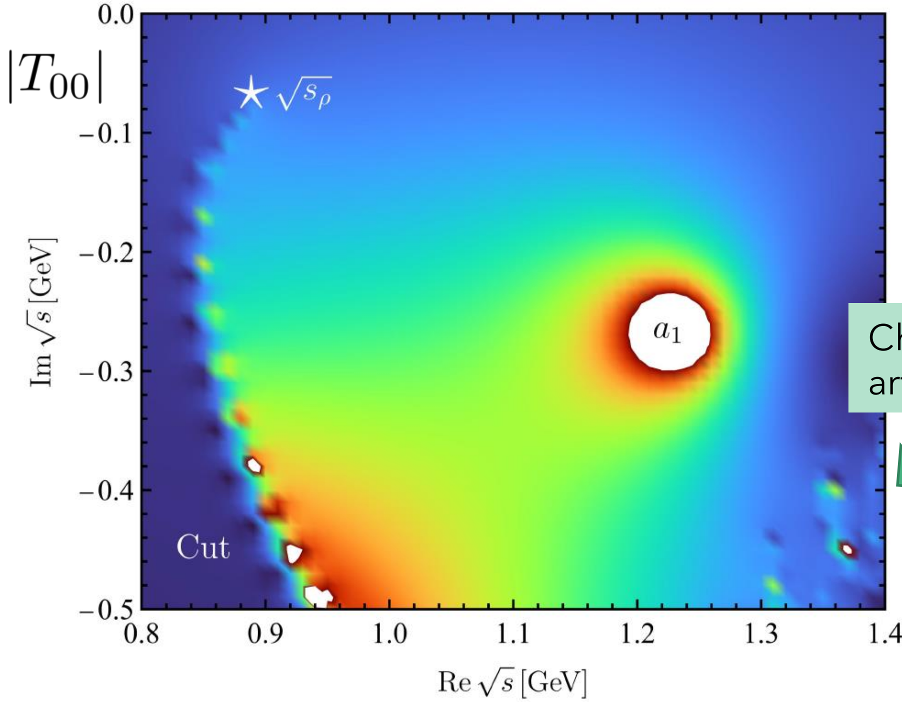
predict

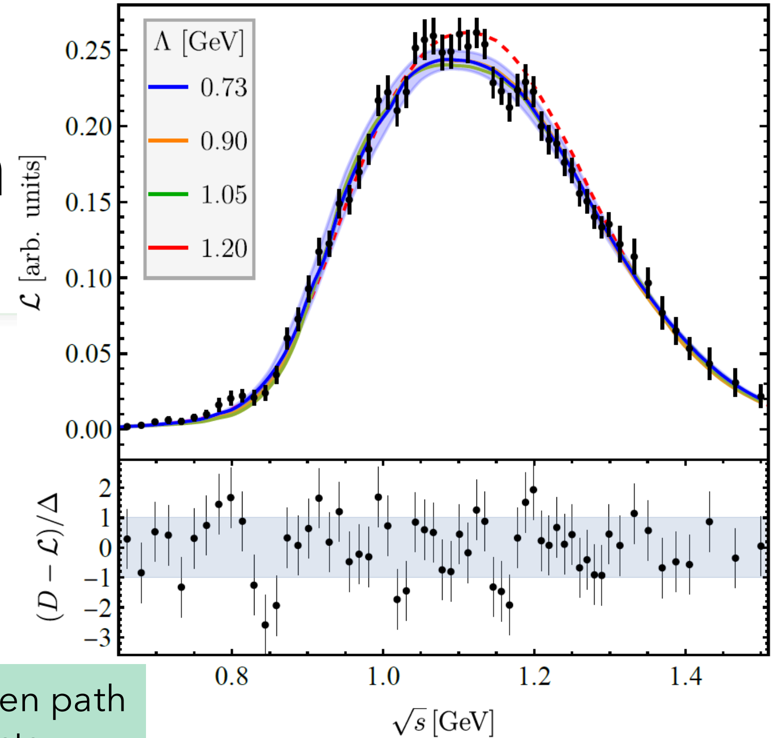


Where is the resonance pole in s? Pole positions & residues are reaction-independent characteristics of resonance mass, width, and branching ratios

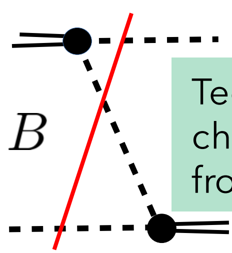
Result: Pole position

Technicalities analytic cont.: contour deformation } [Sadasivan (2021)] [Doering (2009)]



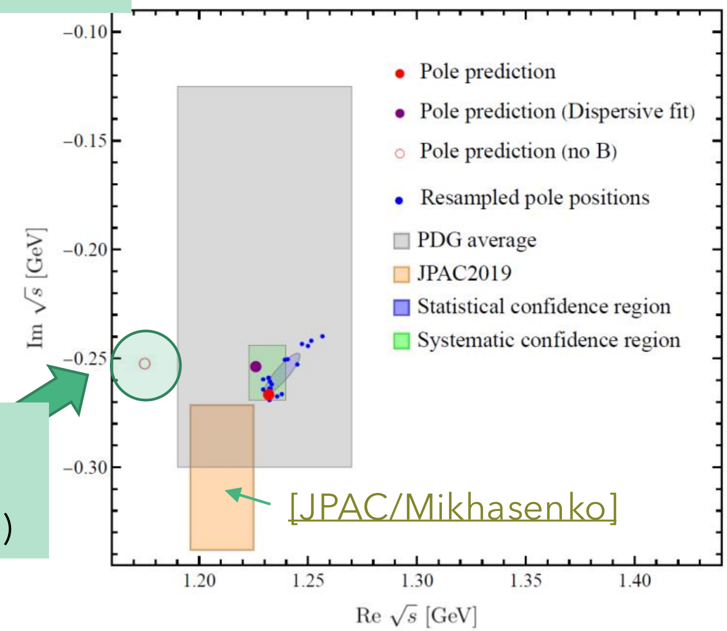


Chosen path artifacts



Technical challenges from 3B cuts

If the B-term is neglected + refit (unitarity violated)

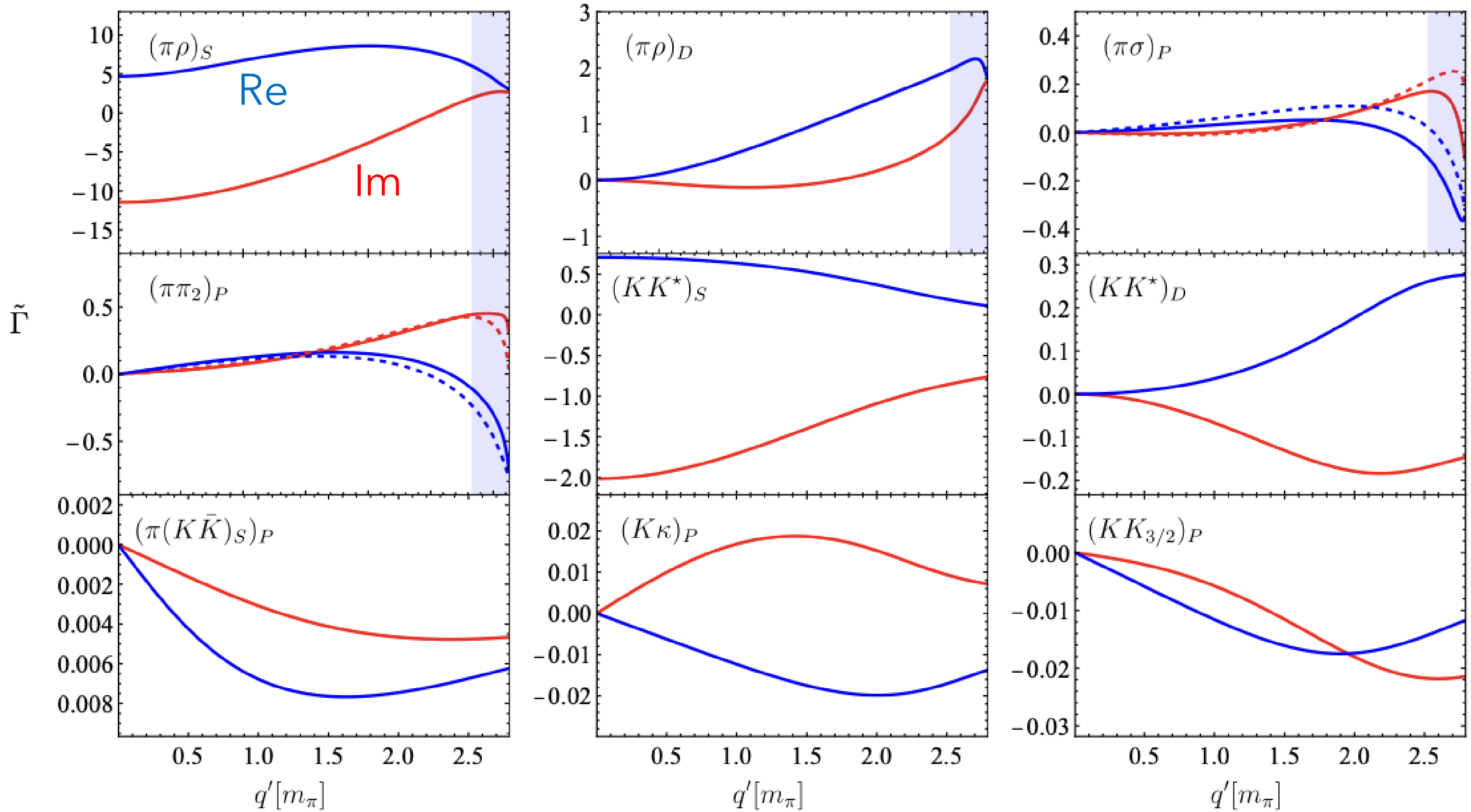


[JPAC/Mikhasenko]

See also: S. M. Dawid [2303.04394 [nucl-th]]

Production amplitude 9-channel model

(Only the (non-trivial) rescattering piece) $3m_\pi < W < 2m_K + m_\pi$

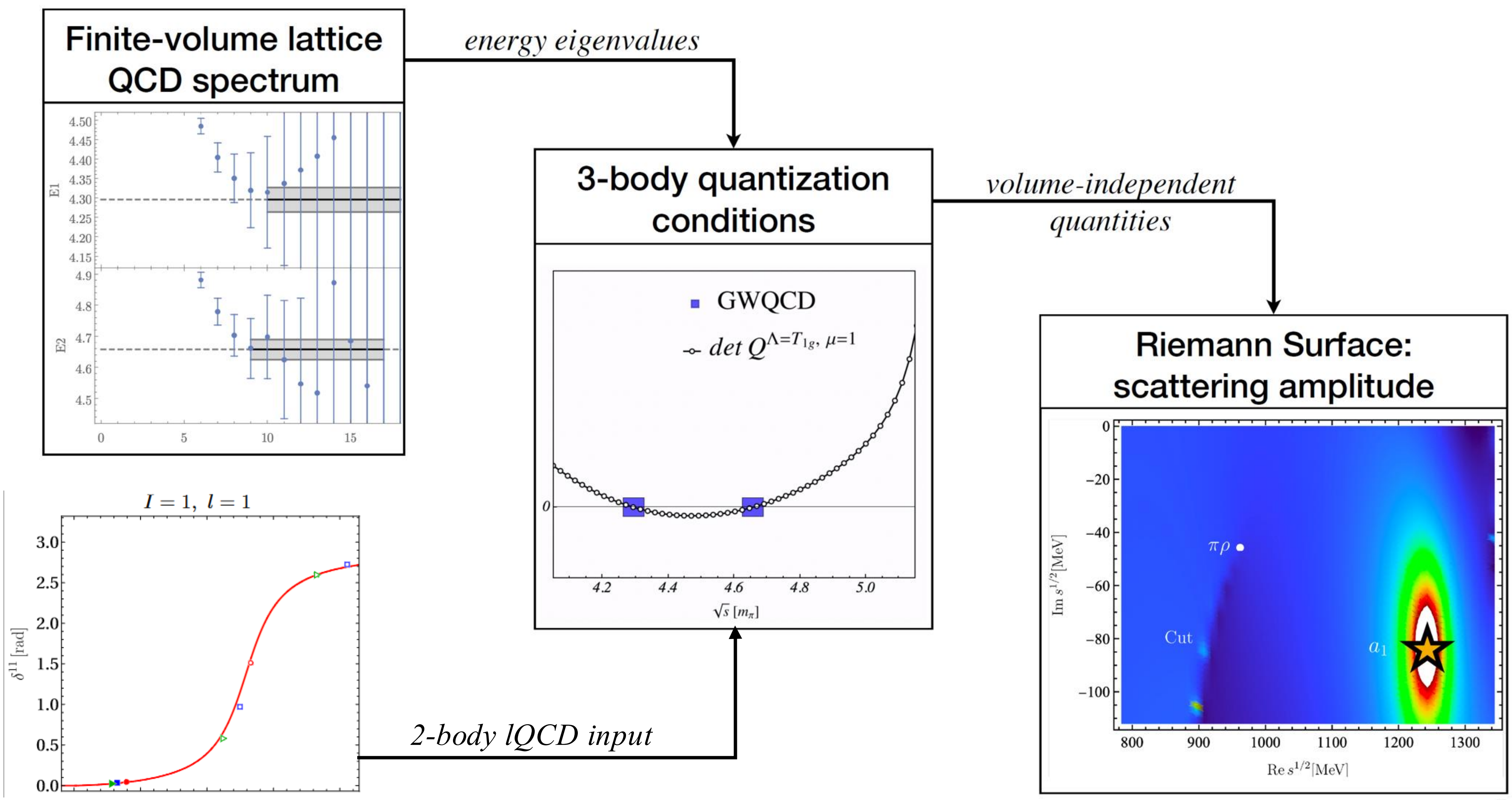


Dashed lines: with **πρ** switched off (influence of coupled channels)

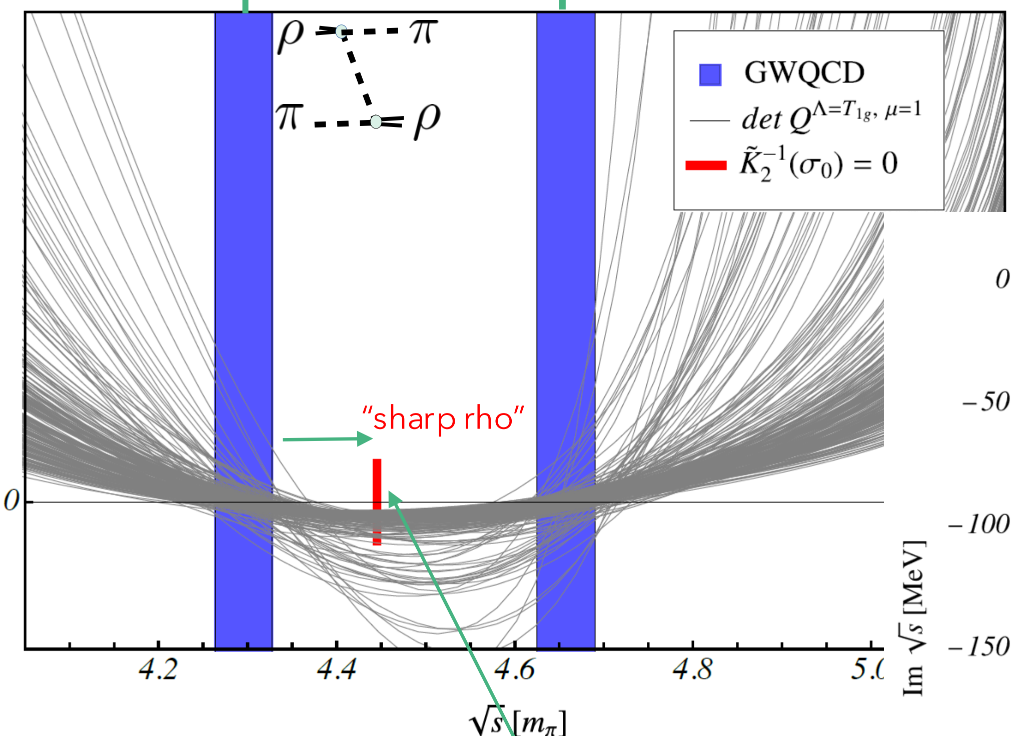
Extraction of $a_1(1260)$ from lQCD

[Mai/MD/GWQCD, PRL 2021]

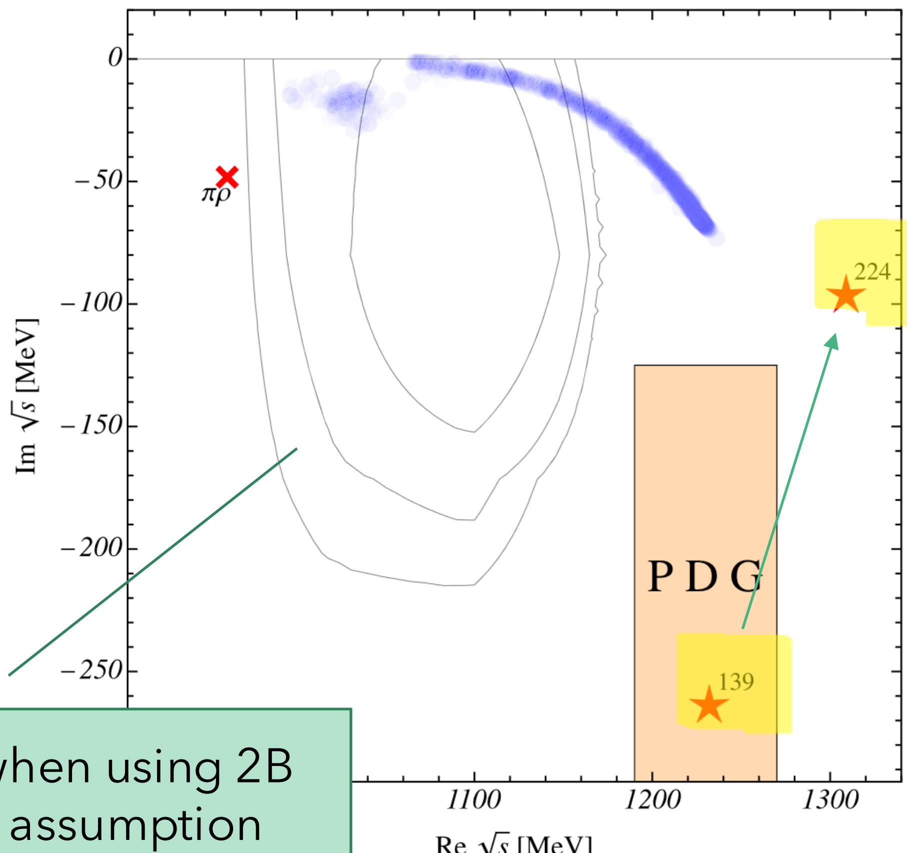
- First-ever three-body resonance from 1st principles (with explicit three-body dynamics).



a_1 results – Details



$$0=\det\left[B(s)+C(s)-E_L\left(\tilde{K}_2^{-1}(s)-\Sigma_2^L(s)\right)\right]_{\substack{(\lambda'\lambda)\\(\boldsymbol{p}'\boldsymbol{p})}}$$



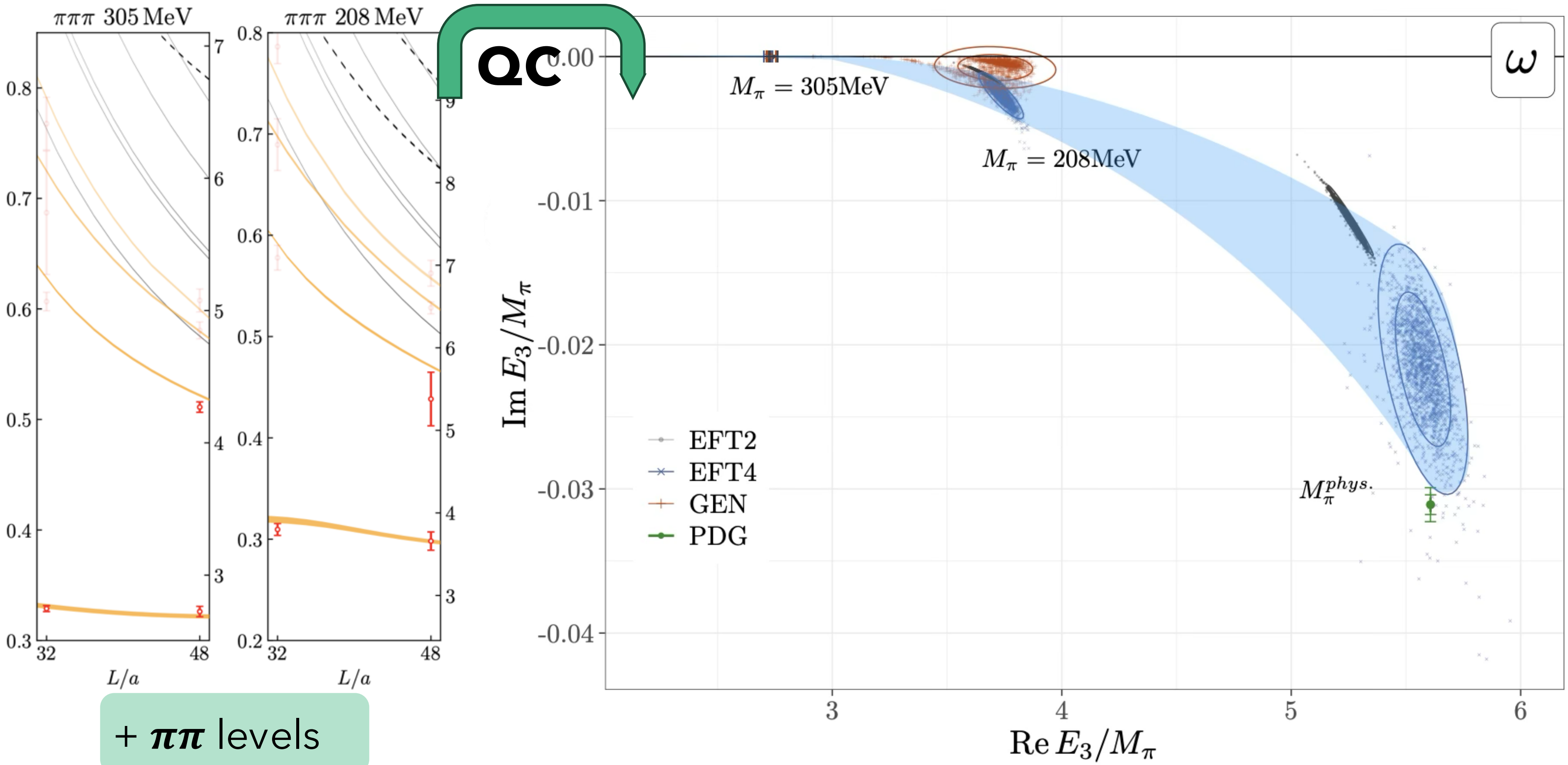
PDF for pole when using 2B Breit-Wigner assumption

$\omega \to \pi\pi\pi$ from Lattice QCD

[Yan, Mai, et al., PRL (2024)]

Lattice QCD
Nf = 2 + 1 Clover fermions
2/3 particle operators
2 pion masses, 2 volumes

Result
- Various EFT based ansatzes
- ω becomes a bound state at ~300 MeV
- at the physical point very close to the EXP value

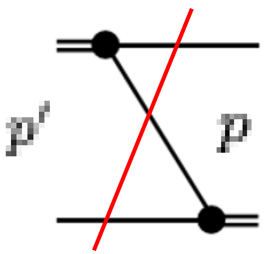


+ $\pi\pi$ levels

How to solve the scattering equation

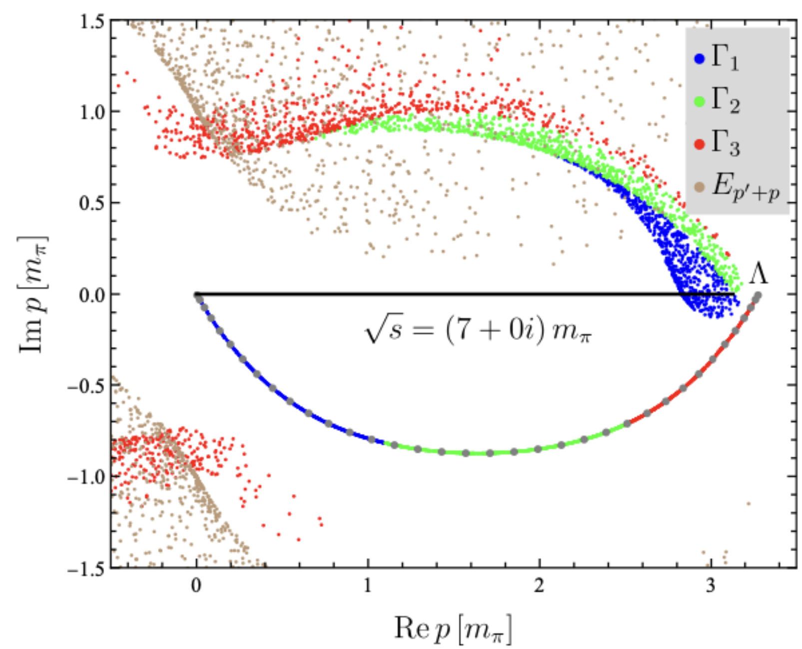
$$\tilde{T}_{ji}(s,p',p) = \boxed{\tilde{B}_{ji}}(s,p',p) + \tilde{C}_{ji}(s,p',p) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3\, 2E_l} \left(\tilde{B}_{jk}(s,p',l) + \tilde{C}_{jk}(s,p',l)\right) \tilde{\tau}_k(\sigma_l)\, \tilde{T}_{kj}(s,l,p)$$

- **Three-body cuts**



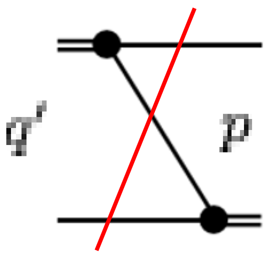
$$\tilde{B}_{ji}(s,\boldsymbol{p}',\boldsymbol{p}) = \frac{(\tilde{\boldsymbol{I}}_F)_{ji}\, v_j^*(p, P-p-p')v_i(p', P-p-p')}{2E_{p'+p}(\sqrt{s} - E_p - E_{p'} - E_{p'+p}) + i\epsilon}$$

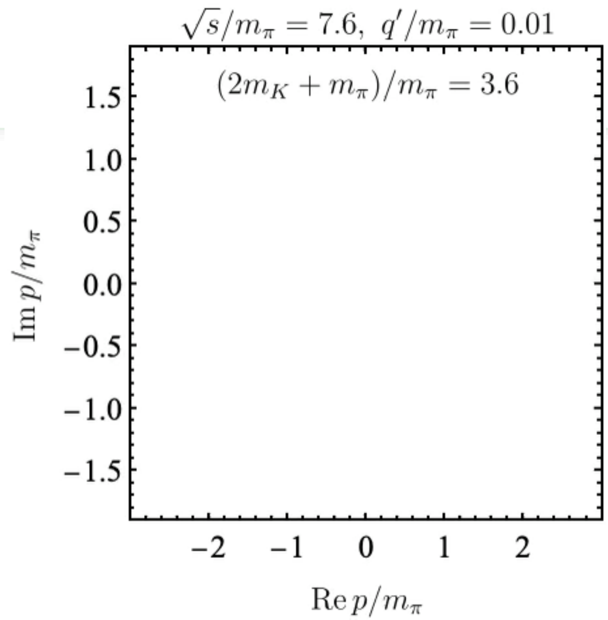
- Angle, energy dependent
- Depend also on p' and p
- Solve LSE for complex momenta on a deformed contour

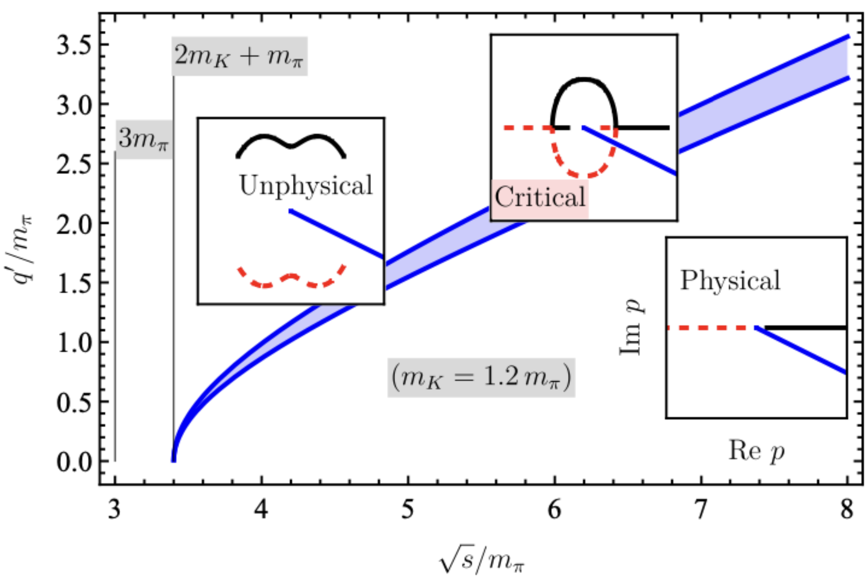


How to get the solution for real, physical momenta

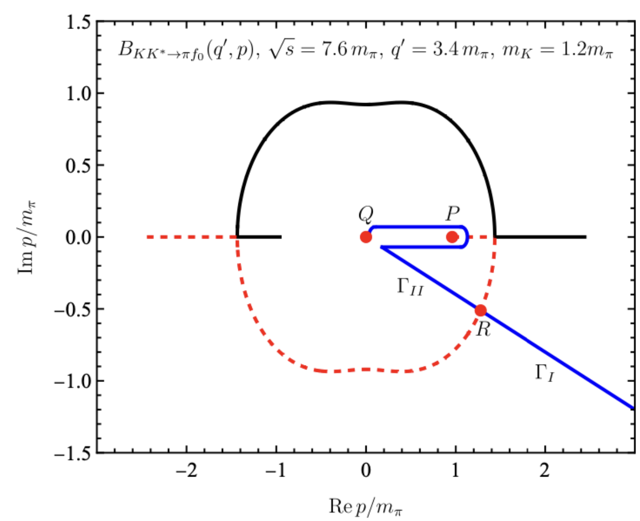
- No solution for real momenta in"critical region"







Solution 1: Contour deformation [Cahill & Sloane]



How to get the solution for real, physical momenta

Solution 2: Direct inversion [Ziegelmann et al.]

Production amplitude
$$\tilde{\Gamma}_j^T(s,q') = D_j(s,q') + \int_0^\Lambda \frac{dq\,q^2}{(2\pi)^3 2E_q}\,\tilde{B}ji(s,q',q)\,\tilde{\tau}_i(\sigma(q))\,\tilde{\Gamma}_i^T(s,q)$$

Ansatz
$$\tilde{\Gamma}^T(q) \approx \sum_{i=1}^N \tilde{\Gamma}^T(q_i)\,H_i(q)$$
with Lagrange polynomials
$$H_i(q) = \frac{\prod_{j\neq i}^N (q-q_j)}{\prod_{j\neq i}^N (q_i-q_j)}$$

Makes integral equation a matrix equation
$$\tilde{\Gamma}^T(q_j) = D(q_j) + A_{ji}\tilde{\Gamma}^T(q_i)$$

With singular integrals
$$A_{ji} = \int_0^\Lambda \frac{dq\,q^2}{(2\pi)^3 2E_q}\,\tilde{B}(q_j,q)\,\tilde{\tau}(\sigma(q))\,H_i(q)$$

… for which many established algorithms exist

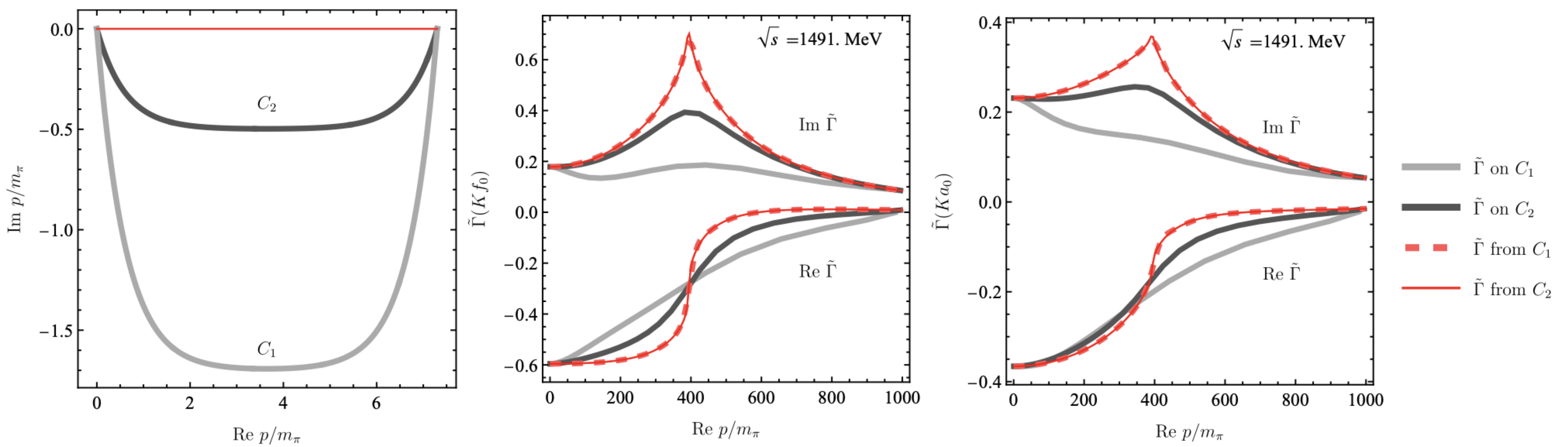
How to get the solution for real, physical momenta

Solution 3: Continued fractions

- Not rigorous as methods 1 and 2, but very good extrapolation properties
- Sample the solution on complex contour and evaluate the continued fraction at real momenta

$$f(x) = f(x_1) + \cfrac{x - x_1}{\rho(x_1, x_2) + \cfrac{x - x_2}{\rho_2(x_1, x_2, x_3) - f(x_1) + \cfrac{x - x_3}{\rho_3(x_1, x_2, x_3, x_4) - \rho(x_1, x_2) + \cdots}}}$$

[example: 2511.02543 ,
MD, K. Khemchandani, A. Martinez



Partial-wave decomposition

- Plane-wave basis

$$T_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) = (B_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1)+C)+$$

$$\sum_{\lambda''}\int\frac{d^3\boldsymbol{l}}{(2\pi)^3 2E_l}(B_{\lambda'\lambda''}(\boldsymbol{p},\boldsymbol{l})+C)\,\tau(\sigma(l))T_{\lambda''\lambda}(\boldsymbol{l},\mathbf{q}_1)$$

$$B^J_{\lambda\lambda'}(q_1,p) = 2\pi\int_{-1}^{+1}\mathrm{d}x\, d^J_{\lambda\lambda'}(x)B_{\lambda\lambda'}(\boldsymbol{q}_1,\boldsymbol{p}) \qquad B^J_{LL'}(q_1,p) = U_{L\lambda}B^J_{\lambda\lambda'}(q_1,p)U_{\lambda'L'}$$

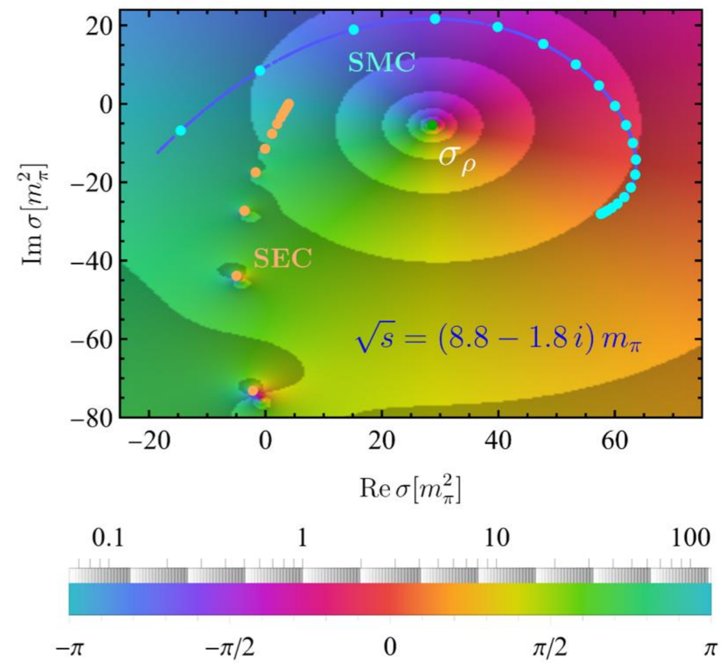
- JLS basis:

$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p)+C_{LL'}(q_1,p)\right)+$$

$$\int_0^\Lambda\frac{\mathrm{d}l\,l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l)+C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

Analytic cont. 3-body

[Sadasivan (2021)]
[Doering (2009)]



SMC

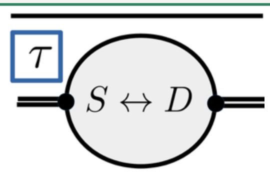
$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3 2E_l} \left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right) \tau(\sigma(l)) T^J_{L''L'}(l,p)$$

$$\tau^{-1}(\sigma) = K^{-1} - \Sigma\,,$$

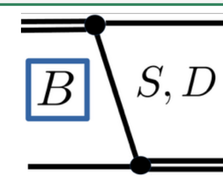
$$\Sigma = \int_0^\infty \frac{\mathrm{d}k\, k^2}{(2\pi)^3} \frac{1}{2E_k} \frac{\sigma^2}{\sigma'^2} \frac{\tilde{v}(k)^* \tilde{v}(k)}{\boxed{\sigma - 4E_k^2 + i\epsilon}}$$

$$B_{\lambda\lambda'}(\boldsymbol{p},\boldsymbol{p}') = \frac{v^*_\lambda(P-p-p',p)\, v_{\lambda'}(P-p-p',p')}{2E_{p'+p}\boxed{(\sqrt{s} - E_p - E_{p'} - E_{p'+p} + i\epsilon)}}$$

SEC



Singularities



- Two contours (SMC and SEC)
- Deform both "adiabatically" to go to complex s
- Set of rules:
    - Contours cannot intersect with each others
    - Contours cannot intersect with (3-body) cuts
- Passing singularities left or right determines sheet