CosmoLattice

— School 2025 —

CosmoLattice School 2025 (IBS, Korea), Sept 22-26

Practice 5: Gravitational waves from hydrodynamic turbulence

Antonino Salvino Midiri & Kenneth Marschall

Outline

- Elements of hydrodynamic turbulence
- Initializing a turbulent fluid in Fourier space
- Gravitational waves from decaying turbulence

Elements of hydrodynamic turbulence

What is hydrodynamic turbulence?

Monin & Yaglom «Statistical Fluid Mechanics: Mechanics of Turbulence»

Elements of hydrodynamic turbulence

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Let us go back to the fluid equations in the non-conservation form

$$\partial_0 \ln \rho = -\frac{1+c_s^2}{1-c_s^2 u^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

$$\partial_0 u_i = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\, u_i - \frac{\mathrm{c}_\mathrm{s}^2}{1+\mathrm{c}_\mathrm{s}^2}\frac{\nabla_i \ln\rho}{\gamma^2} + u_i \frac{c_s^2}{(1-c_s^2 u^2)\gamma^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{1-c_s^2}{1+c_s^2}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\ln\rho\right]$$

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Let us focus (for simplicity) on the subrelativistic limit ($c_s^2 \ll 1$, $u^2 \ll 1$)

The momentum equation, using a simple model for the viscosity, is

$$\partial_0 u_i = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})u_i + \nu\,\nabla^2 u_i - \frac{\boldsymbol{\nabla} p}{\rho}$$

Elements of hydrodynamic turbulence

What is hydrodynamic turbulence?

The evolution of the velocity is strongly dependent on the interplay between two terms

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$$\partial_0 u_i = \underbrace{-(\boldsymbol{u} \cdot \boldsymbol{\nabla}) u_i}_{\text{nonlinearities}} + \underbrace{\nu \, \nabla^2 u_i}_{\text{viscosity}} - \frac{\boldsymbol{\nabla} p}{\rho}$$

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To gain insight on the interplay between them we can consider a fluid motion with a characteristic lenght scale L and a characteristic velocity v_{rms}

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nonlinearities $\propto v_{rms}^2/L$

viscosity $\propto \nu \, v_{rms}/L^2$

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To gain insight on the interplay between them we can consider a fluid motion with a characteristic lenght scale L and a characteristic velocity v_{rms}

We then define the Reynolds number as the ratio between nonlinearities and viscosity

$$Re = \frac{\text{nonlinearities}}{\text{viscosity}} = \frac{v_{rms} L}{\nu}$$

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$$\partial_0 u_i = \underbrace{-(\boldsymbol{u}\cdot\boldsymbol{\nabla})u_i}_{\text{nonlinearities}} + \underbrace{\nu\,\nabla^2 u_i}_{\text{viscosity}} - \frac{\boldsymbol{\nabla} p}{\rho} \qquad Re = \frac{\text{nonlinearities}}{\text{viscosity}} = \frac{v_{rms} L}{\nu}$$

Experiments show that this ratio can be used to distinguish two different regimes

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Small Reynolds number

A small change in the initial conditions causes
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Laminar regime

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Small Reynolds number	Large Reynolds number
A small change in the initial conditions causes a small change in the fluid profiles (ordered flow)	A small change in the initial conditions can cause instabilities and big changes in the fluid profiles (chaotic flow)
Laminar regime	Turbulent regime

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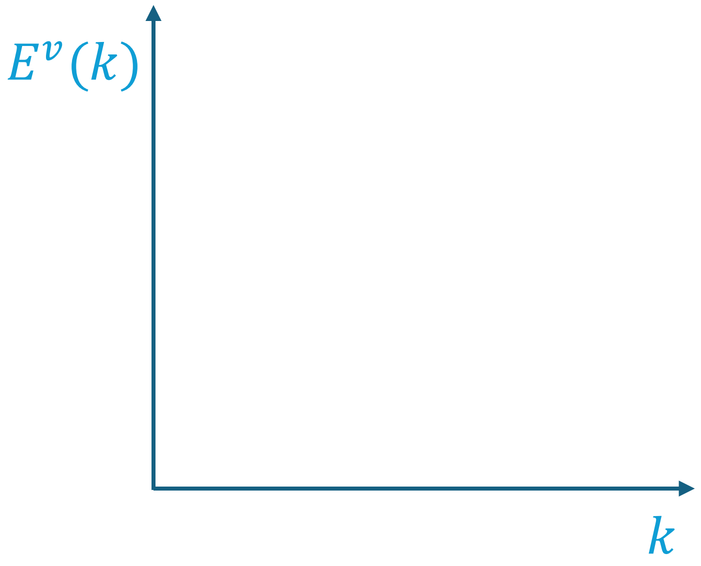
However there are universal statistical (average) properties of turbulence



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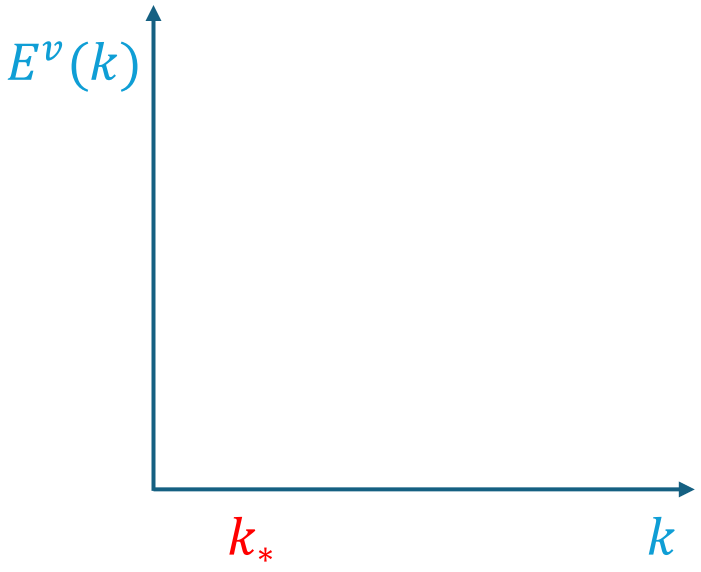
Suppose that turbulence was generated after an injection of energy at a scale k_* in the power spectrum



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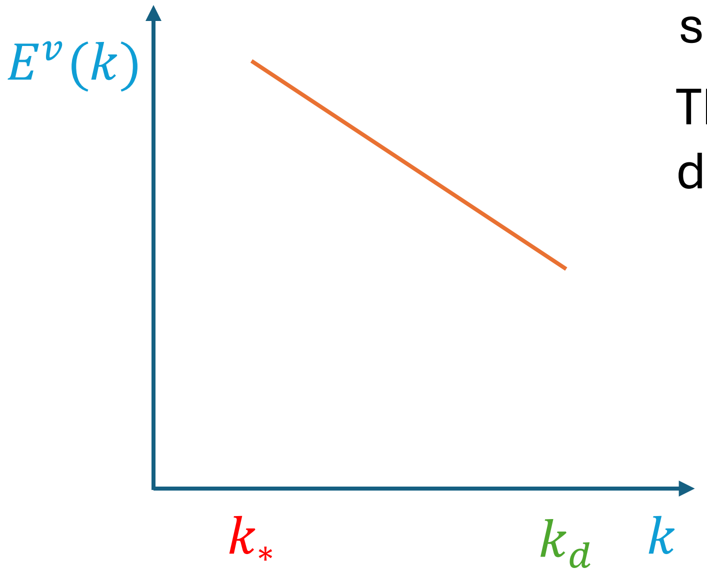

Experimentally we see that in turbulence energy tends to move from larger to smaller scales (nonlinearities allow energy exchange between different scales)

This happens in the inertial range, which is between the injection scale k_* and the dissipation scale (at which viscosity balances nonlinearities) $k_d \approx v_{rms}/\nu$

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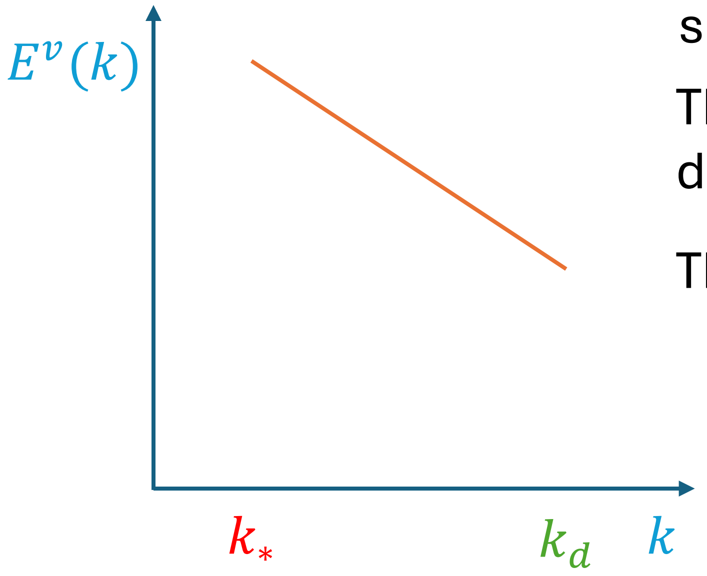

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The energy transfer rate is approximately scale independent

$$const = \ \epsilon = \frac{E}{\delta\tau_{eddy}} \propto \frac{v_{rms}^2}{\frac{1}{v_{rms}\,k}} \rightarrow v_{rms} \propto \epsilon^{\frac{1}{3}}\,k^{-\frac{1}{3}}$$

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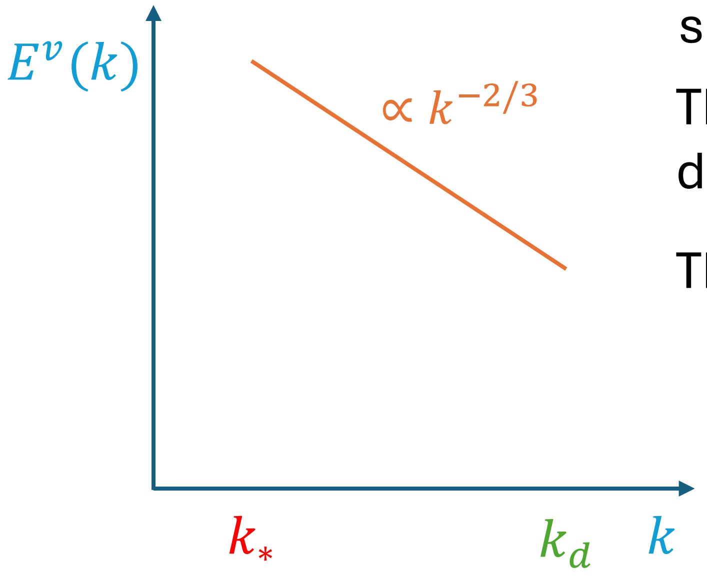

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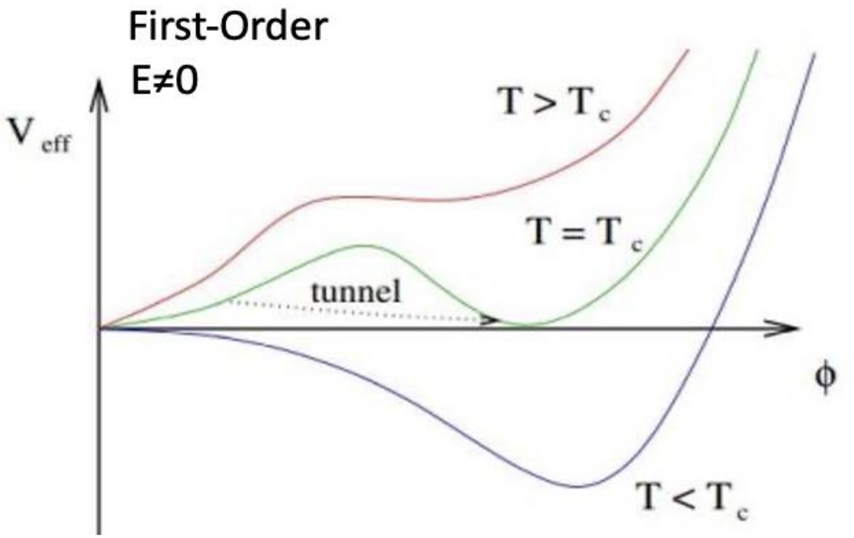
Which implies the Kolmogorov spectrum for $k_* < k < k_d$ $\quad E^v(k)\,\mathrm{d}\ln k \propto v_{rms}^2 \to E^v(k) \propto v_{rms}^2 \propto \epsilon^{\frac{2}{3}}\,k^{-\frac{2}{3}}$

Turbulence in the early Universe: first-order phase transitions

$$V_{\mathrm{eff}}(\phi, T) = D\,(T^2 - T_0^2)\,\phi^2 - E\,T\,\phi^3 + \frac{\lambda}{4}\,\phi^4$$



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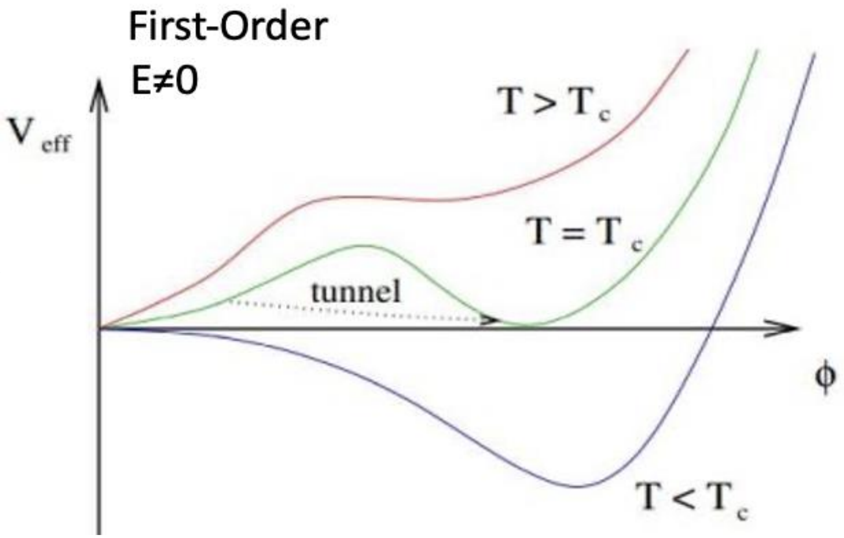
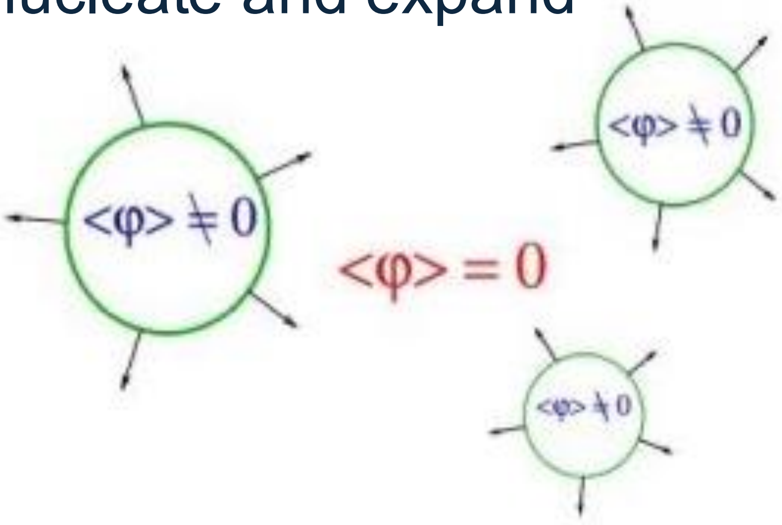


Scalar bubbles
nucleate and expand



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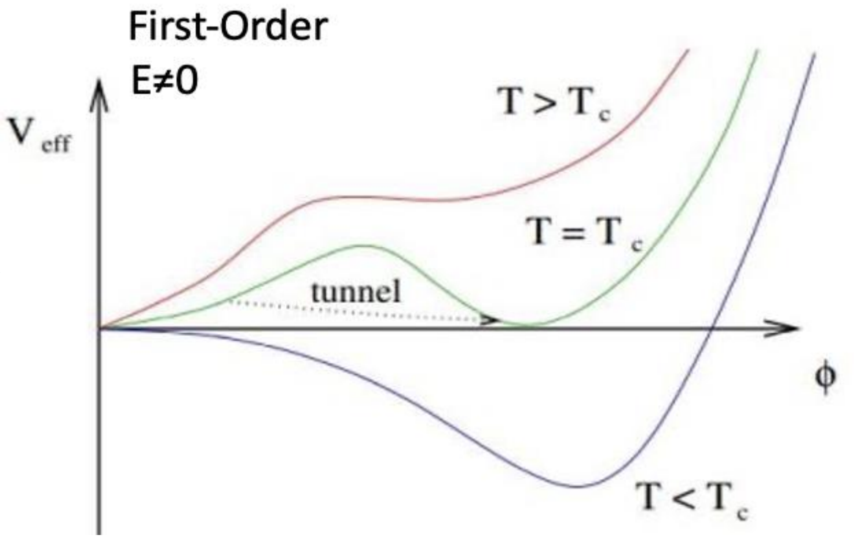

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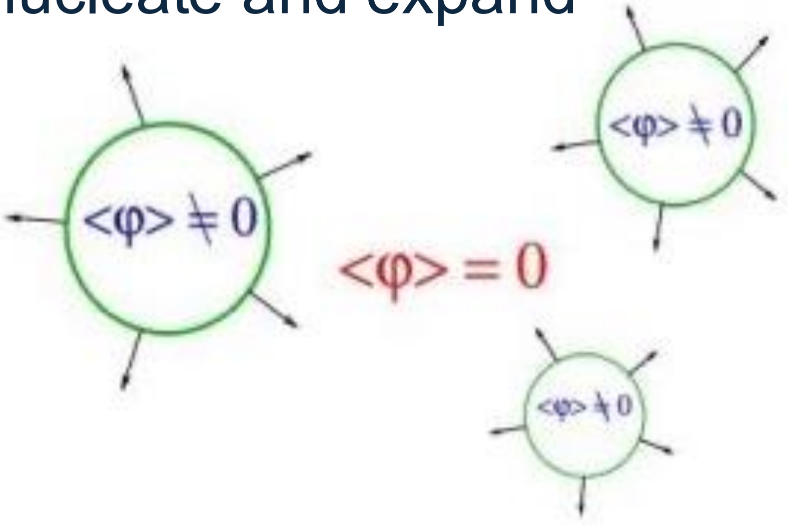
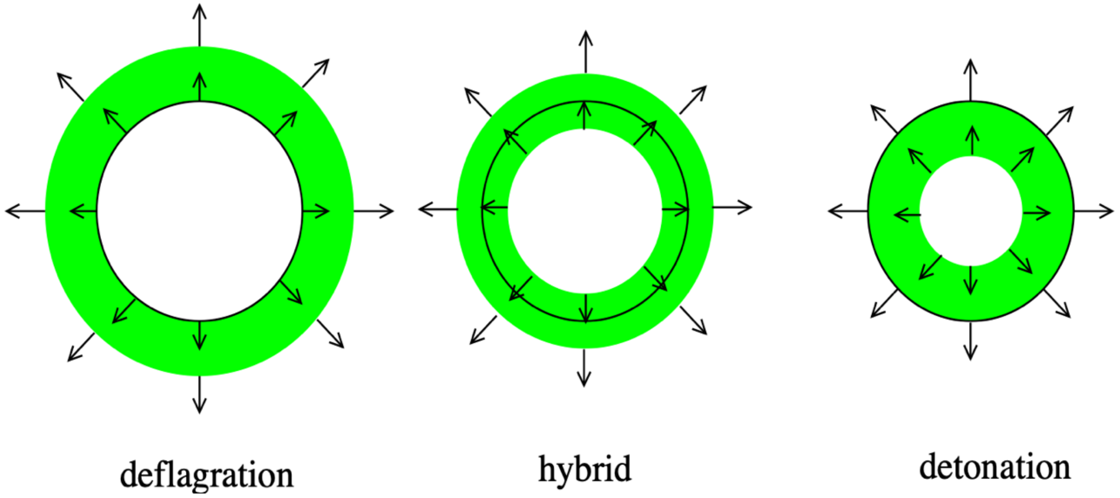


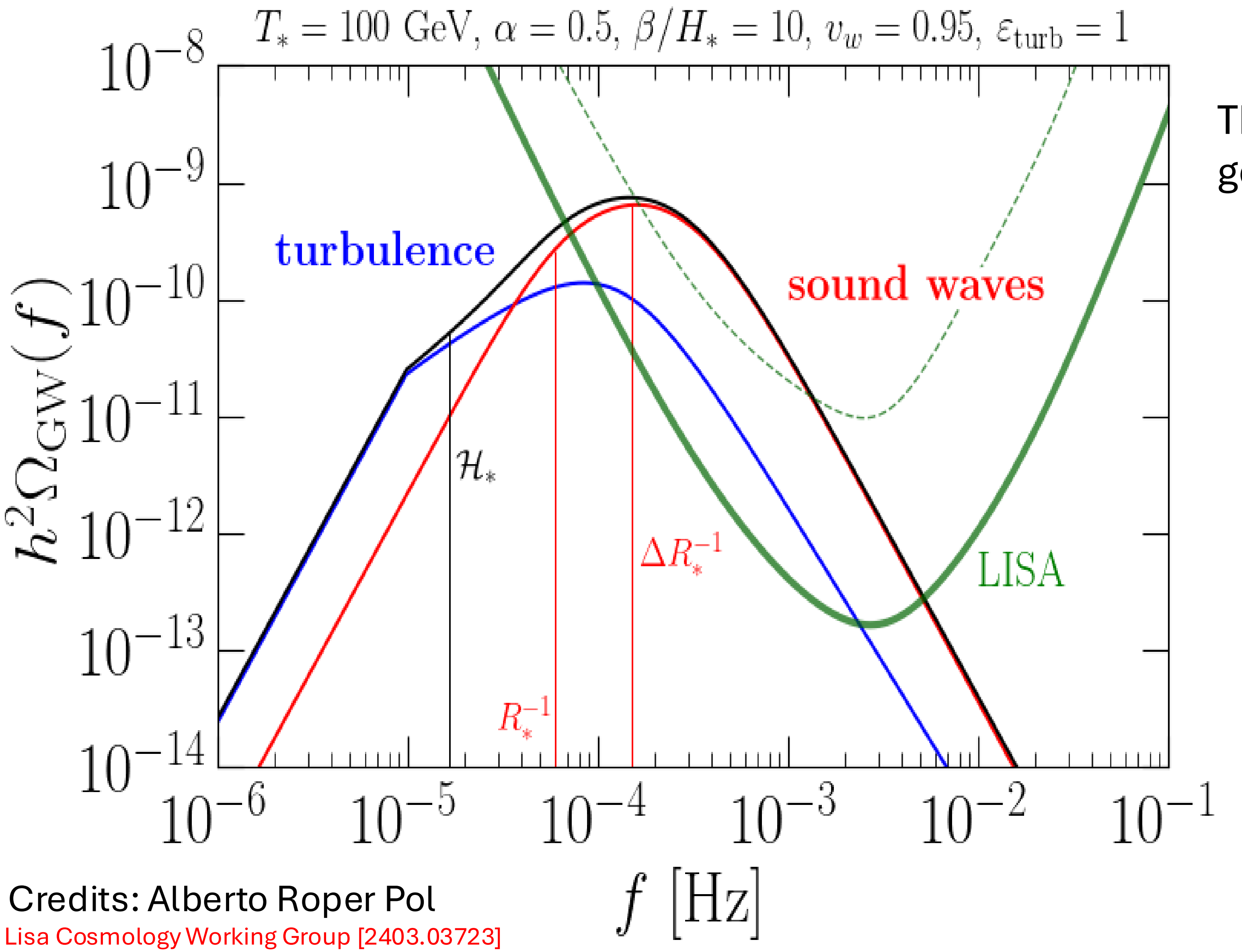
friction between scalar and fluid

Longitudinal fluid perturbations



Espinosa et al. [1004.4187]

Turbulence in the early Universe: first-order phase transitions and gravitational waves



The collisions of the fluid profiles generate sound waves

Credits: Alberto Roper Pol

Lisa Cosmology Working Group [2403.03723]

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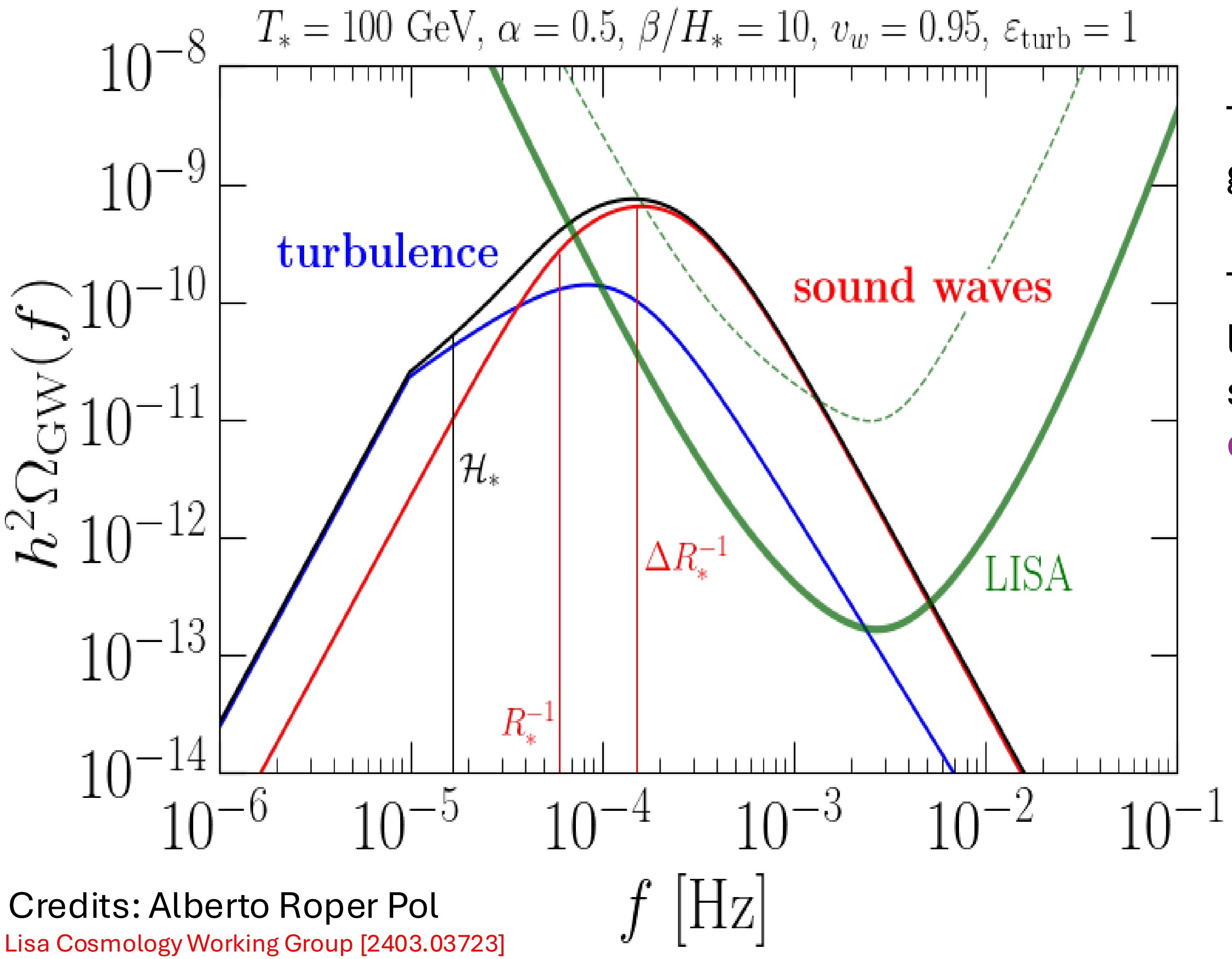

The collisions of the fluid profiles generate sound waves

The development of nonlinearities then leads to turbulence given the extremely small viscosity in the early Universe (Arnold et al. 2003)

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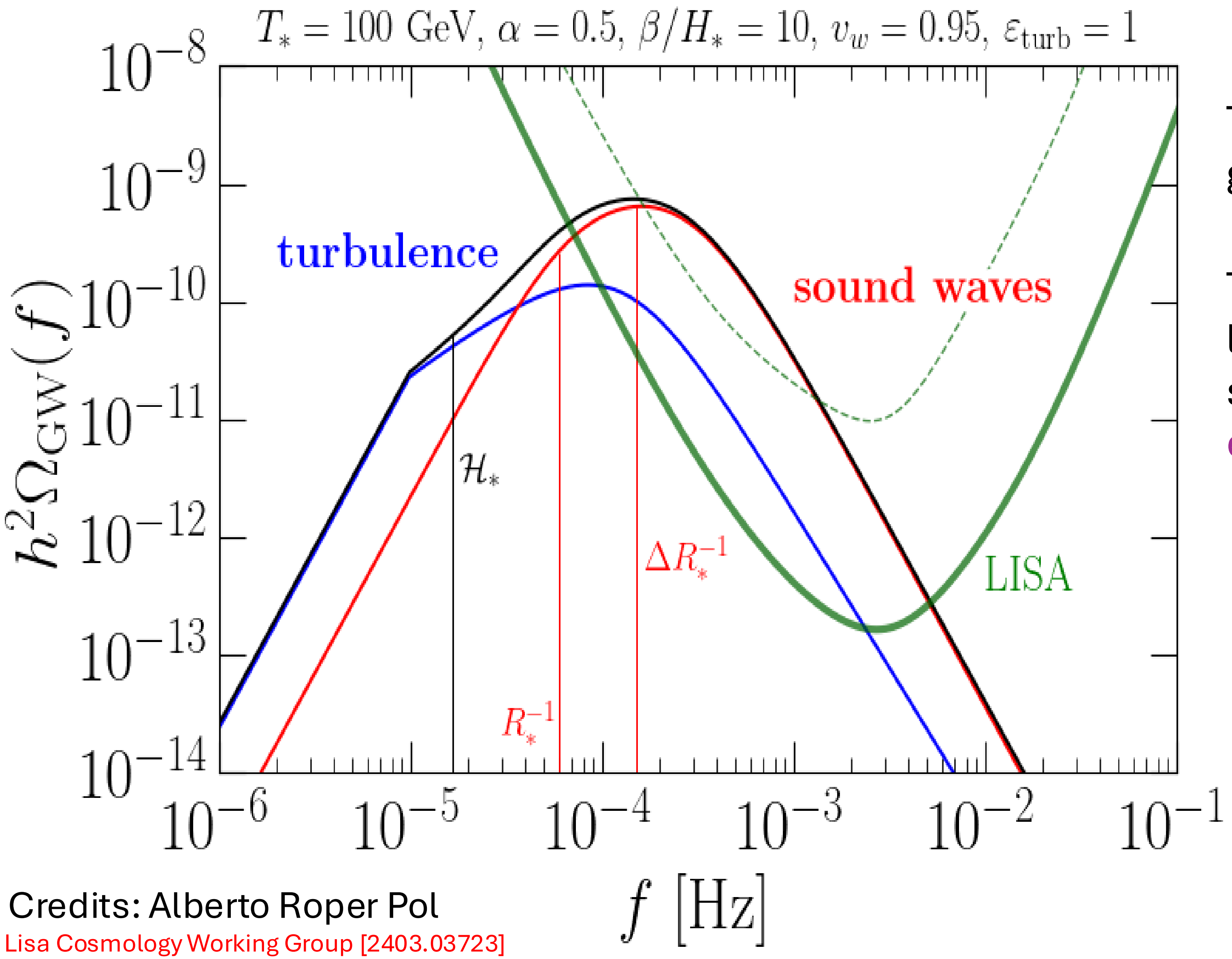

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Magnetic fields can be generated through scalar gradients (Vachaspati et al. 2021) or amplified by turbulence which can lead, given the extremely large conductivity in the primordial plasma (Arnold et al. 2003), to MHD turbulence

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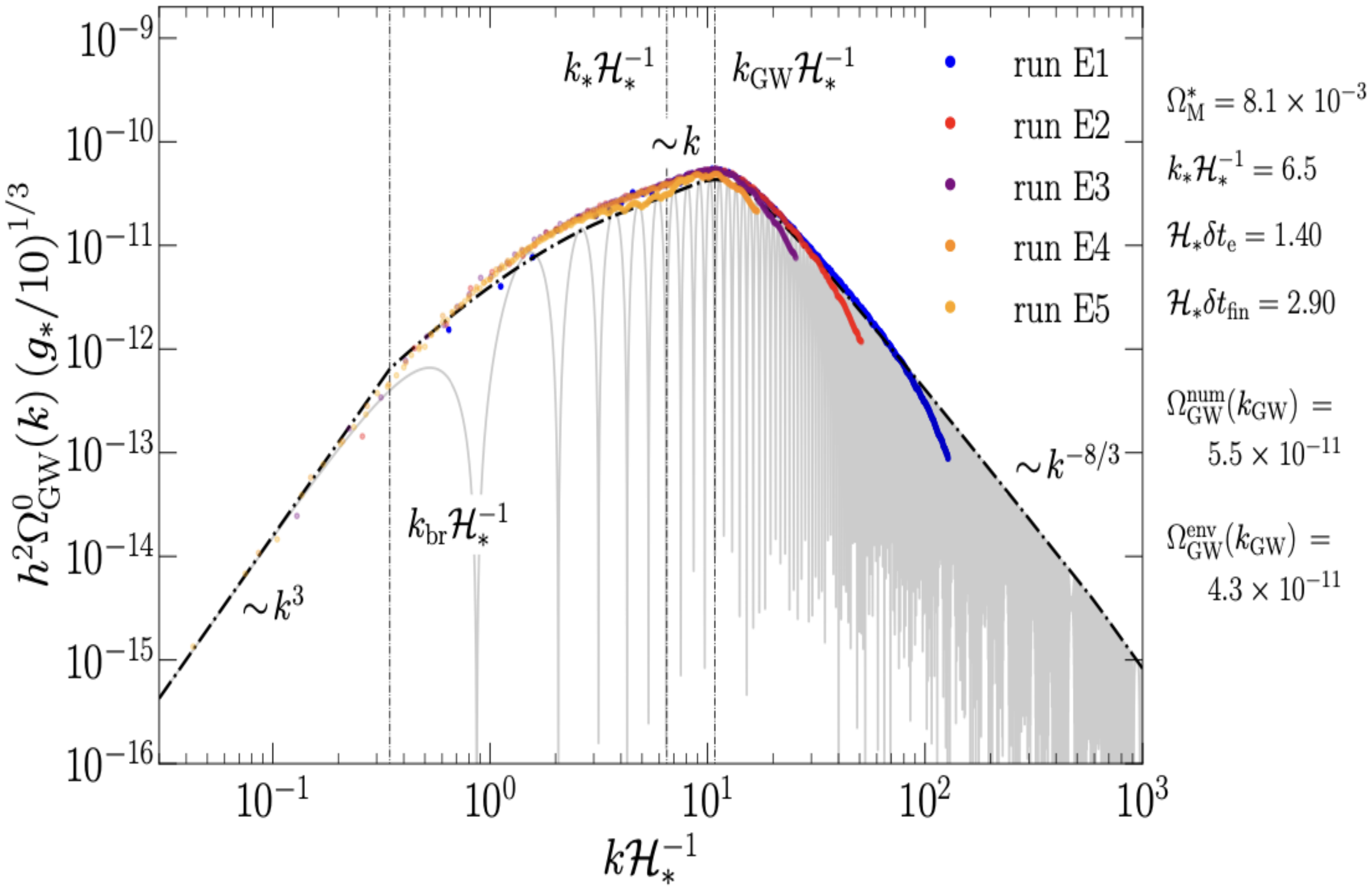
Turbulence in the early Universe: first-order phase transitions and gravitational waves

Simulations of vortical MHD turbulence with the PENCIL code

Roper Pol et al. [2201.05630]



Simulations of vortical hydrodynamic turbulence with the SCOTTS code

Auclair et al. [2205.02588]

Initializing a turbulent fluid in Fourier space

How to initialize a turbulent fluid in the lattice?

Initializing a turbulent fluid in Fourier space

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For a statistically homogeneous and isotropic field we have the following general decomposition of the two-point correlator in Fourier space

$$\langle v_i(\boldsymbol{k}) v_j^*(\boldsymbol{k}')\rangle = (2\pi)^6 \delta^3(\boldsymbol{k}-\boldsymbol{k}')\left[\left(\delta_{ij}-\hat{k}_i\hat{k}_j\right)\frac{E_N^v(k)}{4\pi k^3}\right]$$

vortical

$$E_N^v(k) \neq 0 \rightarrow \nabla \times v \neq 0$$

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vortical compressional

$E_N^v(k) \neq 0 \rightarrow \nabla \times v \neq 0$

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$E_N^v(k) \neq 0 \rightarrow \nabla \times v \neq 0$ $\qquad$ $E_L^v(k) \neq 0 \rightarrow \nabla \cdot v \neq 0$ $\qquad$ $H^v(k) \neq 0 \rightarrow$ *parity violation*

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vortical compressional helical

$E_N^v(k) \neq 0 \rightarrow \nabla \times v \neq 0$ $E_L^v(k) \neq 0 \rightarrow \nabla \cdot v \neq 0$ $H^v(k) \neq 0 \rightarrow$ *parity violation*

Causality (real space two-point correlator of velocity being zero at large scales) implies $E^v(k) \propto k^5$ for $k \to 0$ for a purely vortical ($E_L^v = 0$) or purely compressional ($E_N^v = 0$) field

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$$\left\langle v_i(\boldsymbol{k}) v_j^*(\boldsymbol{k}') \right\rangle = (2\pi)^6 \delta^3(\boldsymbol{k} - \boldsymbol{k}') \left[\left(\delta_{ij} - \hat{k}_i \hat{k}_j\right) \underbrace{\frac{E_N^v(k)}{4\pi k^3}}_{\text{vortical}} + \hat{k}_i \hat{k}_j \underbrace{\frac{E_L^v(k)}{2\pi k^3}}_{\text{compressional}} \right]$$

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A possible solution is to initialize the velocity field in Fourier space as

$$v_i(\boldsymbol{k}) = (2\pi)^3 g_0(k) g_j(\boldsymbol{k}) \left[\sqrt{1-q} \left(\delta_{ij} - \hat{k}_i \hat{k}_j\right) + \sqrt{2q}\, \hat{k}_i \hat{k}_j \right]$$

With $g_j(\boldsymbol{k})$ a random Gaussian number $\langle g_l(\boldsymbol{k}) g_m(\boldsymbol{k}') \rangle = \delta^3(\boldsymbol{k}-\boldsymbol{k}')\, \delta_{lm}$

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Are we sure that $q = 0$ implies $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ and $q = 1$ implies $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$?

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$q = 0 \rightarrow \hat{k}_i v_i = 0 \rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$

$q = 1 \rightarrow \epsilon_{ijl} \hat{k}_j v_l = 0 \rightarrow \boldsymbol{\nabla} \times \boldsymbol{v} = 0$

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Are we sure that $q = 0$ implies $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ and $q = 1$ implies $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$?

$q = 0 \rightarrow \hat{k}_i v_i = 0 \rightarrow \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$

$q = 1 \rightarrow \epsilon_{ijl} \hat{k}_j v_l = 0 \rightarrow \boldsymbol{\nabla} \times \boldsymbol{v} = 0$

The last implications are true only if $\hat{k}_i$ is the Fourier transform of the discrete derivative operator $\rightarrow \hat{k}_i = \hat{k}_{Lat}$

Initializing a turbulent fluid in Fourier space

How to initialize a turbulent fluid in the lattice?

$$v_i(\boldsymbol{k}) = (2\pi)^3 g_0(k) g_j(\boldsymbol{k}) \left[\sqrt{1-q}\left(\delta_{ij} - \hat{k}_i^{Lat}\hat{k}_j^{Lat}\right) + \sqrt{2q}\,\hat{k}_i^{Lat}\hat{k}_j^{Lat}\right]$$

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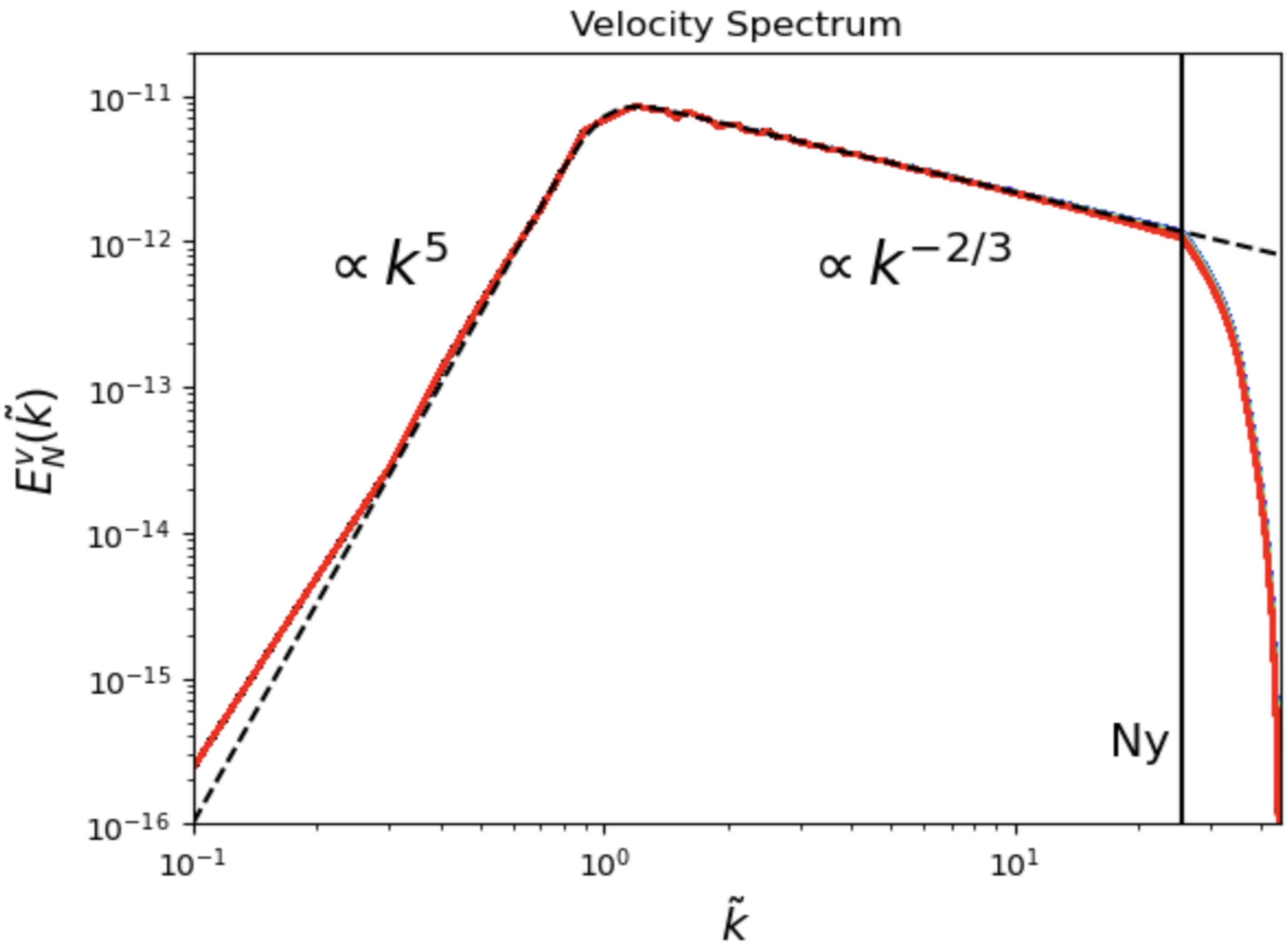
For neutral derivatives of order 2, 4 and 6 we have

$$\boldsymbol{k}_{\boldsymbol{Lat}}^{(2)} = \frac{\sin\left(\frac{2\,\pi\,\tilde{\boldsymbol{n}}}{N}\right)}{\delta x} \qquad \boldsymbol{k}_{\boldsymbol{Lat}}^{(4)} = \frac{4}{3}\frac{\sin\left(\frac{2\,\pi\,\tilde{\boldsymbol{n}}}{N}\right)}{\delta x} - \frac{1}{6}\frac{\sin\left(\frac{4\,\pi\,\tilde{\boldsymbol{n}}}{N}\right)}{\delta x} \qquad \boldsymbol{k}_{\boldsymbol{Lat}}^{(6)} = \frac{3}{2}\frac{\sin\left(\frac{2\,\pi\,\tilde{\boldsymbol{n}}}{N}\right)}{\delta x} - \frac{3}{10}\frac{\sin\left(\frac{4\,\pi\,\tilde{\boldsymbol{n}}}{N}\right)}{\delta x} + \frac{1}{30}\frac{\sin\left(\frac{6\,\pi\,\tilde{\boldsymbol{n}}}{N}\right)}{\delta x}$$

Gravitational Waves from decaying turbulence



<u>Subrelativistic case - numerical setup</u>

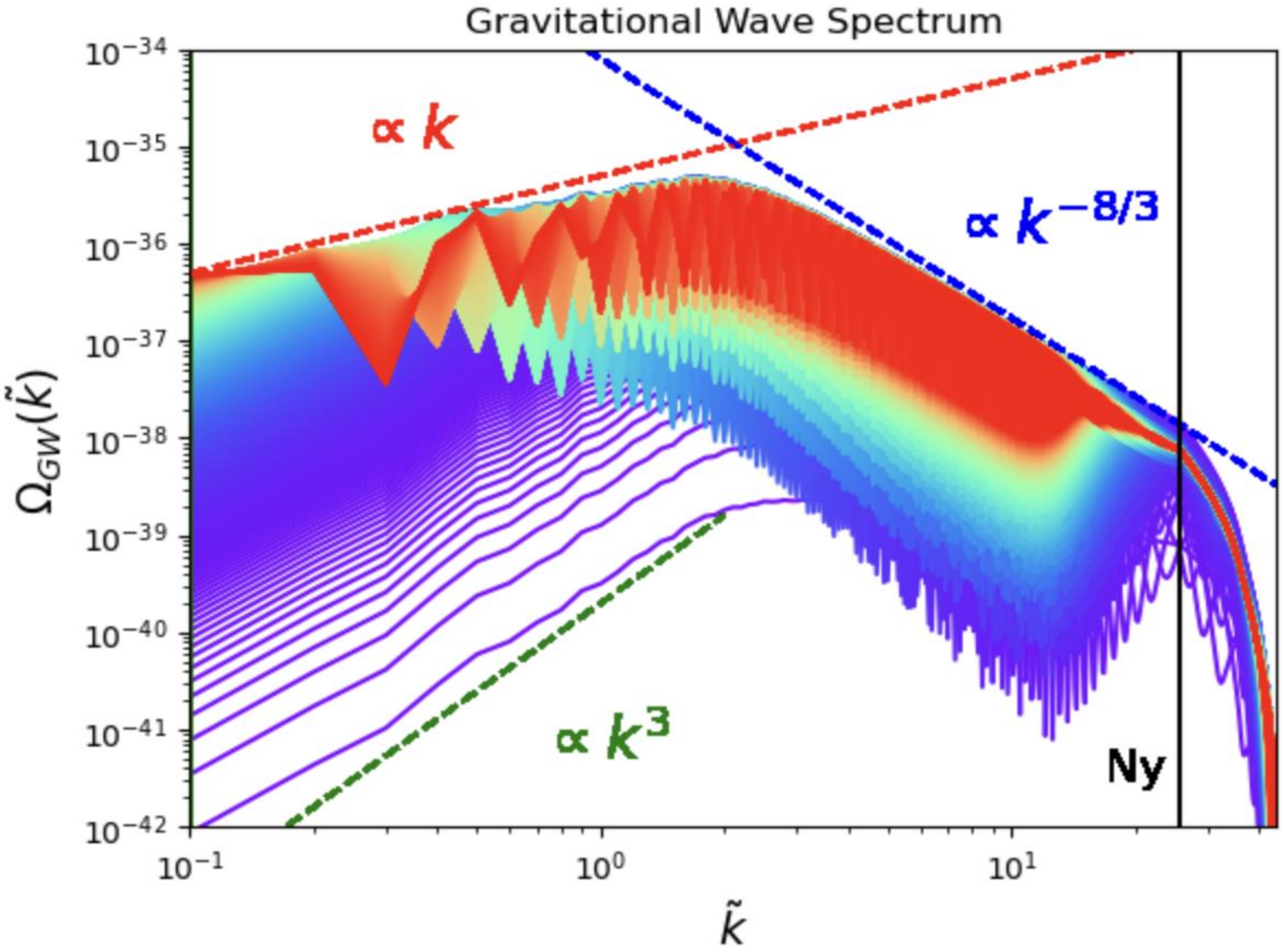
$$N = 512, \quad \tilde{k}_{IR} = 0.1, \quad \tilde{k}_{Ny} = \frac{\tilde{k}_{IR} N}{2} = 25.6,$$

$$\tilde{t}_{fin} - \tilde{t}_* = \frac{1}{\tilde{k}_{IR}} = 10, \quad d\tilde{t} = 10^{-3}, \quad \frac{d\tilde{t}}{d\tilde{x}} < 0.01,$$

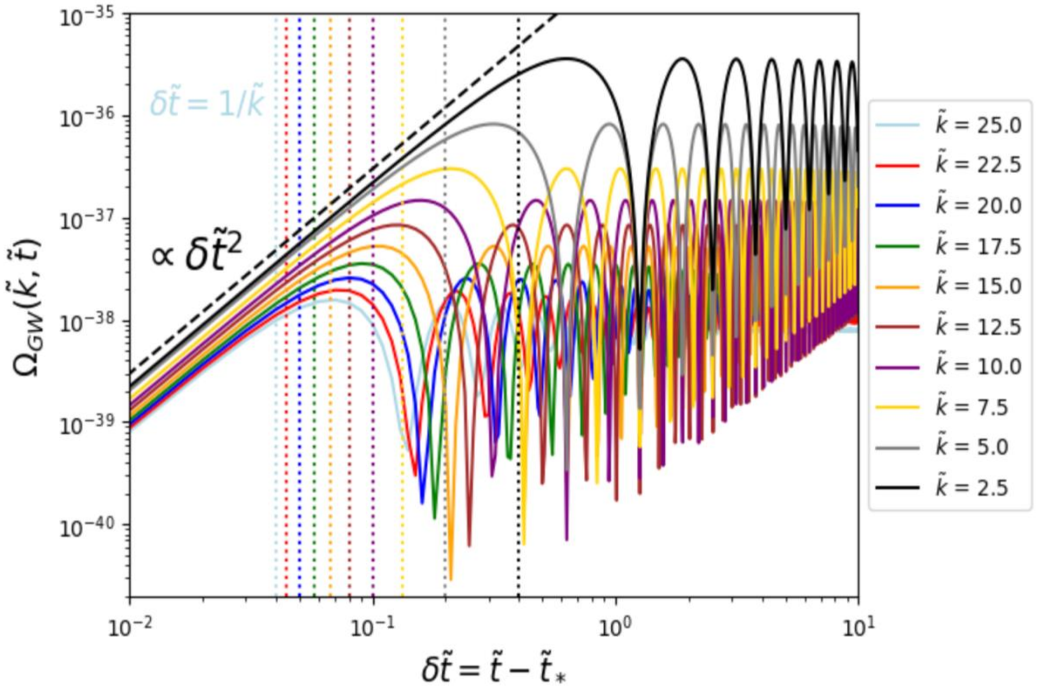
$$v_{rms} \approx 4 \times 10^{-6}, \;\; \underset{(\nu)}{viscosity} = \frac{v_{rms}}{\tilde{k}_{Ny}} \approx 10^{-7}$$

$\delta\tau_{eddy} \sim 10^7$ → almost no decay of the source

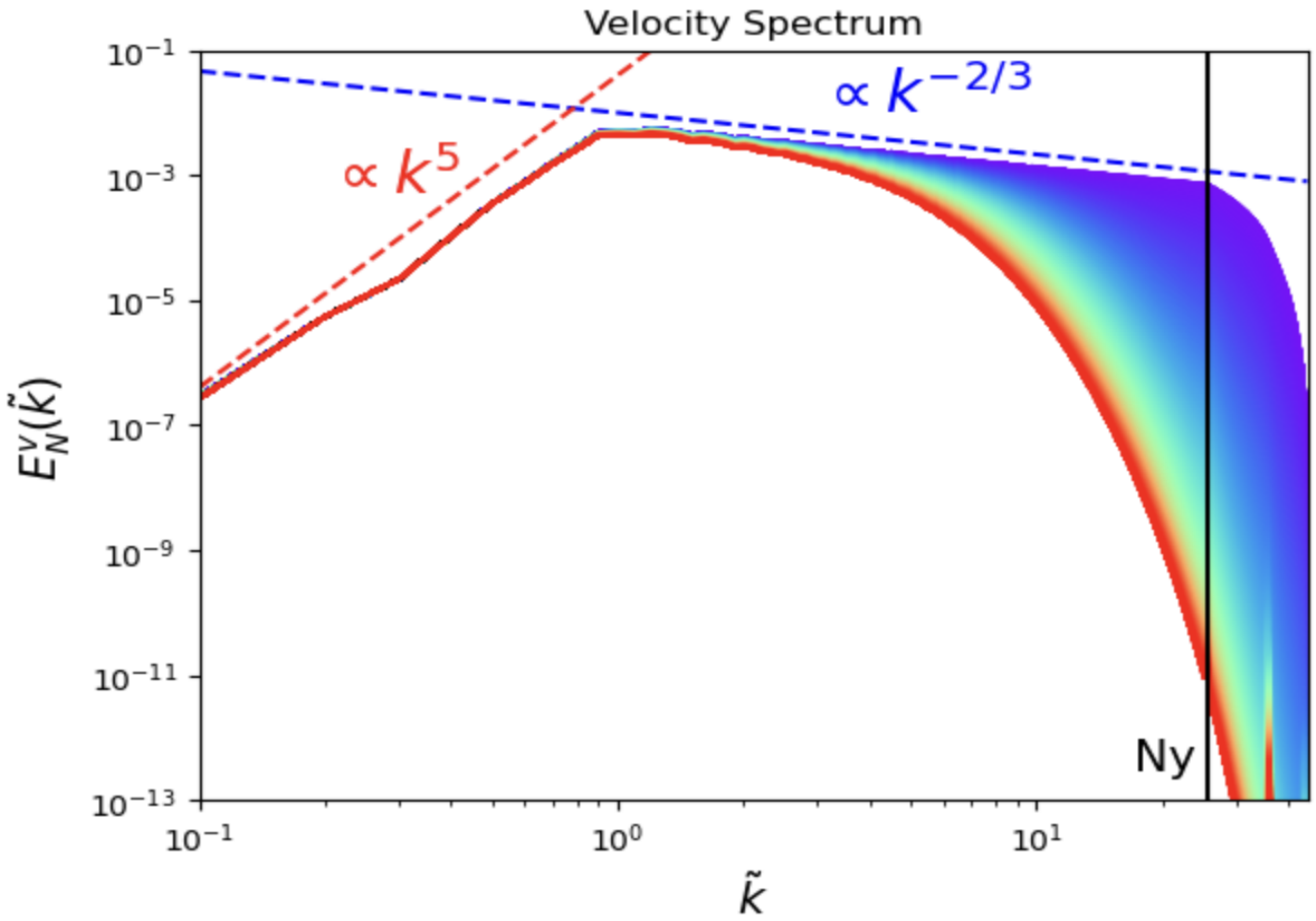
Gravitational Waves from decaying turbulence



modes grow as $\delta\tilde{t}^2$
and saturate when $\tilde{k}\delta\tilde{t} \approx 1$



Gravitational Waves from decaying turbulence



<u>Relativistic case - numerical setup</u>

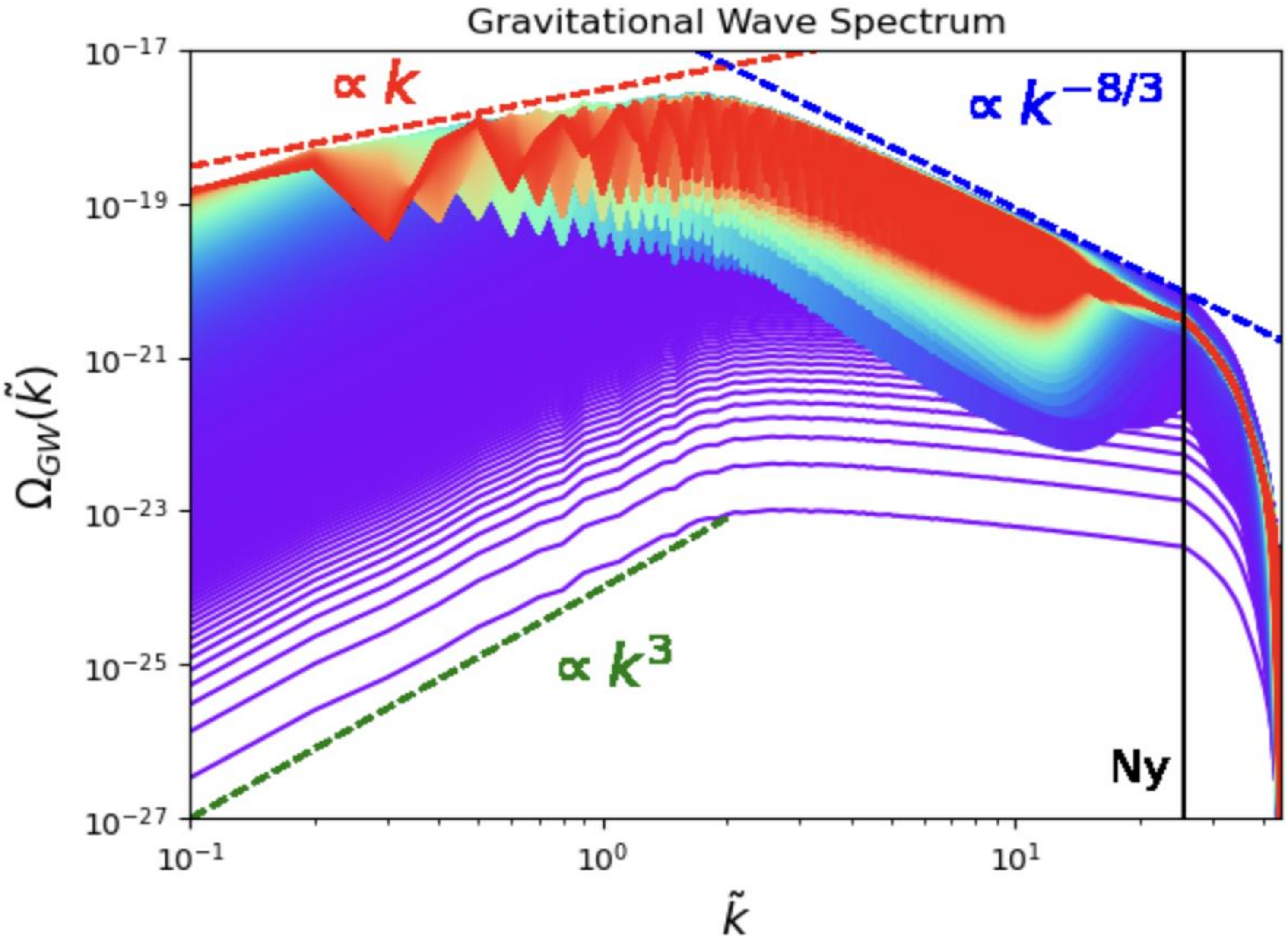
$$N = 512, \qquad \tilde{k}_{IR} = 0.1, \qquad \tilde{k}_{Ny} = \frac{\tilde{k}_{IR} N}{2} = 25.6,$$

$$\tilde{t}_{fin} - \tilde{t}_* = \frac{1}{\tilde{k}_{IR}} = 10, \quad d\tilde{t} = 10^{-4}, \quad \frac{d\tilde{t}}{d\tilde{x}} < 0.001,$$

$$v_{rms} \approx 0.1, \qquad \underset{(\nu)}{viscosity} = \frac{v_{rms}}{\tilde{k}_{Ny}} \approx 10^{-2}$$

$\delta\tau_{eddy} \sim 10^2 \longrightarrow$ visible decay of the source

Gravitational Waves from decaying turbulence



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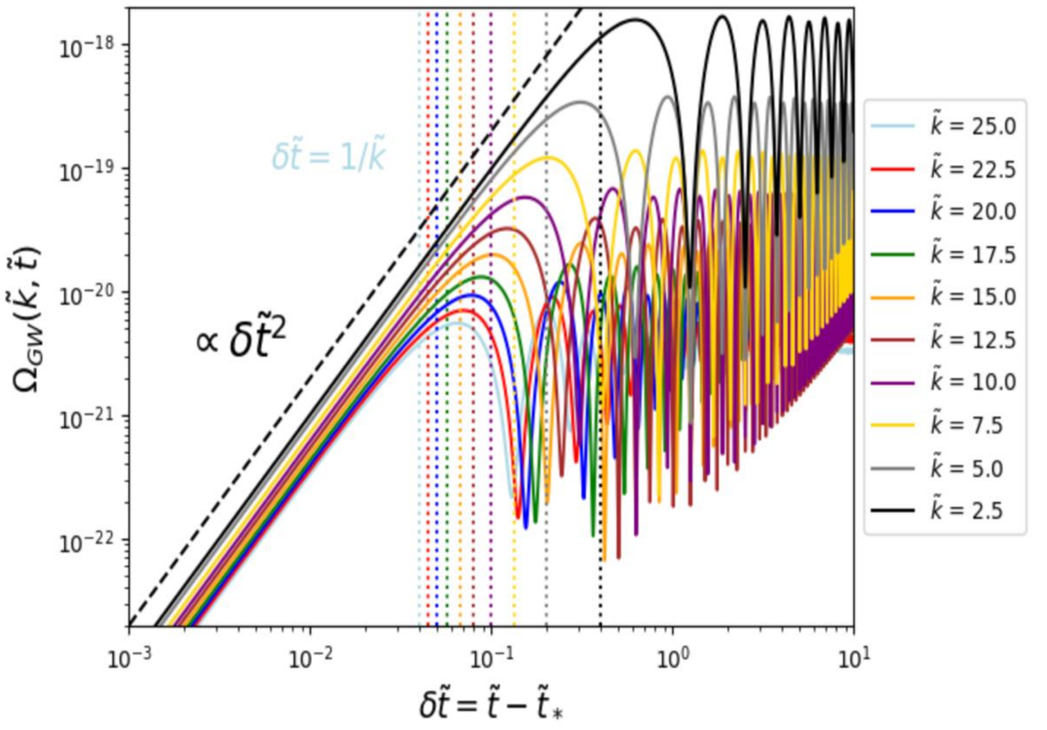
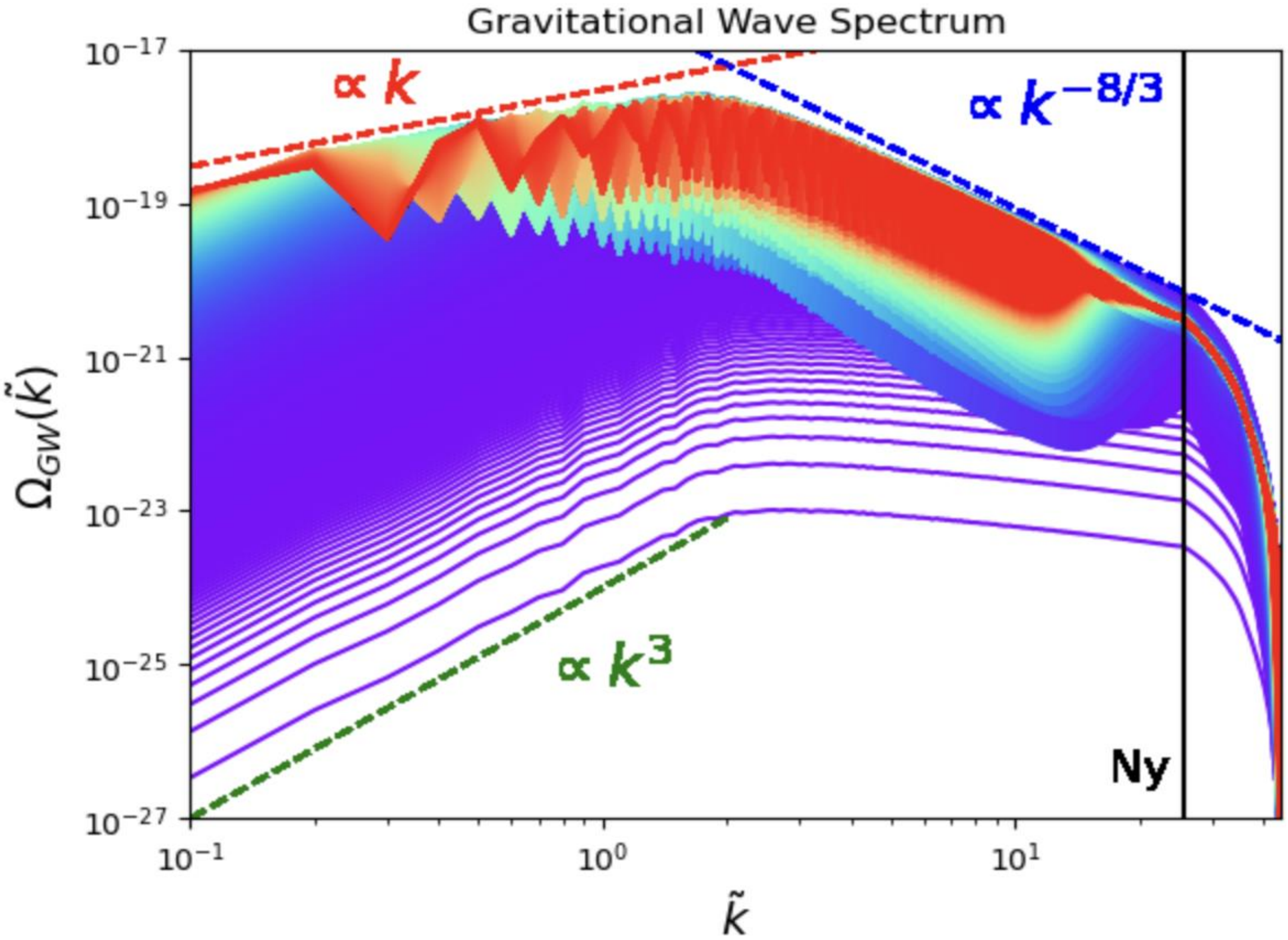


Gravitational Waves from decaying turbulence

This spectral shape and behavior seems quite universal. Is it consistent with theoretical expectations?



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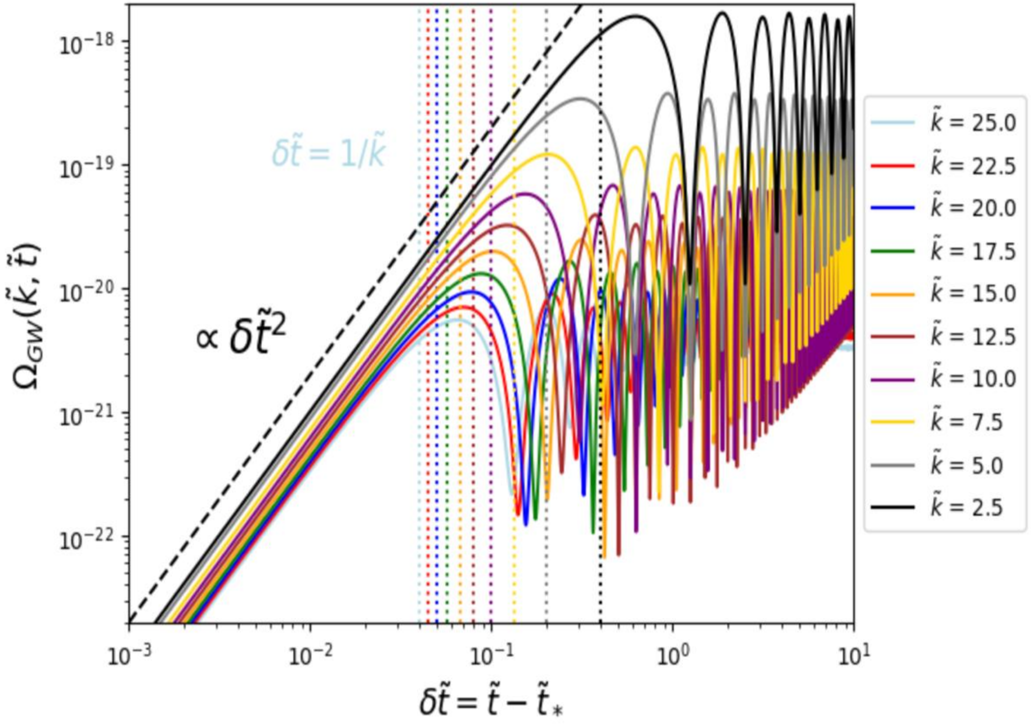

Gravitational Wave Background from stochastic processes in the early Universe

Tensor perturbations over FLRW $\quad ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + \ell_{ij})\, dx^i\, dx^j\right]$

GW equation (radiation domination) $\quad \xrightarrow[a = \tau/\tau_*]{h_{ij} = a\,\ell_{ij}} \quad (\partial_\tau^2 + k^2)\, h_{ij}(\tau, \boldsymbol{k}) = 6\, \Pi_{ij}(\tau, \boldsymbol{k}) \frac{\mathcal{H}_*}{\tau}$

$\Pi_{ij} = a^2 \Lambda_{ijkl} T^{kl} / \rho_{crit}$

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source active for $\tau_* < \tau < \tau_{fin}$

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GW spectrum at present time

$$\Omega_{GW}(t_0) = \frac{\rho_{GW}^0}{\rho_{crit}^0} = \frac{M_{pl}^2}{4\rho_{crit}^0} \langle \left| \partial_t \ell_{ij}(\boldsymbol{x}, t_0) \right|^2 \rangle = \frac{1}{12\, H_0^2} \left(\frac{a_*}{a_0}\right)^4 \langle \left| \partial_\tau h_{ij}(\boldsymbol{x}, \tau_0) - \mathcal{H}_0 h_{ij}(\boldsymbol{x}, \tau_0) \right|^2 \rangle$$

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$$\langle\, \Pi_{ij}(\tau_1, \boldsymbol{k}) \Pi^*_{\mathrm{lm}}(\tau_2, \boldsymbol{k}') \rangle = \frac{1}{2} (2\pi)^6 \delta^3(\boldsymbol{k} - \boldsymbol{k}') [\Lambda_{ijlm}(\hat{\boldsymbol{k}}) \frac{E_\Pi(k, \tau_1, \tau_2)}{4\pi k^3} + i\mathcal{A}_{ijlm}(\hat{\boldsymbol{k}}) \frac{H_\Pi(k, \tau_1, \tau_2)}{4\pi k^3}]$$

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Constant-in-time model for the UETC of the source

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Assuming that the source is slowly decaying* for $\tau_* < \tau < \tau_{fin}$ $\longrightarrow$ $E_\Pi(k, \tau_1, \tau_2) = E_\Pi^*(k)$

*with respect to the light crossing time at wavenumber k

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$$\equiv 3\,\mathcal{T}_{GW} E_\Pi^*(k)\,\Delta^2(k,\tau_0)$$

$$\Delta(k,\tau_0) \equiv \int_{\tau_*}^{\min[\tau_0,\tau_{fin}]} \frac{\mathrm{d}\tilde\tau}{\tilde\tau}\cos k(\tau_0-\tilde\tau)$$

$$\mathcal{T}_{GW} = \left(\frac{a_*}{a_0}\right)^4 \left(\frac{H_*}{H_0}\right)^2 \approx 1.6\times 10^{-5}\left(\frac{g_*}{100}\right)^{-\frac{1}{3}}$$

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Constant-in-time model for the UETC of the source

$k(\tau - \tau_*) = k\delta\tau < 1$

$$\rightarrow \Delta^2(k,\tau) \propto \ln^2\left(1 + \frac{\delta\tau}{\tau_*}\right) \xrightarrow{\delta\tau/\tau_* \ll 1} \approx \left(\frac{\delta\tau}{\tau_*}\right)^2$$

(flat spacetime)

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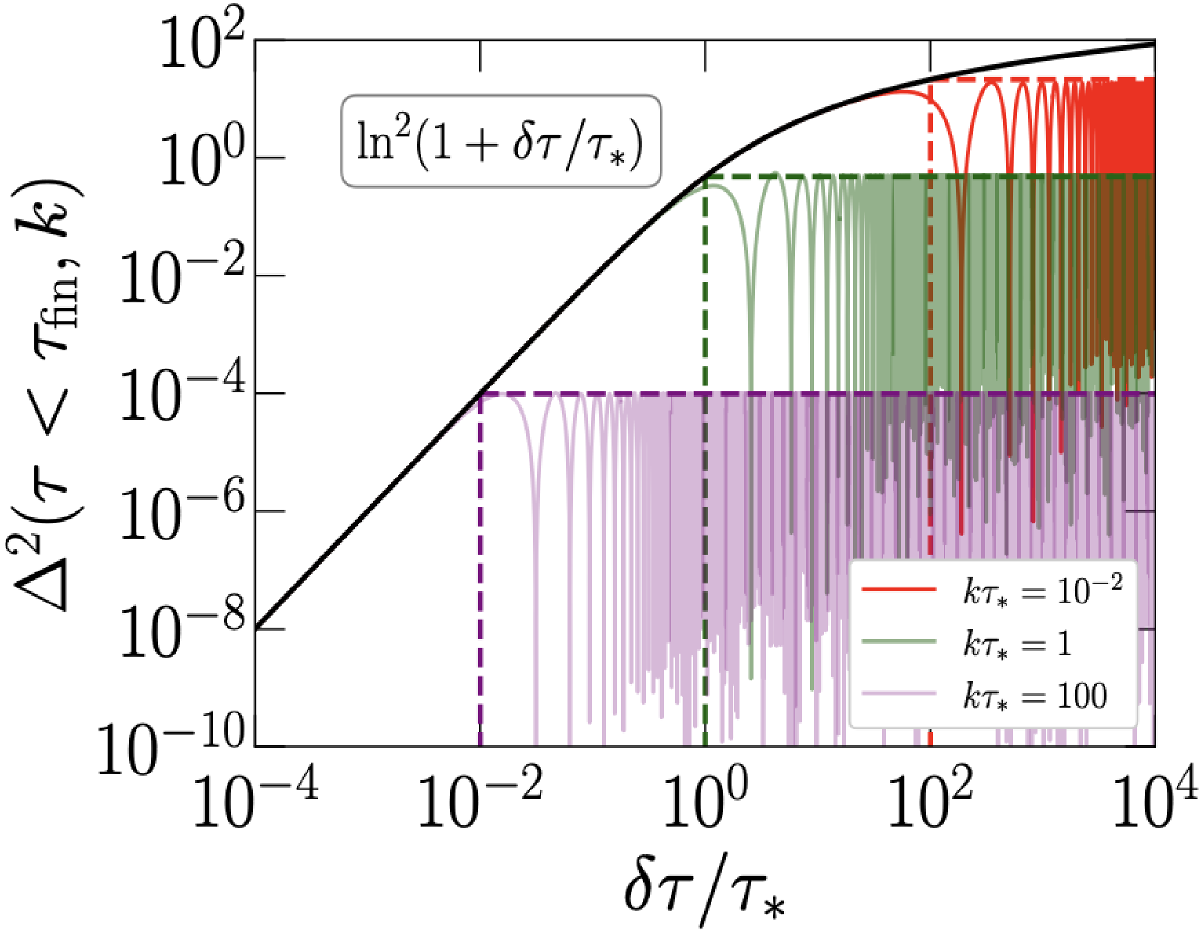

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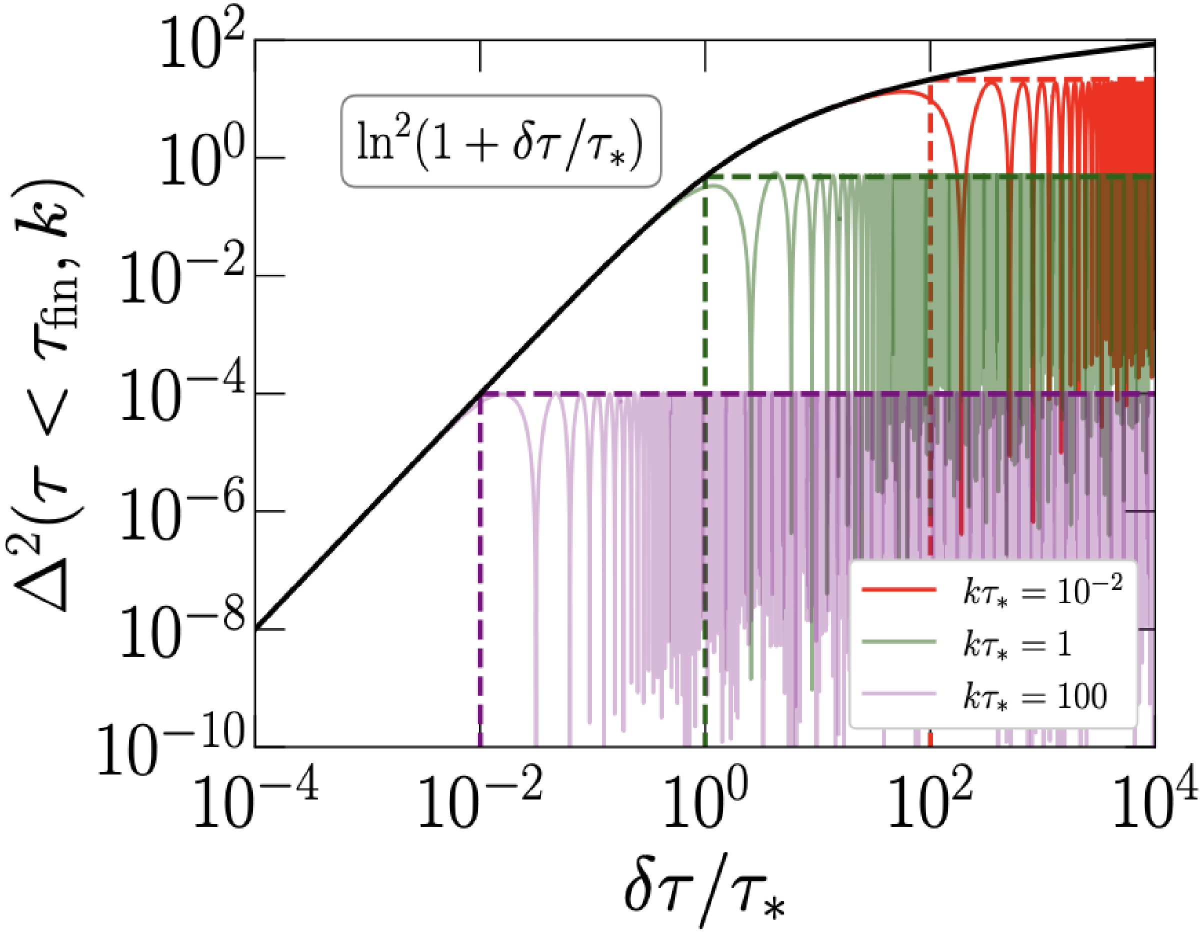

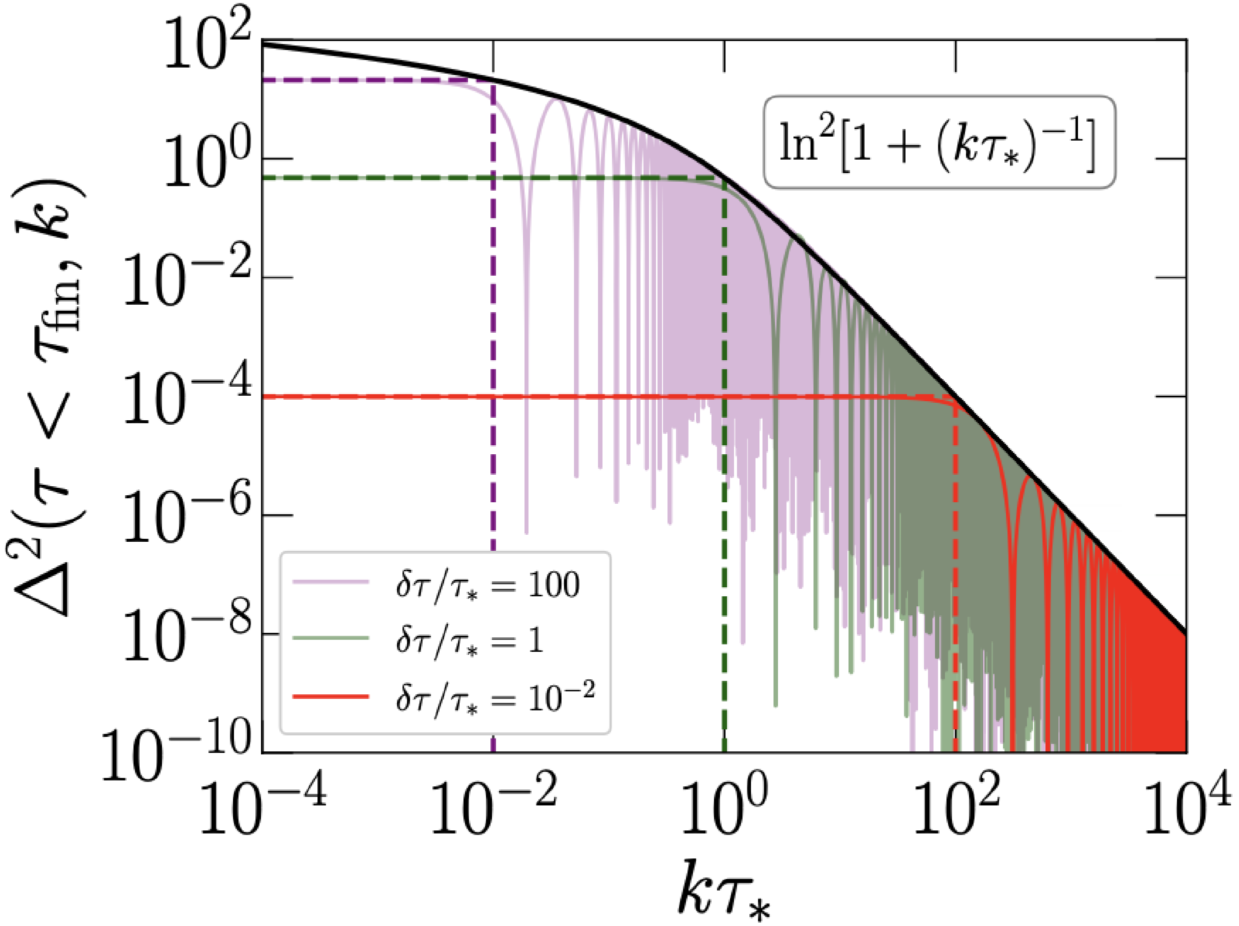

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Modes $k > 1/\delta\tau_{fin}$ saturated with amplitude $\ln^2[1 + (k\tau_*)^{-1}]$

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Modes $k < 1/\delta\tau_{fin}$ stop growing at $\tau = \tau_{fin}$ and have amplitude $\ln^2[1 + \delta\tau_{fin}/\tau_*]$

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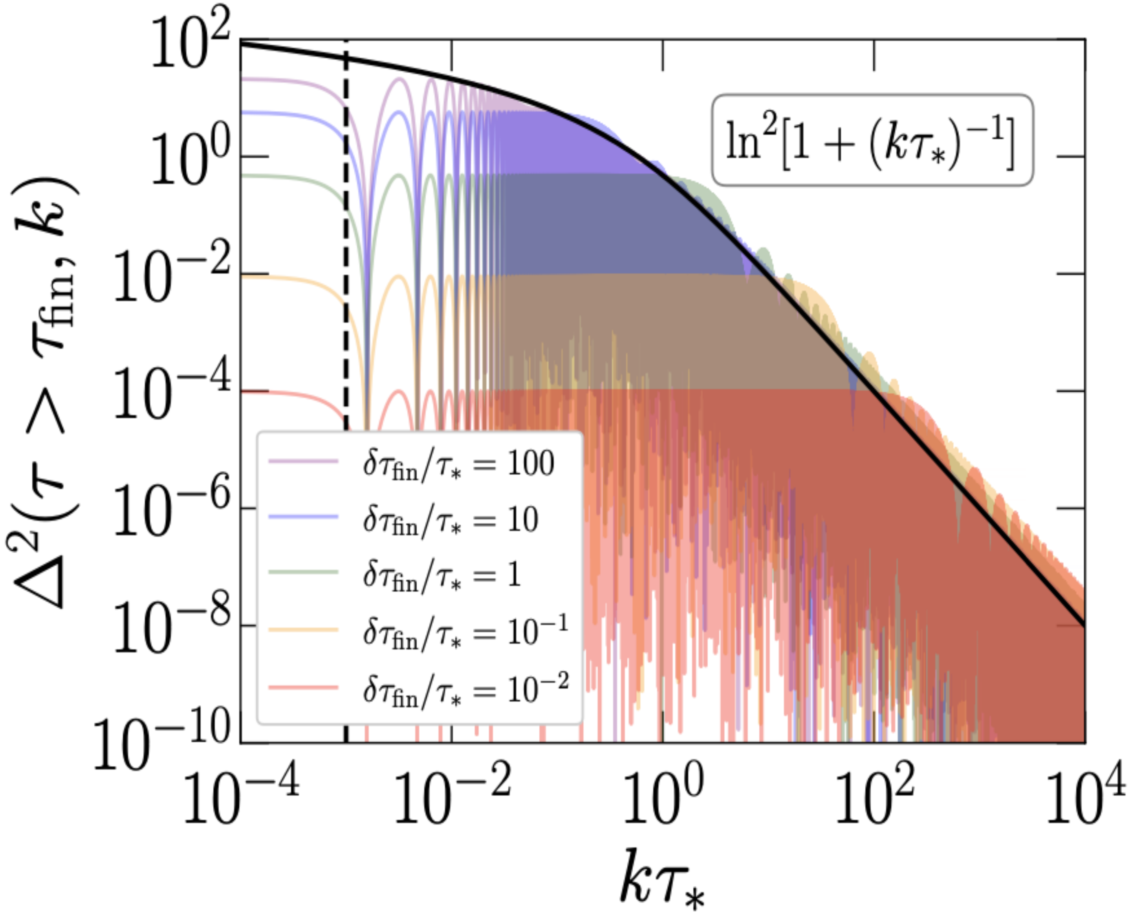
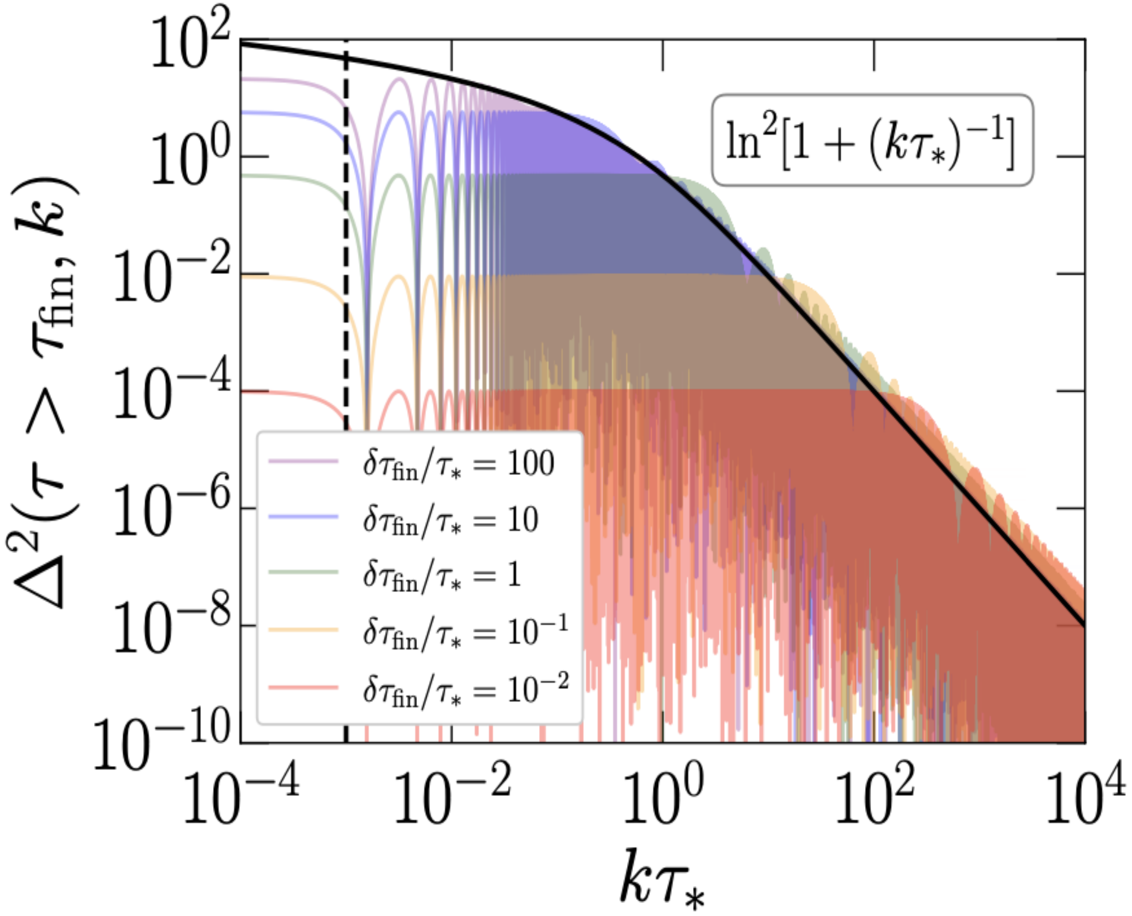


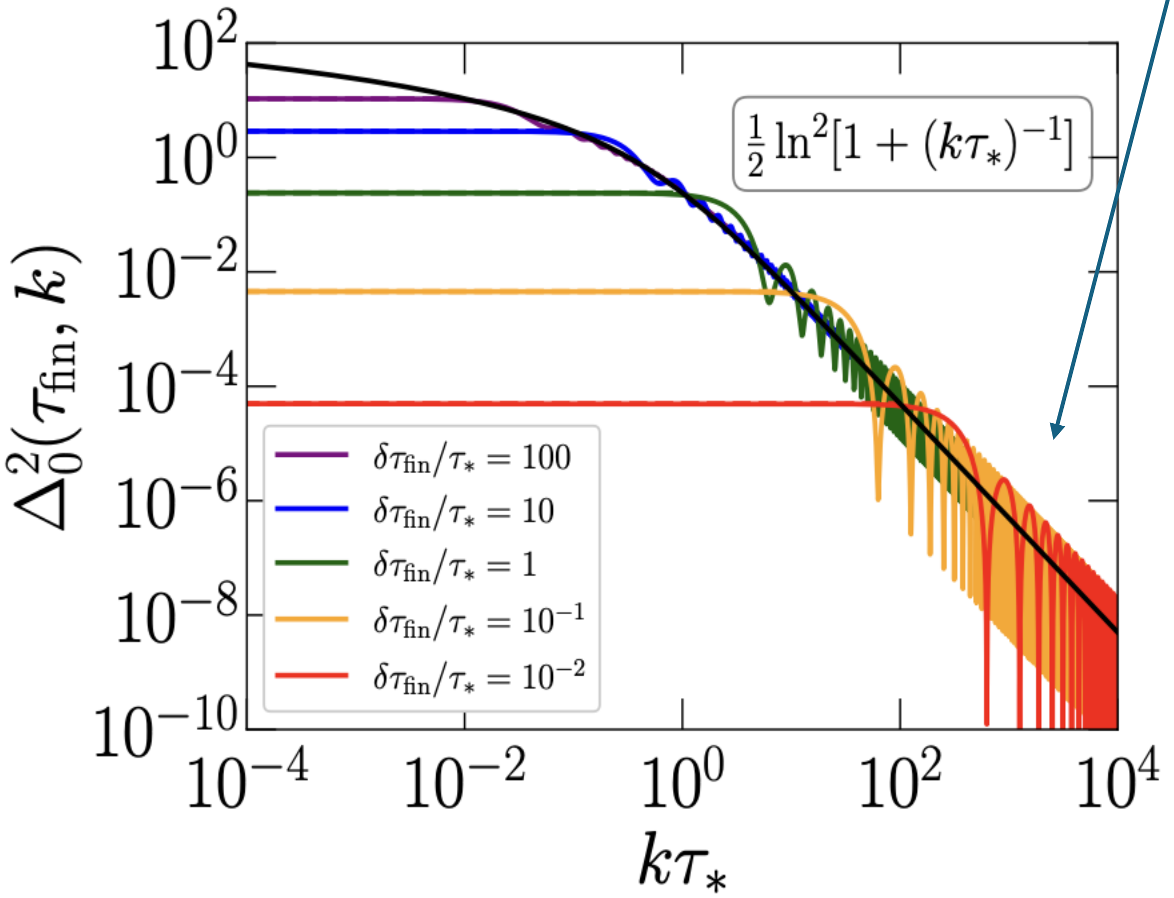
Constant-in-time model for the UETC of the source

Modes $k > 1/\delta\tau_{fin}$ saturated with amplitude $\ln^2[1 + (k\tau_*)^{-1}]$

averaging over fast frequency oscillations at present time

Modes $k < 1/\delta\tau_{fin}$ stop growing at $\tau = \tau_{fin}$ and have amplitude $\ln^2[1 + \delta\tau_{fin}/\tau_*]$



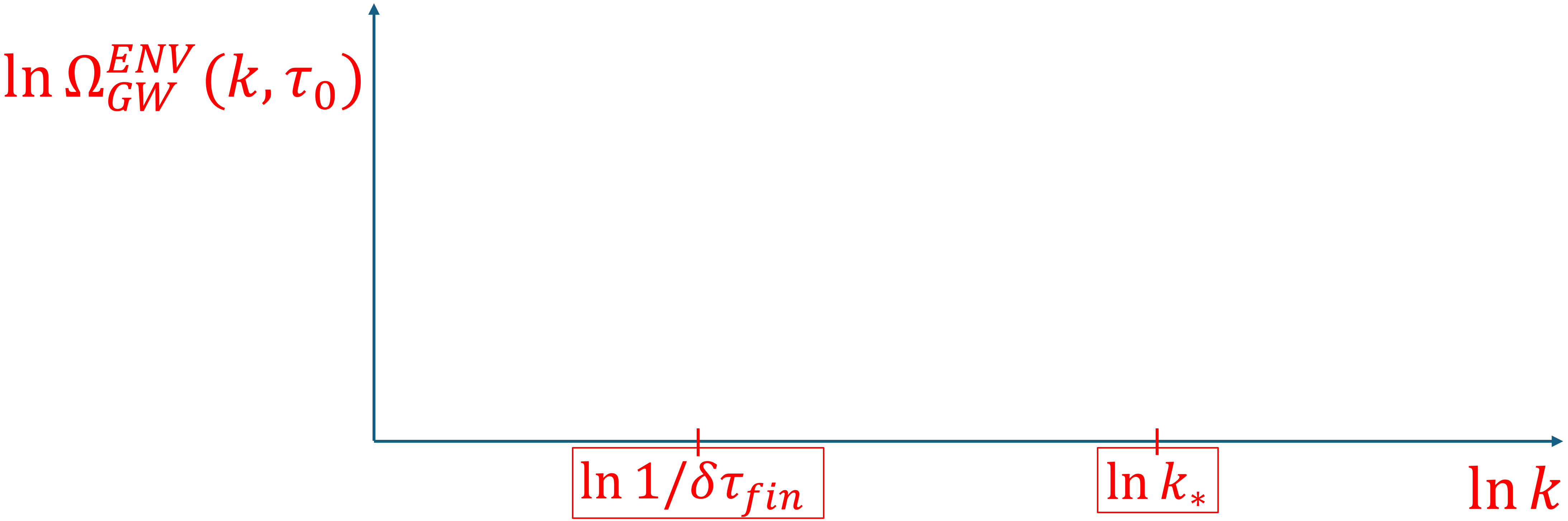


Constant-in-time model for the UETC of the source

$$\Omega_{GW}(k,\tau_0) \equiv 3\,\mathcal{T}_{GW}\,E^*_\Pi(k)\;\Delta_0^2(k,\tau_{fin})$$

causality

Assuming for the UETC $E^*_\Pi(k) \sim \begin{cases} k^3 & k < k_* \\ k^{-b} & k > k_* \end{cases}$



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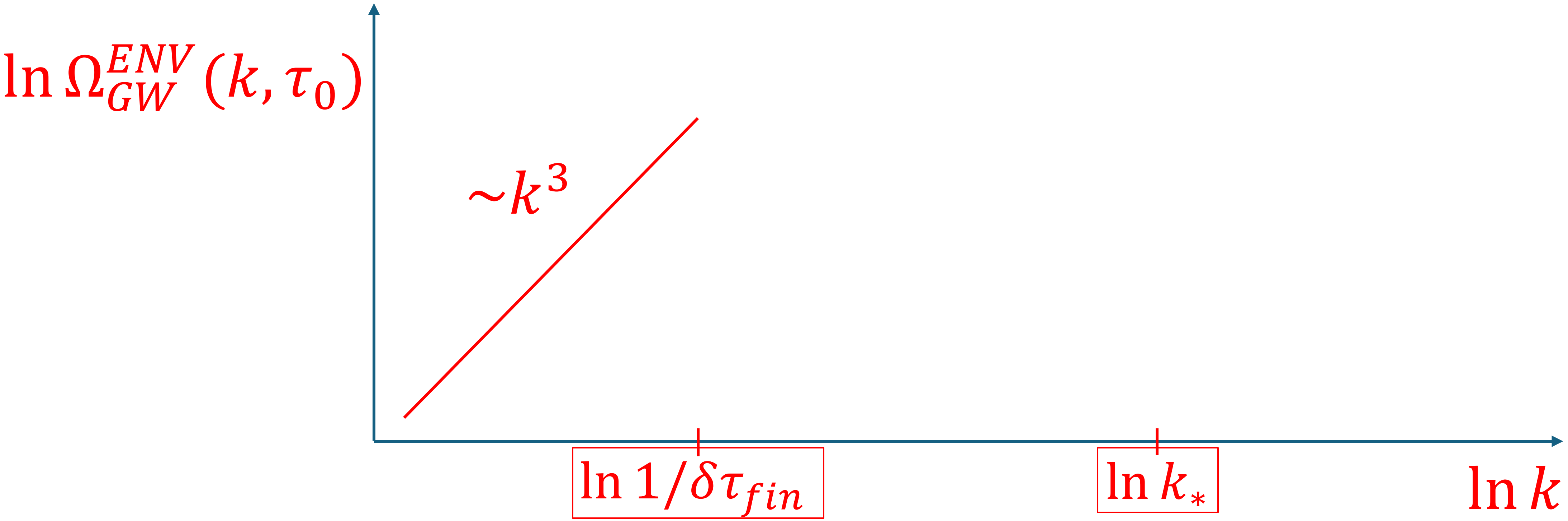

Constant-in-time model for the UETC of the source

$$\Omega_{GW}(k,\tau_0) \equiv 3\, \mathcal{T}_{GW} E_\Pi^*(k)\, \Delta_0^2(k,\tau_{fin})$$

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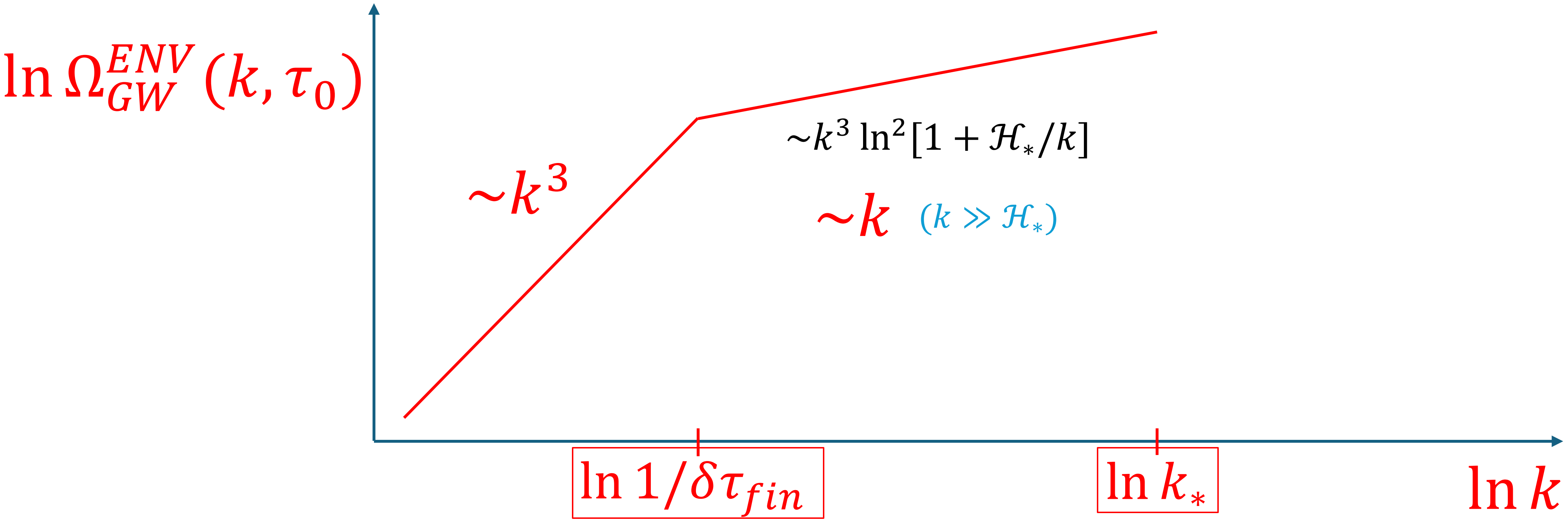

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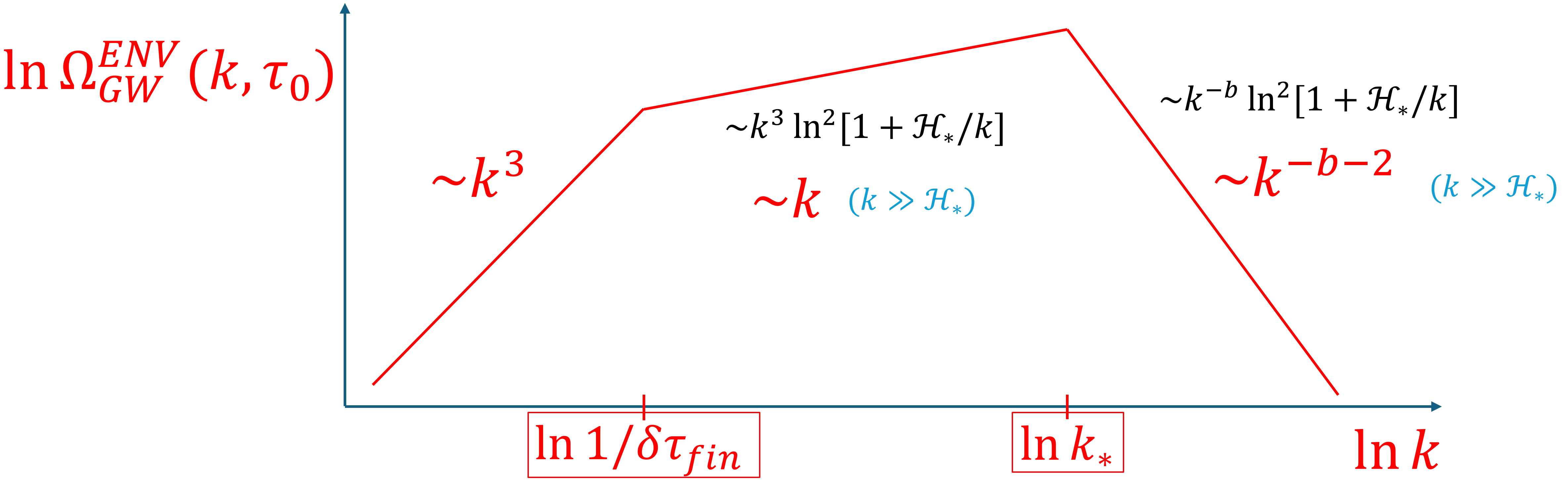

Gravitational Waves from decaying turbulence

For a purely vortical velocity field with a Von Kármán spectrum

$$E_N^v(k) \sim \begin{cases} k^5 & (k/k_{peak} \to 0) \quad Batchelor \\ k^{-2/3} & (k/k_{peak} \to \infty) \quad Kolmogorov \end{cases}$$

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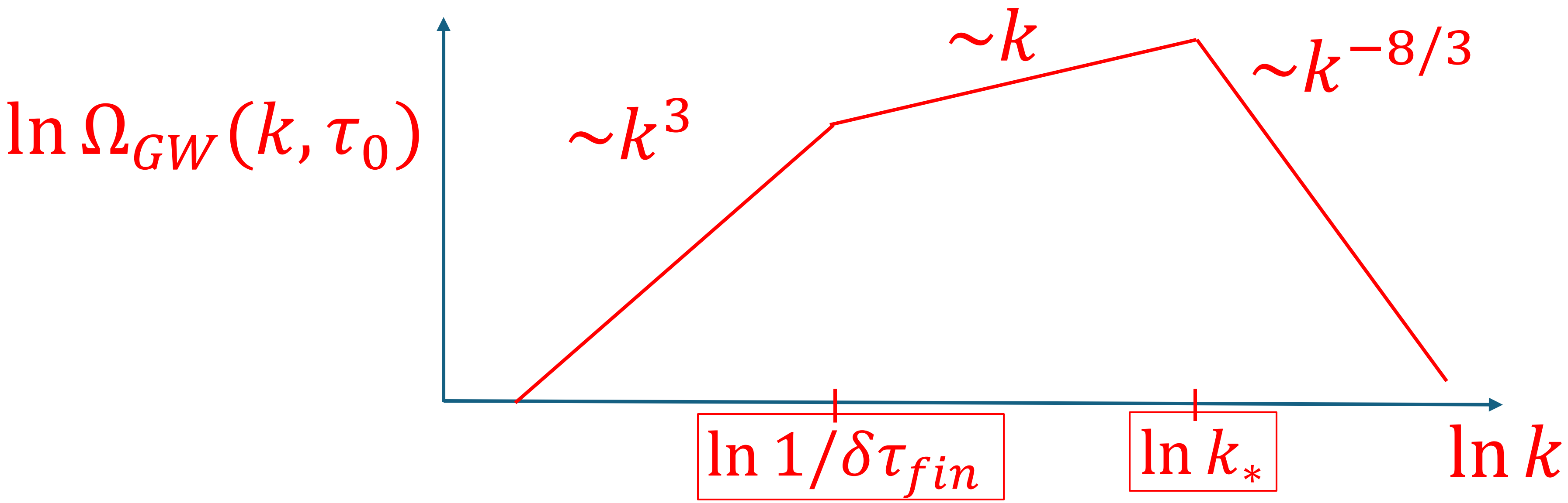
GW spectrum envelope for vortical turbulence in the constant-in-time model (flat spacetime)



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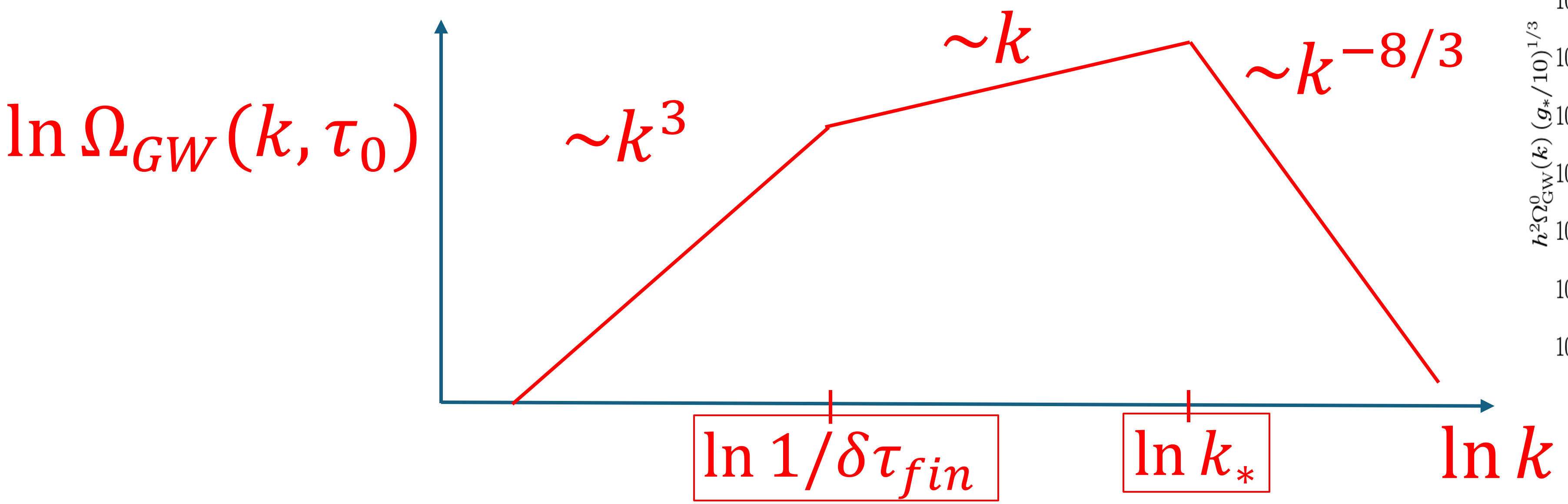
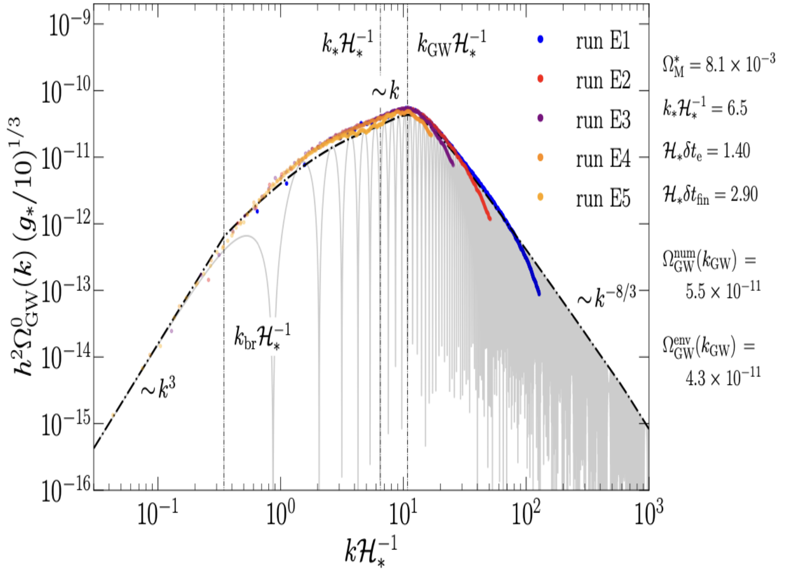
Roper Pol et al. [2201.05630]





Conclusions

- Hydrodynamic and magnetohydrodynamic turbulence are expected to be generated from phase transitions in the early Universe

- The actual sourcing phase of turbulence is relevant for the final Gravitational Wave spectrum but requires full phase transition simulations

- We can simulate decaying turbulence by properly initializing the velocity field in Fourier space

- The Gravitational Wave spectrum from decaying turbulence (in the subrelativistic limit) can be described assuming a constant-in-time unequal time correlator of the anisotropic stresses