NNLO QCD corrections to top-quark pair production in association with a jet

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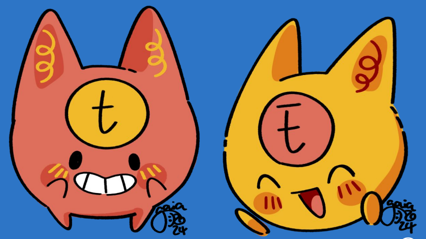
Based on: JHEP 03 (2025) 070, arXiv:2511.xxxxx

With: S. Badger, M. Becchetti, M. Czakon, H. B. Hartanto, R. Poncelet, S. Zoia

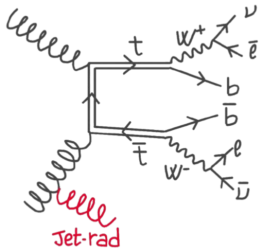
Workshop on top quark mass measurements
-
Valencia

10th November 2025



@qftoons

tt+jet production at the LHC

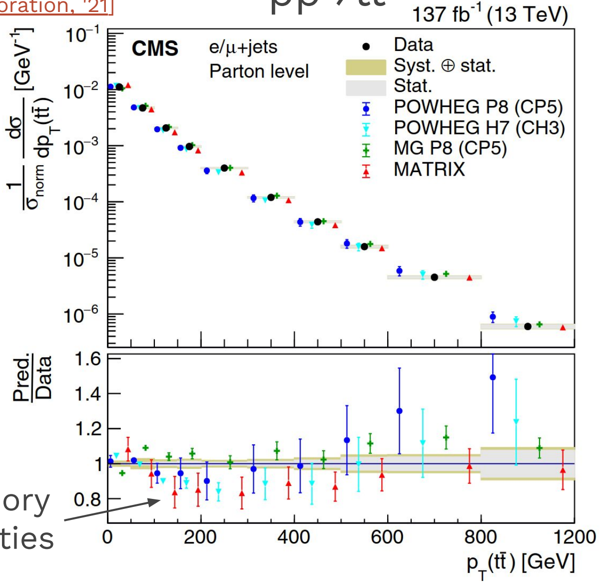


LHC is a top factory!

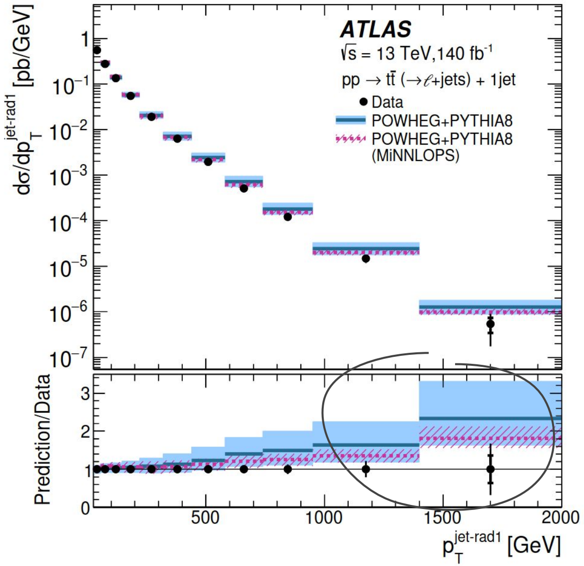
Main production channel: pp→tt

~40% events associated with jet (p_T>40 GeV) [ATLAS Collaboration '24]

[CMS Collaboration, '21]



Large theory uncertainties

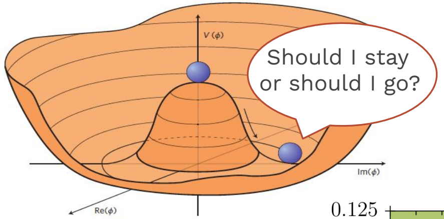


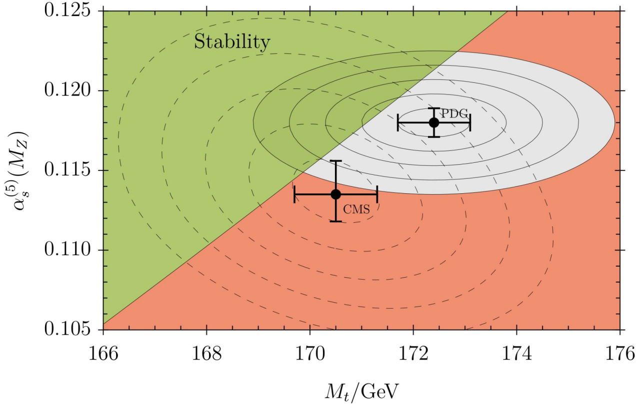
Need N^{3}LO tt → NNLO tt+jet key ingredient

Need NNLO tt+jet

Top mass extraction

Top-quark mass value essential
to asses the EW vacuum stability





[Hiller, Hohne, Litim, Steudtner '24]

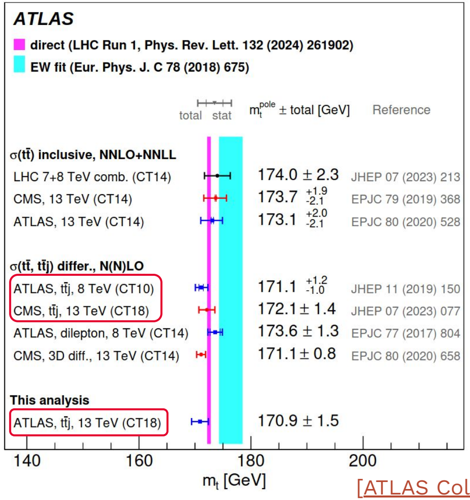
[Alioli, Fernandez, Fuster, Irles, Moch, Uver, Vos '14]
[Bevilacqua, Hartanto, Kraus, Schulze, Worek '17]
[Alioli, Fuster, Garzelli, Gavardi, Irles, Melini, Moch, Uwer, Vos '22]

tt+jet production highly sensitive
to top-quark mass value

$$\mathcal{R}(m_t^R, \rho_s) = \frac{1}{\sigma_{t\bar{t}+1\text{-jet}}} \frac{d\sigma_{t\bar{t}+1\text{-jet}}}{d\rho_s}(m_t^R, \rho_s)$$

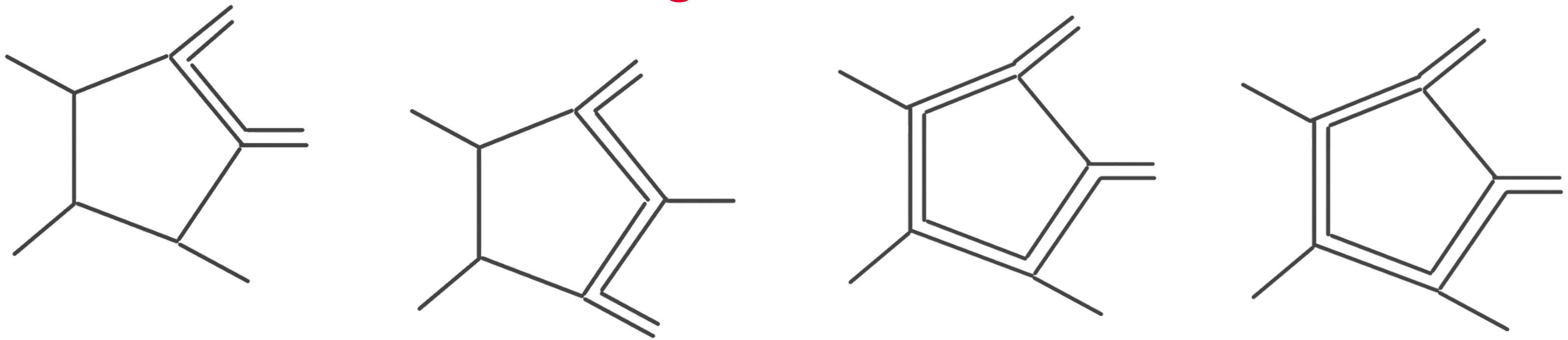
$$\rho_s = \frac{2m_0}{\sqrt{s_{t\bar{t}j}}}$$

top mass from
ttj distributions



[ATLAS Collaboration '25]

Current status of tt+jet corrections



- **NLO QCD** corrections
  [Dittmaier, Uwer, Weinzierl, '07]
- Mixed QCD and EW corrections
  [Gütschow, Lindert, Schönherr '18]
- Full off-shell decays and interfaces with parton shower
  [Melnikov and Schulze '10]
  [Alioli, Moch, Uwer '12]
  [Bevilacqua, Czakon, Hartanto, Kraus, Worek '15-'16]

Nowadays NLO is automated

OpenLoops2
[Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller '19]

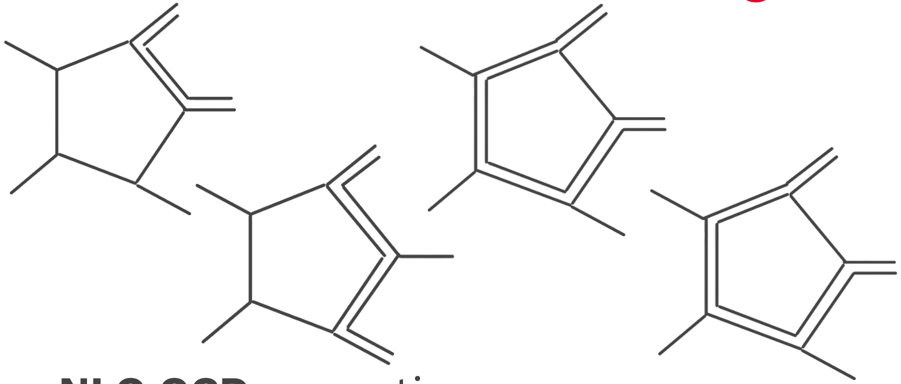
Helac-NLO
[Bevilacqua, Czakon, Garzelli, van Hameren, Kardos, Papadopoulos, Pittau, Worek '11]

MadGraph5_aMC@NLO
[Alwall, Frederix, Frixione, Hirschi, Maltoni, Mattelaer, Shao, Stelzer, Torrielli, Zaro '14]

Implemented within the framework

Powheg-Box
[Alioli, Nason, Oleari, Re '10]

Current status of tt+jet corrections



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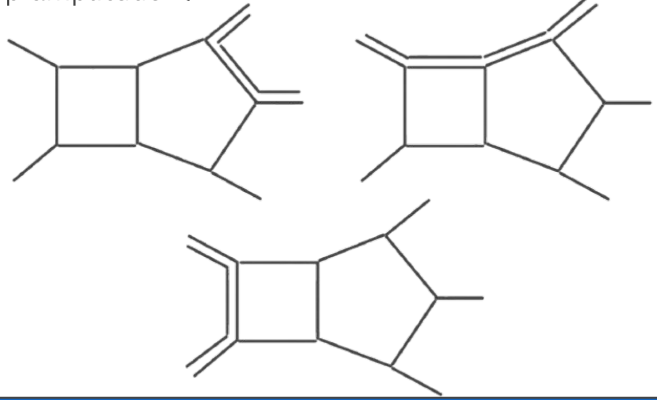
  [Bevilacqua, Czakon, Hartanto, Kraus, Worek ’15-’16]

- **NNLO QCD** corrections needed
  ➡ initial steps towards this challenge

One-loop at $O(\varepsilon^2)$ { [Badger, Becchetti, Chaubey, Marzucca, Sarandrea ’22] [Bera, **CB**, Canko, Hartanto, ‘25]

DEs for two-loop planar topologies { [Badger, Becchetti, Chaubey, Marzucca ’23] [Badger, Becchetti, Giraudo, Zoia ’24] [Becchetti, Dlapa, Zoia ’25]

Numerical evaluation two-loop amplitude { [Badger, Becchetti, **CB**, Hartanto, Zoia ’24]



Towards NNLO QCD corrections tt+jet

Cross section = Amplitude + <u>Phase space integration</u> → STRIPPER

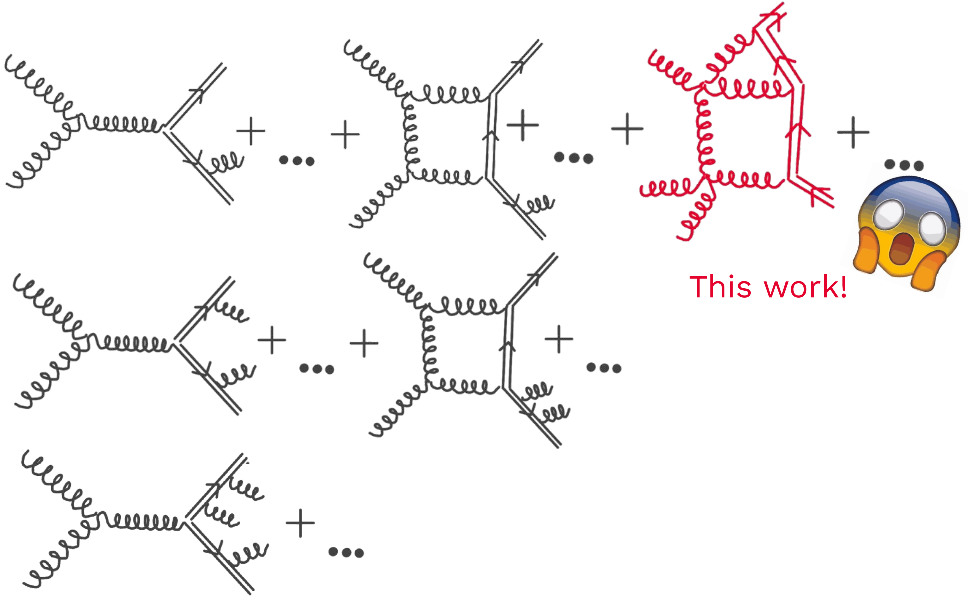
[Czakon '10][Czakon, Heymes '14]
[Czakon, van Hameren, Mitov, Poncelet '19]

This work!

$$\sigma_n^{NNLO} = \int_n d\sigma^B + d\sigma^V + d\sigma^{VV}$$
$$+ \int_{n+1} d\sigma^R + d\sigma^{RV}$$
$$+ \int_{n+2} d\sigma^{RR}$$

OpenLoops2
[Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller '19]

AvH library
[Bury, Hameren '15]



Numerical evaluation of the finite remainder

Less than one year ago at this workshop!

Numerical evaluation of finite remainders up to two loops in the leading color limit for gg → ttg production

Helicity	$R^{(0),[1]}$	$R^{(0),[2]}$	$R^{(0),[3]}$	$R^{(0),[4]}$
$+++$	$0.26326 - 0.0097514\,i$	0	0	0
$+-+$	$5.9619 - 0.16047\,i$	0	0	$-0.31659 - 0.097935\,i$
$++-$	$-5.9575 + 0.0089231\,i$	$-12.606 - 0.067440\,i$	$4.6564 + 0.024911\,i$	$-1.9692 - 0.010535\,i$

Helicity	$R^{(1),[1]}/R^{(0),[1]}$	$R^{(1),[2]}/R^{(0),[1]}$	$R^{(1),[3]}/R^{(0),[1]}$	$R^{(1),[4]}/R^{(0),[1]}$
$+++$	$38.396 - 5.8002\,i$	$71.982 - 4.0653\,i$	$-14.289 + 0.70866\,i$	$17.909 - 0.39528\,i$
$+-+$	$19.221 - 8.4151\,i$	$-4.8506 + 4.8015\,i$	$0.67096 - 0.09959\,i$	$-1.2201 + 2.1594\,i$
$++-$	$20.369 - 19.991\,i$	$41.522 - 41.969\,i$	$-15.990 + 15.739\,i$	$6.2964 - 6.4584\,i$

Helicity	$R^{(2),[1]}/R^{(0),[1]}$	$R^{(2),[2]}/R^{(0),[1]}$	$R^{(2),[3]}/R^{(0),[1]}$	$R^{(2),[4]}/R^{(0),[1]}$
$+++$	$882.48 - 91.619\,i$	$2489.7 - 266.72\,i$	$-492.28 + 8.1003\,i$	$593.35 - 87.569\,i$
$+-+$	$414.16 - 206.87\,i$	$-171.78 + 189.69\,i$	$25.226 - 1.5639\,i$	$-54.820 + 95.716\,i$
$++-$	$332.97 - 646.02\,i$	$623.01 - 1325.1\,i$	$-259.14 + 512.33\,i$	$89.185 - 198.65\,i$

Check out our paper arXiv:2412.13876 😊

pp→ttj cross section

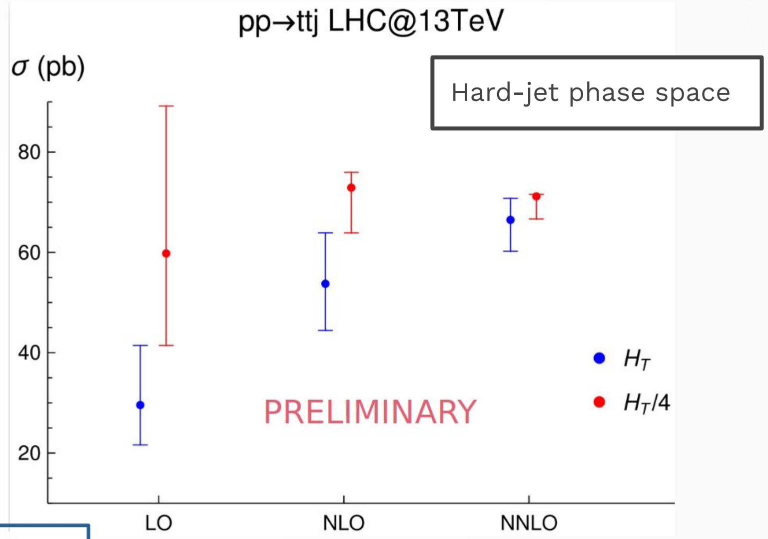
New!

- $\sqrt{s} = 13$ TeV
- PDF4LHC21 PDF set
- Stable top productions
  $$m_t = 172.5 \text{ GeV}$$
- Two fiducial phase spaces:

Hard	Soft
$p_T(j) \geq 120$ GeV	$p_T(j) \geq 30$ GeV
$p_T(j) \geq 0.4 \cdot (p_T(t) + p_T(\bar{t}))$	$\lvert y(j)\rvert \leq 2.5$
$\lvert y(j)\rvert \leq 2.4$	

- Scales:
  $$\mu_R = \mu_F = \mu_0 = H_T/n\ , \quad n \in \{1, 2, 4\}$$
  $$H_T = M_T(t) + M_T(\bar{t}) + p_T(j_1)\ .$$

From H. B. Hartanto's talk at TOP25



$$K_{\text{NLO}} = \frac{\sigma_{\text{NLO}}}{\sigma_{\text{LO}}} \qquad K_{\text{NNLO}} = \frac{\sigma_{\text{NNLO}}}{\sigma_{\text{NLO}}}$$

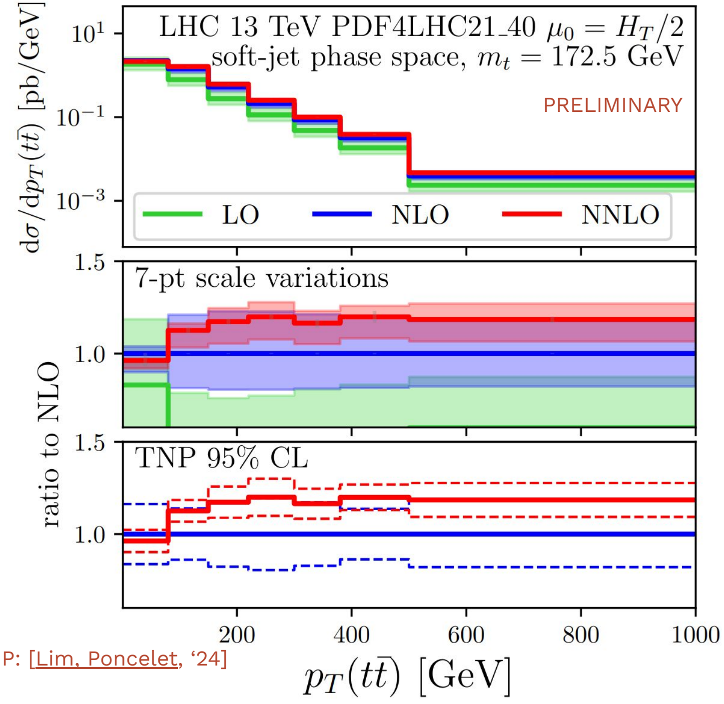
H_T: $K_{\text{NLO}} \sim 1.82$, $K_{\text{NNLO}} \sim 1.24$

$H_T/4$: $K_{\text{NLO}} \sim 1.22$, $K_{\text{NNLO}} \sim 0.98$

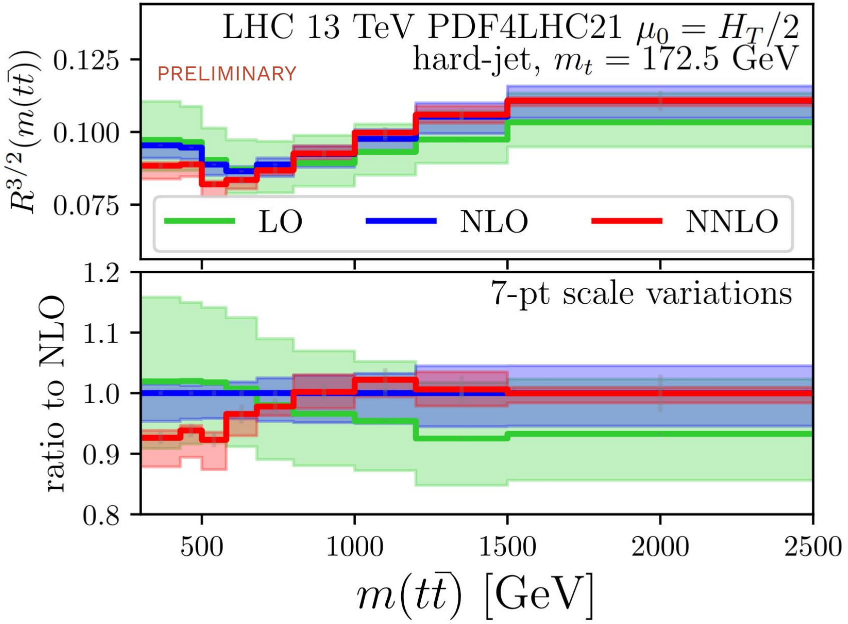
Theory uncertainties from 7-point scale variations

pp→ttj differential distributions

New!



TNP: [Lim, Poncelet, ‘24]



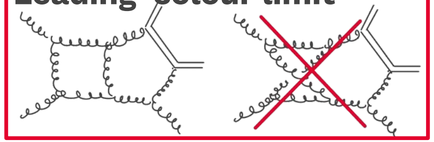
$$R^{3/2}(X) = \frac{d\sigma^{t\bar{t}j}(X)}{d\sigma^{t\bar{t}}(X)}$$

➡ Highly sensitive to m_t and α_s

Amplitude computation: the challenges

Laurent expansion of the MIs and their coefficients around ε=0 (with d=4-2ε):

Leading-colour limit



$$A^{(2L)}(\epsilon,\vec{x}) = \sum_i \sum_{k=-4}^{0} \epsilon^k r_{ki}(\vec{x}) F_i(\vec{x})$$

High **algebraic** complexity

	$r_{g;1}^{(2),N_c^2;\vec{h}}$	$r_{g;2}^{(2),N_c^2;\vec{h}}$	$r_{g;3}^{(2),N_c^2;\vec{h}}$	$r_{g;4}^{(2),N_c^2;\vec{h}}$
master integral coefficients of mass-renormalised amplitude (full ϵ dependence)	404/393	398/389	411/402	421/411
special function coefficients of finite remainder	314/303	305/296	321/312	326/317
linear relations	291/280	287/278	299/293	304/299
denominator matching #1	291/0	287/0	299/0	304/0
partial fraction decomposition in x_{5123}	44/40	55/51	57/54	58/54
denominator matching #2	44/0	54/0	54/0	56/0

Tamed with finite-field approach

[von Manteuffel, Schabinger '14]
[Peraro '16][Peraro '19]

High **analytic** complexity

- Minimised impact of non-polylogarithmic functions
- Derived a (potential over-complete) special function basis allowing for:
  1. **Analytic UV/IR pole subtraction**
  2. **Simplification of finite remainders**

⬇

For the first time in case of elliptic functions

[Badger, Becchetti, **CB**, Hartanto, Zoia '24]

See also: [Badger, Becchetti, Giraudo, Zoia '24]
[Badger, Becchetti, Chaubey, Marzucca '23]
[Becchetti, Dlapa, Zoia '25]

Numerical evaluation

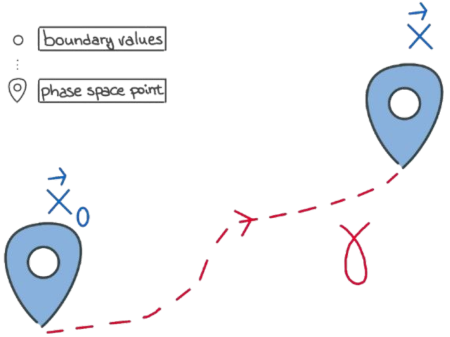
Numerical integration of the DEs for the special functions

→ **Bulirsch–Stoer algorithm**
→ Previously used for pp→tt production

[Caffo, Czyz, Remiddi, 02]
[Boughezal, Czakon, Schutzmeier, '07]
[Czakon, '08]

→ new implementation for the ttj – highly promising for phenomenological applications

Check also [Petit Rosàs, Torres Bobadilla, '25]



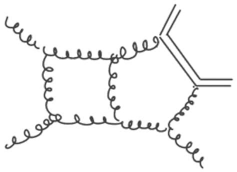
Numerical C++ implementation
~20 seconds per point
(double-double for rational coefficients,
double for special functions)

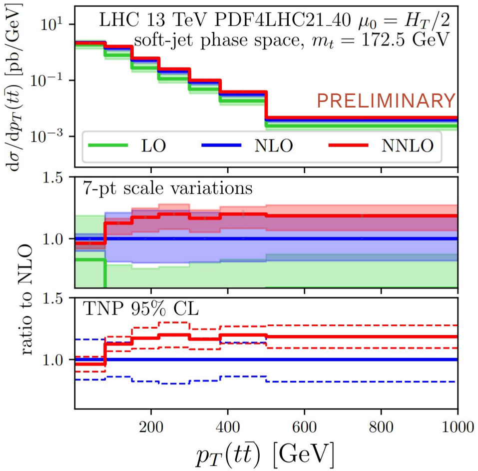
→ interface to STRIPPER
→ on-the-fly evaluation

dominates the total evaluation time

Conclusions

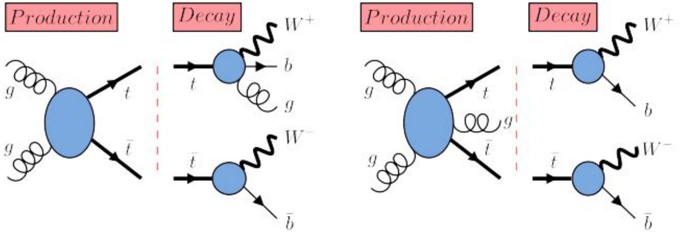
- **Analytic expressions** of **two-loop leading-colour helicity amplitudes** for **pp→ttj**
- **NNLO QCD predictions**: cross section ad differential distributions for **pp→ttj** at LHC@13 TeV
- **NNLO precision is essential** to reach **sub-10%** theoretical accuracy





Outlook

- In-depth **phenomenological studies**
  ➡ m_t and α_s extraction
- NNLO QCD predictions including top-quark decay via NWA



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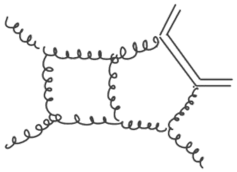

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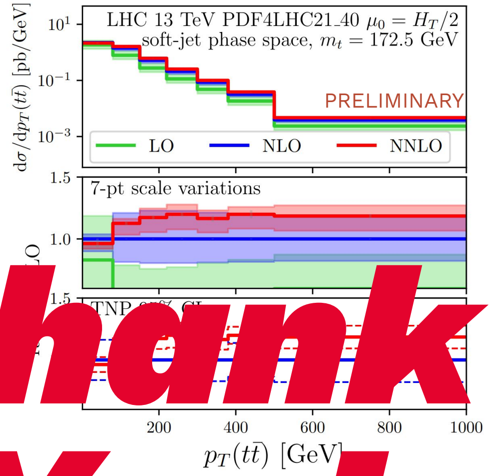

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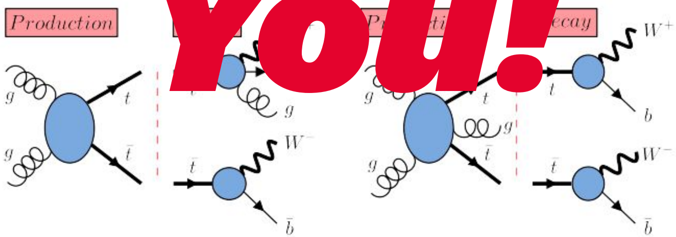


Thank You!

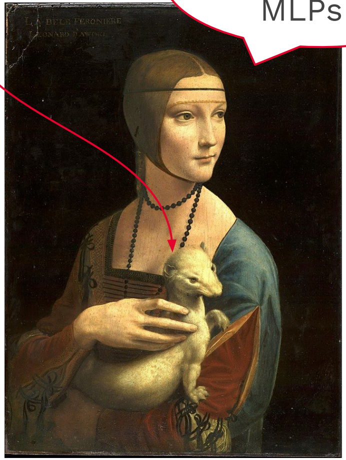
BACKUP

Growing challenges for phenomenology

Massless amplitudes

Massless internal propagators

Logs, Classical Polylogs, MLPs ...



- Analytic cancellation of the poles
- Compact expressions
- Fast and stable numerics

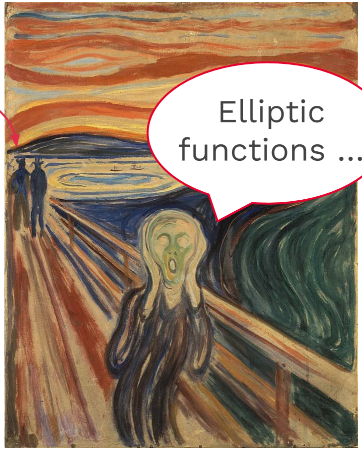
↓

Pentagon functions approach

See [Gehrmann, Henn, Lo Presti '18] [Chicherin, Sotnikov '20] [Chicherin, Sotnikov, Zoia '22] [Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia '24]

Internal masses

Massive internal propagators



Elliptic functions ...

- Functions under investigation

↓

How to obtain the same advantages of the massless case?

Helicity amplitudes – useful definitions 1

Colour decomposition:

$$\mathcal{M}^{(L)}(1_{\bar{t}}, 2_t, 3_g, 4_g, 5_g) = \sqrt{2}\,\bar{g}_s^3 \left[(4\pi)^\epsilon e^{-\epsilon\gamma_E} \frac{\bar{\alpha}_s}{4\pi}\right]^L \times \sum_{\sigma\in Z_3} \left(t^{a_{\sigma(3)}} t^{a_{\sigma(4)}} t^{a_{\sigma(5)}}\right)_{i_2}^{\bar{i}_1} A_g^{(L)}(1_{\bar{t}}, 2_t, \sigma(3)_g, \sigma(4)_g, \sigma(5)_g)$$

$$\boxed{A_x^{(2)} = N_c^2\, A_x^{(2),N_c^2} + N_c n_f\, A_x^{(2),N_c n_f} + n_f^2\, A_x^{(2),n_f^2}}$$

$$x \in \{g, q\}$$

Finite reminder:

$$\boxed{\begin{aligned}\mathcal{F}_x^{(2)} &= \mathcal{A}_{x,\mathrm{ren}}^{(2)} \\ &- \mathbf{Z}_x^{(1)} \mathcal{A}_{x,\mathrm{ren}}^{(1)} \\ &- \left[\mathbf{Z}_x^{(2)} - \left(\mathbf{Z}_x^{(1)}\right)^2\right] \mathcal{A}_x^{(0)}\end{aligned}}$$

$$\mathcal{A}_{x;\mathrm{mren}}^{(2)} = \mathcal{A}_x^{(2)} - \delta Z_m^{(2)} \mathcal{A}_{x;\mathrm{mct}}^{(0)} + (\delta Z_m^{(1)})^2 \mathcal{A}_{x;2\mathrm{mct}}^{(0)} - \delta Z_m^{(1)} \mathcal{A}_{x;\mathrm{mct}}^{(1)}$$

$$\begin{aligned}\mathcal{A}_{x,\mathrm{ren}}^{(2)} = \mathcal{A}_{x,\mathrm{mren}}^{(2)} &+ \left(\frac{5}{2}\delta Z_{\alpha_s}^{(1)} + \delta Z_t^{(1)}\right) \mathcal{A}_{x,\mathrm{mren}}^{(1)} \\ &+ \left(\frac{3}{2}\delta Z_{\alpha_s}^{(2)} + \frac{5}{2}\delta Z_{\alpha_s}^{(1)}\delta Z_t^{(1)} + \delta Z_t^{(2)} + \frac{3}{8}(\delta Z_{\alpha_s}^{(1)})^2\right) \mathcal{A}_x^{(0)}\end{aligned}$$

Helicity amplitudes – useful definitions 2

Hard functions:

$$\mathcal{R}^{(L)}(1_{\bar{t}}, 2_t, 3_g, 4_g, 5_g) = \sqrt{2}\, g_s^3 \left[(4\pi)^\epsilon e^{-\epsilon\gamma_E} \frac{\alpha_s}{4\pi}\right]^L \times \sum_{\sigma \in Z_3} \left(t^{a_{\sigma(3)}} t^{a_{\sigma(4)}} t^{a_{\sigma(5)}}\right)_{i_2}^{\bar{i}_1} \mathcal{F}_g^{(L)}(1_{\bar{t}}, 2_t, \sigma(3)_g, \sigma(4)_g, \sigma(5)_g)$$

$$\boxed{\begin{aligned} \mathcal{H}^{(0)} &= \overline{\sum_{\text{colour}}}\,\overline{\sum_{\text{pol.}}} |\mathcal{R}^{(0)}|^2 , \\ \mathcal{H}^{(1)} &= 2\,\mathrm{Re} \overline{\sum_{\text{colour}}}\,\overline{\sum_{\text{pol.}}} \mathcal{R}^{(0)*}\mathcal{R}^{(1)} , \\ \mathcal{H}^{(2)} &= 2\,\mathrm{Re} \overline{\sum_{\text{colour}}}\,\overline{\sum_{\text{pol.}}} \mathcal{R}^{(0)*}\mathcal{R}^{(2)} + \overline{\sum_{\text{colour}}}\,\overline{\sum_{\text{pol.}}} |\mathcal{R}^{(1)}|^2 \end{aligned}}$$

Amplitudes in terms of massive spinor form factors:

$$\mathcal{A}_x^{(L)h_3h_4h_5} = \sum_{i=1}^{4} \Psi_i\, \mathcal{G}_{x;i}^{(L)h_3h_4h_5}$$

$$\boxed{\Omega_{ij} = \sum_{\text{pol.}} \Psi_i^\dagger \Psi_j}$$

$$\mathcal{G}_{x;i}^{(L)h_3h_4h_5} = \sum_{j=1}^{4} \left(\Omega^{-1}\right)_{ij} \tilde{\mathcal{A}}_{x;j}^{(L)h_3h_4h_5}$$

$$\tilde{\mathcal{A}}_{x;i}^{(L)h_3h_4h_5} = \sum_{\text{pol.}} \Psi_i^\dagger\, \mathcal{A}_x^{(L)h_3h_4h_5}$$

$$\begin{aligned} \Psi_1 &= m_t^2\, \bar{u}(p_2) v(p_1) , \\ \Psi_2 &= m_t\, \bar{u}(p_2) \not{p}_3 v(p_1) , \\ \Psi_3 &= m_t\, \bar{u}(p_2) \not{p}_4 v(p_1) , \\ \Psi_4 &= \bar{u}(p_2) \not{p}_3 \not{p}_4 v(p_1) . \end{aligned}$$

BACKUP

Univariate partial fraction decomposition

Step 1 Look for **linear relations** among the coefficients

$$\sum_i c_i\, r_i(\vec{x}) = 0 \quad \forall \vec{x}$$

Step 2 Make the ansatz

$$r_i(\vec{x}) = \frac{N_i(\vec{x})}{\prod_j D(\vec{x})_j^{q_{ij}}}, \qquad q_{ij} \in \mathcal{Z}$$

Step 3 Perform **univariate partial fraction decomposition** [Badger, Hartanto, Zoia ‘21]

Example

$$r_i(\vec{x}) = \frac{x^3 + 2x^2y + 2xy^2 + 4y^3}{y^2(x^2+y^2)} \longrightarrow \frac{\overset{x}{q_1(x)}}{y^2} + \frac{\overset{2}{q_2(x)}}{y} + \frac{\overset{x}{q_3(x)} + \overset{2}{q_4(x)}y}{x^2+y^2}$$

max. numerator|denominator degrees 4|5 $\longrightarrow$ max. numerator|denominator degrees 1|0

BACKUP

Momentum twistor variables

Univariate partial fraction decomposition
+
Momentum twistor variables
=
feasible analytic reconstruction

$$s_{34} \to 2d_{34}\,, \quad t_{45} \to \frac{d_{45}}{d_{34}}\,, \quad t_{12} \to \frac{d_{12}+m_t^2}{d_{34}}\,, \quad t_{23} \to \frac{d_{23}}{d_{34}}\,, \quad t_{51} \to \frac{d_{15}}{d_{34}}\,,$$

$$x_{5123} \to \frac{-4d_{12}(d_{15}+d_{23}) + 4d_{15}d_{45} - 4d_{34}d_{45} + 4d_{23}(d_{34}-m_t^2) - 4d_{15}m_t^2 + \mathrm{tr}_5}{8(d_{12}+m_t^2)(d_{12}-d_{34}-d_{45}+m_t^2)}$$

with $d_{ij} = p_i \cdot p_j$ and $\mathrm{tr}_5 = 4\,\mathrm{i}\,\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$

- Degrees reduction of ~80%
- no x_{5123} dependence

Differential equations for MIs

MIs satisfy the following **differential equation**:

$$\boxed{\mathrm{d}\vec{f}(\vec{x},\epsilon) = \mathrm{d}A(\vec{x},\epsilon)\ \vec{f}(\vec{x},\epsilon)}$$

[Kotikov '91-'93]
[Bern, Dixon, Kosower '94]
[Remiddi '97]
[Gehrmann, '00]
[Henn, '13]

For **PBA** and **PBC**:

[Badger, Becchetti, Chaubey, Marzucca '23]
[Badger, Becchetti, Giraudo, Zoia '24]

$$\mathrm{d}A_{\tau,\sigma}(\vec{x}) = \epsilon \sum_i c_i \, \mathrm{dlog}\, W_i(\vec{x})$$

ϵ factorised

dlog-form

For **PBB**:

[Badger, Becchetti, Giraudo, Zoia '24]

$$\mathrm{d}A(\vec{x},\epsilon) = \sum_{k=0}^{2} \epsilon^k \sum_i c_{ki}\, \omega_i(\vec{x})$$

DEs quadratic in ϵ

one-form

solution in terms of elliptic functions

Pentagon functions derivation: the idea

- Solution of the canonical DEs in terms of **Chen iterated integrals**
- Write the MIs components in terms of **symbols**
- Select the MI components for the **basis at symbol level**

Include products of lower-weight functions

$$\left\{f_i^{(0)}\right\} \longrightarrow \{1\}$$

$$\left\{f_i^{(1)}\right\} \longrightarrow \left\{F_k^{(1)}\right\}$$

$$\left\{f_i^{(2)}\right\} \cup \left\{F_i^{(1)} \times F_j^{(1)}\right\} \longrightarrow \left\{F_k^{(2)}\right\}$$

ATTENTION!
Pentagon functions in progress

Tune ordering to streamline the amplitude

Preferred ordering:

- lower-weight functions over higher-weight functions
- 1-loop functions over 2-loop functions

BACKUP

Special functions basis: non-canonical case

The DEs for the coefficients $\vec{f}^{(k)}(\vec{x})$ are

$$\mathrm{d}\vec{f}^{(k)}(\vec{x}) = \cancel{\Omega^{(0)}(\vec{x})\,\vec{f}^{(k)}(\vec{x})} + \Omega^{(1)}(X)\,\vec{f}^{(k-1)}(\vec{x}) + \cancel{\Omega^{(2)}(X)\,\vec{f}^{(k-2)}(\vec{x})}$$

with $k \geq 0$

$$\vec{f}^{(-2)}(\vec{x}) = \vec{f}^{(-1)}(\vec{x}) = 0$$

$k=0$: $\mathrm{d}\vec{f}^{(0)}(\vec{x}) = 0 \longrightarrow \vec{f}^{(0)}(\vec{x})$ Constant

$k=1$: $\mathrm{d}\vec{f}^{(1)}(\vec{x}) = \sum_i A_{1,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(0)} \longrightarrow$ Canonical

$k=2$: $$\mathrm{d}f_{15}^{(2)}(\vec{x}) = \frac{1}{24}\underbrace{\left[12\,f_{103}^{(1)}(\vec{x}_0)+8\,f_{110}^{(1)}(\vec{x}_0)+4\,f_{111}^{(1)}(\vec{x}_0)+3\,f_{118}^{(1)}(\vec{x}_0)-48\,f_{63}^{(1)}(\vec{x}_0)\right]}_{\text{vanishing for a conspiracy of the boundary values}}\omega_2(\vec{x})+\ldots\,.$$

$$\mathrm{d}\vec{f}^{(2)}(\vec{x}) = \sum_i A_{2,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(1)} \longrightarrow \text{Canonical}$$

BACKUP

Special functions basis: non-canonical case

The DEs for the coefficients $\vec{f}^{(k)}(\vec{x})$ are

$$\mathrm{d}\vec{f}^{(k)}(\vec{x}) = \Omega^{(0)}(\vec{x})\,\vec{f}^{(k)}(\vec{x}) + \Omega^{(1)}(X)\,\vec{f}^{(k-1)}(\vec{x}) + \Omega^{(2)}(X)\,\vec{f}^{(k-2)}(\vec{x})$$

with $k \geq 0$

$$\vec{f}^{(-2)}(\vec{x}) = \vec{f}^{(-1)}(\vec{x}) = 0$$

$k=0$ $\quad \mathrm{d}\vec{f}^{(0)}(\vec{x}) = 0 \longrightarrow \vec{f}^{(0)}(\vec{x})$ Constant

$k=1$ $\quad \mathrm{d}\vec{f}^{(1)}(\vec{x}) = \sum_i A_{1,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(0)} \longrightarrow$ Canonical

$k=2$ $\quad \mathrm{d}\vec{f}^{(2)}(\vec{x}) = \sum_i A_{2,i}\,\mathrm{dlog}\,W_i(\vec{x})\,\vec{f}^{(1)} \longrightarrow$ Canonical

$k=3$ $\quad$ Same as for $k=2$

$k=4$ $\quad \vec{f}_i^{(4)}(\vec{x}) = F_i^{(4^*)}$

$F^{(4^*)} \longrightarrow$ Non-polylogarithmic special functions

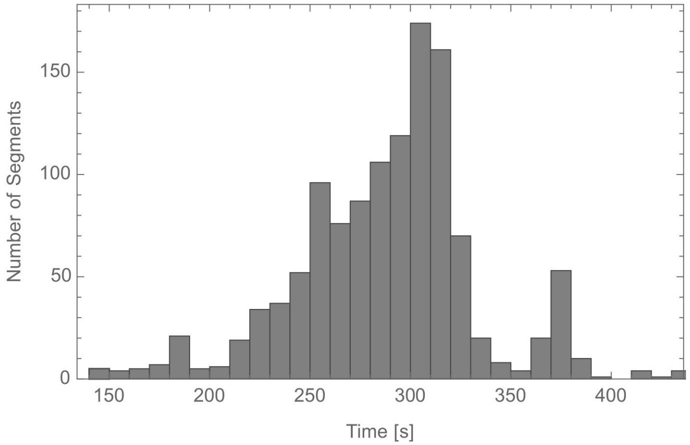
$F_i \longrightarrow$ MIs of the elliptic sectors

BACKUP

Numerical evaluation of special functions

Generalised power series method (as implemented in *DiffExp*) [Moriello ’20] [Hidding ’21]

Evaluation time per segment - Special functions



	PB_A	PB_B	PB_C	all MIs	special func. (all)
$\langle T \rangle$	$43\,s$	$77\,s$	$66\,s$	$309\,s$	$297\,s$
σ	$7\,s$	$17\,s$	$14\,s$	$27\,s$	$65\,s$

Numerical integration with **Bulirsch–Stoer** algorithm

From M. Czakon’s talk at 2nd HOCTools-II

```
Apple M1 Max
Compiler g++-15 -O2
Quadruple precision using boost::multiprecision::cpp_bin_float_quad

                     double precision                      quadruple precision
system          avg_time        der_time           avg_time        der_time

1282            2.61978         0.00110528         114.946         0.0793199
1282_mt1        0.462443        0.000299987        75.9921         0.0525661

366             0.0905311       8.57285e-05        12.8849         0.0120771
366_mt1         0.0640641       6.00082e-05        9.97814         0.00933499
366_mt1_div     0.0812877       7.56775e-05        13.2533         0.0136928
366_mt1_mul     0.126833        7.02523e-05        16.3394         0.0119685

Intel(R) Core(TM) i9-10980XE CPU @ 3.00GHz
Compiler g++-13 -O2
Quadruple precision using dd_real from the QD library

                     double precision                      quadruple precision
system          avg_time        der_time           avg_time        der_time

1282            5.65543         0.00222672         20.7156         0.0144029
1282_mt1        1.11583         0.000747801        14.05           0.00972801

366             0.201802        0.000188134        2.82972         0.00266182
366_mt1         0.150381        0.000140963        1.99558         0.00186848
366_mt1_div     0.1814          0.000171836        2.43324         0.0025114
366_mt1_mul     0.274803        0.000150671        3.40485         0.00249252
```