Lecture 6: Lattice Formulation of Abelian and non-Abelian gauge theories

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Based on previous lectures by F. Torrenti and A. Florio

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Deutsches Elektronen
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Motivation

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- Grand Unified Theories comprise larger gauge groups, e.g., $SU(5)$

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- Broad parametric resonance of gauge fields coupled to oscillating complex scalars (minimal gauge coupling, **this lecture**)
- Gauge field production during axion inflation (**tomorrow on lecture 7**!)

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Caveat: lattice formulation must preserve **gauge invariance**!

Overview

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U(1) theory in the continuum

Consider a simple model with a **complex scalar and a U(1) gauge field**

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[(D_\mu\varphi)^*(D^\mu\varphi) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + V(|\varphi|)\right]$$

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- Complex scalar field:

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$$

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$g \equiv$ gauge coupling

$Q \equiv$ charge of φ under U(1)

U(1) gauge sector:

- Abelian gauge field:

$$A_\mu = (A_0, A_1, A_2, A_3)$$

- Covariant derivative:

$$D_\mu = \partial_\mu - igQA_\mu$$

- Field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- Electric field: $E_i \equiv F_{0i}$
- Magnetic field: $B_i \equiv \frac{1}{2}\varepsilon_{ijk}F^{jk}$

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Action is invariant under **gauge transformations**

$$\varphi(x) \longrightarrow \exp\left[-igQ\boldsymbol{\lambda(x)}\right] \times \varphi(x)$$

$$A_\mu(x) \longrightarrow A_\mu(x) - \partial_\mu\boldsymbol{\lambda(x)}$$

$\boldsymbol{\lambda(x)}$ arbitrary

this implies

$$F_{\mu\nu} \longrightarrow F_{\mu\nu} \qquad \text{(Invariant)}$$

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We will work in the **temporal gauge** ($A_0 = 0$)

Equations of motion in flat spacetime

Equations of motion in flat spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$)

$$\textbf{Complex scalar}: \quad D_\mu D_\mu \varphi = -\frac{1}{2}\frac{\partial V}{\partial|\varphi|}\frac{\varphi}{|\varphi|}$$

$$\textbf{Gauge field}: \quad \partial_\nu F_{i\nu} = J_i \equiv 2gQ\,\mathrm{Im}[\varphi^*(D_i\varphi)]$$

where J_i is the **U(1) current** and J_0 **the charge density**

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Gauss law must be preserved at all times

$$\partial_i F_{0i} = J_0 \quad \longrightarrow \nabla \cdot \vec{E} = J_0$$

Equations of motion in expanding background

Consider a **FLRW metric**

$$ds^2 = -a^{2\alpha}(\eta)d\eta^2 + a^2(\eta)\delta_{ij}dx^i dx^j$$

The **equations of motion of the fields in an expanding background** are

$$\boldsymbol{\varphi} : \qquad \varphi'' - a^{-2(1-\alpha)} D_i D_i \varphi + (3-\alpha)\frac{a'}{a}\varphi' = -\frac{a^{2\alpha}}{2}\frac{\partial V}{\partial|\varphi|}\frac{\varphi}{|\varphi|}$$

$$\boldsymbol{A_i} : \qquad F'_{0i} - a^{-2(1-\alpha)}\partial_j F_{ji} + (1-\alpha)\frac{a'}{a}F_{0i} = a^{2\alpha} J_i$$

$$\textbf{Gauss' law} : \qquad \partial_i F_{0i} = a^2 J_0$$

Notation: $f' \equiv \frac{\partial f}{\partial \eta}$

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Exercise: derive them!

Hint: impose $A_0 = 0$ after computing the EOMs to obtain Gauss' law

Equation of motion of the scale factor

Self-consistent expansion: a evolves according to **second Friedman equation**

$$a'' = \frac{a^{2\alpha-1}}{3m_{\mathrm{p}}^2}\langle(\alpha-2)K_\varphi + \alpha G_\varphi + (\alpha+1)V + (\alpha-1)(K_{\mathrm{U}(1)} + G_{\mathrm{U}(1)})\rangle$$

Sourced by the **average energy components of scalar and gauge fields**

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Sourced by the **average energy components of scalar and gauge fields**

First Friedman equation provides a constrain for **energy conservation**

$$\textbf{Hubble constraint}: \quad \left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3m_{\rm p}^2}\underbrace{\langle K_\varphi + G_\varphi + V + K_{\rm U(1)} + G_{\rm U(1)}\rangle}_{\textbf{Total energy density}}$$

Scalar and gauge energy components

Energy components for the **scalar field** and **gauge field**:

Kinetic : $K_\varphi = \frac{1}{a^{2\alpha}}(D_0\varphi)^*(D_0\varphi)$

Gradient : $G_\varphi = \frac{1}{a^2}\sum_i (D_i\varphi)^*(D_i\varphi)$

Electric : $K_{\mathrm{U}(1)} = \frac{1}{2a^{2(1+\alpha)}}\sum_i F_{0i}^2 = \frac{1}{2a^{2(1+\alpha)}}\sum_i E_i^2$

Magnetic : $G_{\mathrm{U}(1)} = \frac{1}{2a^4}\sum_{i,j<i} F_{ij}^2 = \frac{1}{2a^4}\sum_i B_i^2$

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Caution: some authors define physical electric/magnetic fields

$$E_i^{\mathrm{phys}} = a^{1+\alpha}E_i \qquad B_i^{\mathrm{phys}} = a^2 B_i$$

Equations of motion in the continuum

We want to simulate the evolution of $\boldsymbol{\varphi} + \boldsymbol{A_i} + \boldsymbol{a}$

$$\textbf{Complex scalar}: \quad (a^{3-\alpha}\varphi')' - a^{1+\alpha} D_i D_i \varphi = -\frac{a^{3+\alpha}}{2}\frac{\partial V}{\partial |\varphi|}\frac{\varphi}{|\varphi|}$$

$$\textbf{Gauge field}: \quad (a^{1-\alpha}F_{0i})' - a^{\alpha-1}\partial_j F_{ji} = a^{2\alpha} J_i$$

$$\textbf{Scale factor}: \quad a'' = \frac{a^{2\alpha-1}}{3m_{\rm p}^2}\Big\langle(\alpha-2)K_\varphi + \alpha G_\varphi + (\alpha+1)V + (\alpha-1)(K_{\rm U(1)} + G_{\rm U(1)})\Big\rangle$$

Evolution must obey **two constraints**:

$$\textbf{Gauss' law}: \quad \partial_i F_{0i} = a^2 J_0$$

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Overview

Discretizing the theory

- For scalar theories, we discretized the EOMs by substituting (spacial) derivatives with finite-volume differences (**hybrid approach**)

$$\partial_i \varphi \longrightarrow \nabla_i^+ \varphi \equiv \frac{\varphi(n+\hat{i}) - \varphi(n)}{\delta x} = \partial_i \varphi \left(n + \frac{1}{2}\hat{i} \right) + \mathcal{O}(\delta x^2)$$

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Discretizing the theory

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➤ **Can we do the same for gauge theories?** $\longrightarrow$ **NO**

Naive discretization of the EOMs breaks gauge invariance, and propagates spurious degrees of freedom

$$D_i \varphi \equiv \nabla_i^+ \varphi - igQA_i \varphi$$

$$\longrightarrow \nabla_i^+ \left[\mathrm{e}^{-igQ\lambda(n)} \varphi(n) \right] - igQ(A_i(n) - \nabla_i^+ \lambda(n)] \mathrm{e}^{-igQ\lambda(n)} \varphi \neq \mathrm{e}^{-igQ\lambda(n)} D_i \varphi(n)$$

- Leibniz rule $(fg)' = f'g + fg'$ does not hold for finite differences

Gauge-invariant discretization

We must use **links and plaquettes** to **preserve gauge invariance**

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- **Parallel transporter** connects two points in spacetime in a gauge invariant form

$$U(x,y) = \underbrace{P}_{\text{Path ordering}}\exp\left[-igQ\int_x^y \mathrm{d}x^\mu A_\mu(x)\right] \xrightarrow[\text{transform}]{\text{Gauge}} \mathrm{e}^{-ie\lambda(x)}U(x,y)\mathrm{e}^{ie\lambda(y)}$$

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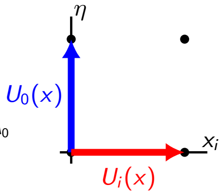
P → Path ordering

- Define spacial and temporal **link variables** between neighboring lattice sites

$$U_i(x) \equiv \exp\left[-igQ\int_x^{x+\delta x\hat{i}} \mathrm{d}x' A_i(x')\right] \approx e^{-igQ\delta x A_i}$$

$$U_0(x) \equiv \exp\left[-igQ\int_\eta^{\eta+\delta\eta\hat{0}} \mathrm{d}\eta' A_0(\eta')\right] \approx e^{-igQ\delta\eta A_0}$$



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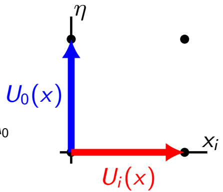


- Link variables **live at $n + \hat{\mu}/2$**
- **Notation**: $U_\mu(n) \equiv U_\mu\left(n + \frac{1}{2}\hat{\mu}\right) \quad U_{-\mu} = U_\mu^\dagger(n-\hat{\mu})$

Gauge-invariant discretization of kinetic terms

Covariant derivative includes **link variable**

$$
\begin{aligned}
D_\mu^+ \varphi\left(n+\tfrac{\hat{\mu}}{2}\right) &= \frac{1}{\delta x_\mu}\left[U_\mu(n)\varphi(n+\hat{\mu})-\varphi(n)\right] \\
D_\mu^- \varphi\left(n-\tfrac{\hat{\mu}}{2}\right) &= \frac{1}{\delta x_\mu}\left[\varphi(n)-U_\mu^\dagger(n-\hat{\mu})\varphi(n-\hat{\mu})\right]
\end{aligned}
$$

- **Expansion**: $D_\mu^\pm \varphi = D_\mu \varphi + \mathcal{O}(\delta x)$ at $n+\dfrac{\hat{\mu}}{2}$

  but $D_\mu^- D_\mu^+ \varphi = D_\mu D_\mu \varphi + \mathcal{O}(\delta x^2)$ at $n+\dfrac{\hat{\mu}}{2}$

- **Gauge transformation**: $D_\mu^\pm \varphi \longrightarrow e^{igQ\lambda} D_\mu^\pm \varphi$

Gauge-invariant discretization of field-strength tensor

Two formulations to discretize F_{ij}^2 for Abelian gauge theories

Gauge-invariant discretization of field-strength tensor

Two formulations to discretize F_{ij}^2 for Abelian gauge theories

➤ **Non-compact formulation** based on gauge fields

$$F_{ij} = \nabla_i^+ A_j - \nabla_j^+ A_i$$

Note: $F_{ij} \equiv F_{ij}\left(n + \frac{\hat{i}}{2} + \frac{\hat{j}}{2}\right)$ lives in the center of the plaquette

The **discretized EOM** reads

$$\partial_0(a^{1-\alpha}\partial_0 A_i) - a^{\alpha-1}\left[\nabla_j^- \nabla_i^+ A_j - \nabla_j^- \nabla_j^+ A_i\right] = 2a^{1+\alpha} gQ\mathrm{Im}[\varphi^*(D_i^+\varphi)]$$

Exercise: Derive EOM from the discretized action.

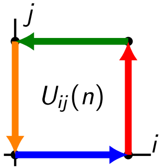
Approach used in $\mathcal{C}$osmo$\mathcal{L}$attice for U(1) theories

Gauge-invariant discretization of field-strength tensor

Two formulations to discretize F_{ij}^2 for Abelian gauge theories

- **Compact formulation** based on **link variables and plaquettes**

Plaquette : $U_{ij} = U_i(n) U_j(n+\hat{i}) U_i^\dagger(n+\hat{j}) U_j^\dagger(n)$



Note: U_{ij} lives in the center of the plaquette

The action can be rewritten following

$$\frac{1}{2} F_{ij} F_{ij} \approx \frac{1}{g^2 Q^2 \delta x^4} \mathrm{Re}[1 - U_{ij}]$$

Only possibility for non-Abelian gauge theories

Gauss' law in a finite volume

Discrete Gauss' law is preserved to machine precision

(control over discretization effects)

$$-\nabla_i^- \partial_0 A_i = 2e\mathrm{Im}[\varphi^* \partial_0 \varphi] = J_0$$

On a periodic volume (discrete or continuous), **Gauss' law imposes $\boldsymbol{Q = 0}$**

$$Q = \sum_V J_0 = -\sum_V \sum_i \nabla_i^- \partial_0 A_i = 0$$

Note: other boundary conditions allow $Q \neq 0$, but are not implemented in $\mathcal{C}$osmo$\mathcal{L}$attice (let me know if you are interested to help!)

Note II: in an expanding universe simulation, $Q \neq 0$ on each Hubble patch

Discrete equations of motion

We discretized the EOM as

$$\varphi : \quad \partial_0(a^{3-\alpha}\partial_0\varphi) - a^{1+\alpha} D_i^- D_i^+ \varphi = -\frac{a^{3+\alpha}}{2}\frac{\partial V}{\partial|\varphi|}\frac{\varphi}{|\varphi|}$$

$$A_i : \quad \partial_0^-(a^{1-\alpha}\partial_0 A_i) - a^{\alpha-1}\left(\nabla_j^-\nabla_j^+ A_i - \nabla_j^-\nabla_i^+ A_j\right) = a^{2\alpha} J_i$$

Discrete current

$$a : \quad \partial_0^2 a = \frac{a^{2\alpha-1}}{3m_{\rm p}^2}\langle(\alpha-2)K_\varphi + \alpha G_\varphi + (\alpha+1)V + (\alpha-1)(K_{\rm U(1)} + G_{\rm U(1)})\rangle$$

Discrete version of energies

Evolution must obey **two constraints**:

Gauss' law : $\quad -\nabla_i^-\partial_0 A_i = a^2 J_0$

Hubble constraint : $\quad \left(\frac{\partial_0 a}{a}\right)^2 = \frac{a^{2\alpha}}{3m_{\rm p}^2}\langle K_\varphi + G_\varphi + V + K_{\rm U(1)} + G_{\rm U(1)}\rangle$

Overview

EOM in program variables

To simulate the dynamics of gauge fields, we need to rewrite the EOM in **dimensionless program variables**

We use **two scales** in $\mathcal{C}$osmo$\mathcal{L}$attice:

$$\tilde{\eta} \equiv a^{-\alpha}\omega_\star t\,, \qquad \tilde{x}^i \equiv \omega_\star x^i\,, \qquad \tilde{\varphi} \equiv \varphi/f_\star$$

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➤ **What about gauge fields?** $\longrightarrow$ We need link variables to be dimensionless

$$U_\mu = \mathrm{e}^{-i\delta x^\mu A_\mu} = \mathrm{e}^{-i\delta\tilde{x}^\mu \tilde{A}_\mu} \longrightarrow \begin{aligned} \tilde{A}_\mu &\equiv A_\mu/\omega_\star \\ \tilde{F}_{\mu\nu} &\equiv F_{\mu\nu}/\omega_\star^2 \\ \tilde{D}_\mu &\equiv D_\mu/\omega_\star \end{aligned}$$

Dimensionless equations of motion in the continuum

We rewrite the **equations of motion** as

$$\varphi: \quad \tilde{\partial}_0(a^{3-\alpha}\tilde{\partial}_0\tilde{\varphi}) - a^{1+\alpha}\tilde{D}_i^-\tilde{D}_i^+\tilde{\varphi} = -\frac{a^{3+\alpha}}{2}\frac{\partial\tilde{V}}{\partial|\tilde{\varphi}|}\frac{\tilde{\varphi}}{|\tilde{\varphi}|}$$

$$A_i: \quad \tilde{\partial}_0(a^{1-\alpha}\tilde{\partial}_0\tilde{A}_i) - a^{\alpha-1}\left(\tilde{\nabla}_j^-\tilde{\nabla}_j^+\tilde{A}_i - \tilde{\nabla}_j^-\tilde{\nabla}_i^+\tilde{A}_j\right) = a^{2\alpha}\tilde{J}_i$$

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Introduce **conjugate momenta**:

$$\tilde{\pi}_\varphi \equiv a^{3-\alpha}\tilde{\partial}_0\tilde{\varphi} \qquad\qquad (\tilde{\pi}_A)_i \equiv a^{1-\alpha}\tilde{\partial}_0\tilde{A}_i$$

Dimensionless equations of motion

- Rewrite EOM as a system of **first order ODEs**

$$\varphi: \quad \tilde{\partial}_0(\tilde{\pi}_\varphi) = \mathcal{K}_\varphi[a, \tilde{\varphi}, \tilde{A}_j] \qquad \tilde{\partial}_0\tilde{\varphi} = a^{\alpha-3}\tilde{\pi}_\varphi$$

$$A_i: \quad \tilde{\partial}_0(\tilde{\pi}_A)_i = \mathcal{K}_{A_i}[a, \tilde{\varphi}, \tilde{A}_j] \qquad \tilde{\partial}_0\tilde{A}'_i = a^{\alpha-1}(\tilde{\pi}_A)_i$$

with **kernels**

$$\mathcal{K}_\varphi[a, \tilde{\varphi}, \tilde{A}_j] = -\frac{a^{3+\alpha}}{2}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}|\tilde{\varphi}|}\frac{\tilde{\varphi}}{|\tilde{\varphi}|} + a^{1+\alpha}\tilde{D}_i^-\tilde{D}_i^+\tilde{\varphi}$$

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In $\mathcal{C}$osmo$\mathcal{L}$attice, we use a **non-compact discetization**

- Some **simplifications**

$$\tilde{J}_i = 2e\,\mathrm{Im}\left[\tilde{\varphi}^*(\tilde{D}_i\tilde{\varphi})\right] = \frac{2e}{\delta\tilde{x}}\frac{f_\star^2}{\omega_\star^2}\mathrm{Im}\left[\tilde{\varphi}^* e^{-ie\delta\tilde{x}\tilde{A}_i}\tilde{\varphi}(n+\hat{i})\right]$$

Friedman equation in program variables

The field EOM are complemented by the **evolution of the scale factor**

$$\begin{aligned}\tilde{\partial}_0 a = &\quad b \qquad \text{(conjugate momentum)}\\ \tilde{\partial}_0 b = &\quad \mathcal{K}_a[a, \tilde{K}_\varphi, \tilde{G}_\varphi, \tilde{K}_{\mathrm{SU(1)}}, \tilde{G}_{\mathrm{SU(1)}}, \tilde{V}]\\ = &\quad \frac{a^{2\alpha+1}}{3}\frac{\boldsymbol{f}_\star^{\mathbf{2}}}{m_{\mathrm{p}}^2}\left\langle(\alpha-2)\tilde{K}_\varphi + \alpha\tilde{G}_\varphi + (\alpha+1)\tilde{V} + (\alpha-1)(\tilde{K}_A + \tilde{G}_A)\right\rangle\end{aligned}$$

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$$= \quad \frac{a^{2\alpha+1}}{3}\frac{\boldsymbol{f_\star^2}}{m_{\mathrm{p}}^2}\left\langle(\alpha-2)\tilde{K}_\varphi + \alpha\tilde{G}_\varphi + (\alpha+1)\tilde{V} + (\alpha-1)(\tilde{K}_A + \tilde{G}_A)\right\rangle$$

Caution: some energies get a $\boldsymbol{\omega_\star^2/f_\star^2}$ **factor**

$$\tilde{K}_{\mathrm{SU(1)}} = \frac{1}{2a^4}\frac{\boldsymbol{\omega_\star^2}}{\boldsymbol{f_\star^2}}\sum_i(\tilde{\pi}_A)_i^2 \qquad \tilde{G}_{\mathrm{SU(1)}} = \frac{1}{2a^4}\frac{\boldsymbol{\omega_\star^2}}{\boldsymbol{f_\star^2}}\sum_{i,j<i}\tilde{F}_{ij}^2 \tag{1}$$

Be aware: other quantities also get this factor (e.g. currents, source of GWs...)

Evolving the equations of motion

Numerical evolution is analogous to scalar models!

Example: Staggered leapfrog algorithm

- **ICs**: $\left\{a, \tilde{\varphi}, \tilde{A}_i\right\}$ at $\tilde{\eta}_0$ and $\{b, \tilde{\pi}_\varphi, (\tilde{\pi}_A)_i\}$ at $\tilde{\eta}_0 - \delta\tilde{\eta}/2$
- **Evolution**:

$$
\begin{cases}
(\tilde{\pi}_\varphi)_{+1/2} = (\tilde{\pi}_\varphi)_{-1/2} + \delta\tilde{\eta}\mathcal{K}_\varphi[a, \tilde{\varphi}, \tilde{A}_i] \\
(\tilde{\pi}_A)_{i,+1/2} = (\tilde{\pi}_A)_{i,-1/2} +, \delta\tilde{\eta}\mathcal{K}_{A_i}[a, \tilde{\varphi}, \tilde{A}_j] \\
b_{+1/2} = b_{-1/2} + \delta\tilde{\eta}\mathcal{K}_a[a, \tilde{K}_\varphi, \tilde{G}_\varphi, \tilde{K}_{\mathrm{SU}(1)}, \tilde{G}_{\mathrm{SU}(1)}, \tilde{V}] \\
a_{+0} = a + \delta\tilde{\eta} b_{+1/2} \longrightarrow a_{+1/2} = (a_{+0} + a)/2 \\
\tilde{\varphi}_{+0} = \tilde{\varphi} + \delta\tilde{\eta} a_{+1/2}^{-(3-\alpha)} (\tilde{\pi}_\varphi)_{+1/2} \\
\tilde{A}_{i,+0} = \tilde{A}_i + \delta\tilde{\eta} a_{+1/2}^{-(1-\alpha)} (\tilde{\pi}_A)_{i,+1/2}
\end{cases}
$$

Kick

Drift

- Complemented by **Gauss' law** and **Hubble constraint**

Initial conditions

➤ We choose ICs for **complex scalar** analogously to scalar singlet

Fluctuations

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \longrightarrow \begin{array}{rcl} \varphi_i(\boldsymbol{x},\eta_0) & = & |\varphi_0| + \delta\varphi_i(\boldsymbol{x}) \\ \varphi_i'(\boldsymbol{x},\eta_0) & = & |\varphi_0'| + \delta\varphi_i'(\boldsymbol{x}) \end{array}$$

Homogeneous mode

Same spectrum of fluctuations as for scalar singlets

$$\delta\varphi_i(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left[|\delta\varphi_i^{(\mathrm{l})}(\boldsymbol{k})|e^{i\theta_i^{(\mathrm{l})}(\boldsymbol{k})} + |\delta\varphi_i^{(\mathrm{r})}(\boldsymbol{k})|e^{i\theta_i^{(\mathrm{r})}(\boldsymbol{k})}\right] \qquad \omega_{i,k}^2 \equiv k^2 + a^2\frac{\partial^2 V}{\partial\varphi_i^2}$$

$$\delta\varphi_i'(\boldsymbol{k}) = \frac{1}{a^{1-\alpha}}\left[\frac{i\omega_{i,k}}{\sqrt{2}}\left(|\delta\varphi_i^{(\mathrm{l})}(\boldsymbol{k})|e^{i\theta_i^{(\mathrm{l})}(\boldsymbol{k})} + |\delta\varphi_i^{(\mathrm{r})}(\boldsymbol{k})|e^{i\theta_i^{(\mathrm{r})}(\boldsymbol{k})}\right)\right] - \mathcal{H}\delta\varphi(\boldsymbol{k})$$

A priori, there are **eight random functions**

$\delta\varphi_i^{(\mathrm{l})}(\boldsymbol{k})$: Rayleigh dist. $\qquad$ $\theta_i^{(\mathrm{l})}(\boldsymbol{k})$: Random phase

Imposing Gauss law

Initial conditions must obey Gauss' law

- **Zero electric charge** in a periodic lattice

$$Q = \int_V \mathrm{d}\boldsymbol{x} J_0(\boldsymbol{x}) \propto \int \mathrm{d}\boldsymbol{k}\, \mathrm{Re}\left[\varphi_0^*(\boldsymbol{k})\varphi_1'(\boldsymbol{k}) - \varphi_0'(\boldsymbol{k})\varphi_1^*(\boldsymbol{k})\right] = 0$$

In $\mathcal{C}$osmo$\mathcal{L}$attice, we impose this with **three constraints**

$$|\delta\varphi_i^{(\mathrm{l})}(\boldsymbol{k})| = |\delta\varphi_i^{(\mathrm{r})}(\boldsymbol{k})| \qquad \theta_1^{(\mathrm{r})} = \theta_0^{(\mathrm{r})} + \theta_1^{(\mathrm{l})} - \theta_0^{(\mathrm{l})}$$

Initial conditions for gauge field

- For **gauge fields**, we set **minimal ICs**

$$\begin{aligned} A_i(\boldsymbol{x}, \eta_0) &= 0 \\ A_i'(\boldsymbol{x}, \eta_0) &= \delta A_i'(\boldsymbol{x}) \end{aligned} \longrightarrow$$

Initially, we set **longitudinal electric field** and no magnetic field

- Electric field **needed to obey Gauss' law**

$$\partial_i A_i'(\boldsymbol{x}) = J_0(\boldsymbol{x}) \xrightarrow[\textit{transform}]{\textit{Fourier}} k_i A_i'(\boldsymbol{k}) = J_0(\boldsymbol{k})$$

Initial spectrum:

$$\delta A_i'(\boldsymbol{k}) = i \frac{k_i}{k^2} J_0(\boldsymbol{k})$$

- Discretize using **lattice momentum**

$$\sum_i \nabla_i^- \partial_0 A_i(\boldsymbol{x}) = J_0(\boldsymbol{x}) \longrightarrow$$

$$\partial_0 A_i(\boldsymbol{k}) = i \frac{k_{\mathrm{L},i}^-}{(k_{\mathrm{L}}^-)^2} J_0(\boldsymbol{k})$$

$$k_{\mathrm{L},i}^- = \frac{\sin(2\pi \tilde{n}_i / N)}{\delta x} - i \frac{1 - \cos(2\pi \tilde{n}_i / N)}{\delta x}$$

Summary on U(1) gauge theories

- Theories of physical interest are gauge theories
- Lattice discretization must **preserve gauge invariance**
  - Hybrid discretization in **temporal gauge**
  - Gauge-covariant finite differences using **link variables**
  - Two options for gauge field-strength tensor:
    - **Non-compact formulation** in terms of A_i
    - **Compact formulation** in terms of **plaquettes**
- Numerical implementation similar to scalar fields
  - **Impose Gauss' invariance** on initial conditions

Any questions?

Overview

SU(2) theory in the continuum

Realistic physics models must include **non-Abelian gauge groups**

Example: Simulations of the electroweak phase transition, SU(2) monopoles...

SU(2) theory in the continuum

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Reminder of group theory:

SU(2) group: $U^\dagger = U^{-1} \qquad \det(U) = 1$

$\mathfrak{su}$**(2) algebra**:

$$M = M^\dagger \qquad \mathrm{Tr}(M) = 0$$

$$M = M^a T^a \quad T^a = \frac{\sigma^a}{2} \quad \mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta_{ab}$$

SU(2) theory in the continuum

Consider a simple model with a complex scalar and a U(1) gauge field

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[(D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] + V(|\Phi|)\right]$$

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Scalar sector:

- Scalar doublet in fundamental irrep:

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_0 + i\varphi_1 \\ \varphi_2 + i\varphi_3\end{pmatrix}$$

- Scalar potential: $V(|\Phi|)$

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SU(2) gauge sector:

- Non-Abelian gauge field:

$$B_\mu = (B_0, B_1, B_2, B_3)$$

$$B_\mu = B_\mu^a T^a \in \mathfrak{su}(2)$$

- Covariant derivative:

$$D_\mu = \partial_\mu \mathbb{1} - ig_B Q_B B_\mu$$

- Field strength:

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - i[B_\mu, B_\nu]$$

- Electric field: $E_i^{SU(2)} \equiv G_{0i}$
- Magnetic field: $B_i^{SU(2)} \equiv \frac{1}{2}\varepsilon_{ijk}G^{jk}$

SU(2) theory in the continuum

Consider a simple model with a complex scalar and a U(1) gauge field

$$S = -\int d^4x \sqrt{-g}\left[(D_\mu\Phi)^\dagger(D^\mu\Phi) + \frac{1}{2}\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}] + V(|\Phi|)\right]$$

Action is invariant under **gauge SU(2) transformations** ($\boldsymbol{\Omega(x)} \in \mathrm{SU}(2)$)

$$\Phi(x) \longrightarrow \boldsymbol{\Omega(x)}\Phi(x) \qquad \textbf{(Fundamental irrep)}$$

$$B_\mu(x) \longrightarrow \boldsymbol{\Omega(x)}B_\mu(x)\boldsymbol{\Omega^\dagger(x)} - \frac{i}{g_B Q_B}[\partial_\mu\boldsymbol{\Omega(x)}]\boldsymbol{\Omega^\dagger(x)}$$

this implies

$$G_{\mu\nu}(x) \longrightarrow \boldsymbol{\Omega(x)}G_{\mu\nu}(x)\boldsymbol{\Omega^\dagger(x)}$$

$$D_\mu\Phi(x) \longrightarrow \boldsymbol{\Omega(x)}D_\mu\Phi(x)$$

As for the U(1) theory, we work in the **temporal gauge** ($B_0 = 0$)

Equations of motion

Equations of motion in the continuum are **similar to the U(1) case**

$$\boldsymbol{\Phi}: \quad \Phi'' - a^{-2(1-\alpha)} D_i D_i \Phi + (3-\alpha)\frac{a'}{a}\Phi' = -\frac{a^{2\alpha}}{2}\frac{\partial V}{\partial |\Phi|}\frac{\Phi}{|\Phi|}$$

$$\boldsymbol{B_i}: \quad \mathcal{D}_0 G_{0i} - a^{-2(1-\alpha)} \mathcal{D}_j G_{ji} + (1-\alpha)\frac{a'}{a} G_{0i} = a^{2\alpha} J_i$$

$$\textbf{Gauss' law}: \quad \mathcal{D}_i G_{0i} = a^2 J_0$$

complemented by **Friedman equations**

Equations of motion

Equations of motion in the continuum are **similar to the U(1) case**

$$
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\textbf{Gauss' law}: &\quad \mathcal{D}_i G_{0i} = a^2 J_0
\end{aligned}
$$

complemented by **Friedman equations**

- Some **new elements**

$$
\begin{aligned}
J_i^a &= 2 g_B Q_B \mathrm{Im}[\Phi^\dagger T_a (D_0 \Phi)] \\
[\mathcal{D}_\mu G_{\mu\nu}]^a &= (\mathcal{D}_\mu)_{ab} G^b_{\mu\nu} \equiv (\delta_{ab}\partial_\mu - f_{abc} B^c_\mu) G_{\mu\nu,b}
\end{aligned}
$$

Overview

Discretizing the SU(2) gauge theory

We discretize using **links and plaquettes** (**compact formulation**)

$$\textbf{Links}: \quad U_\mu(n) = \mathrm{e}^{-ie\delta x B_\mu} \in \mathrm{SU}(2)$$

$$\textbf{Plaquette}: \quad U_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n)$$

- **Continuum limit**:

$$U_{\mu\nu} \approx \mathbb{1} - ig_B Q_B \delta x^2 G_{\mu\nu} + \frac{1}{2} g_B^2 Q_B^2 \delta x^4 G_{\mu\nu} G_{\mu\nu} + \ldots \quad \text{at } n + \frac{\hat{\mu}}{2} + \frac{\hat{\nu}}{2}$$

Exercise: prove this!

Hint: expand around the center of the plaquette

Discretizing the SU(2) gauge theory

➤ We use **links to discretize the covariant derivative**

$$D_\mu^+ \Phi \left(n + \frac{\hat{\mu}}{2} \right) = \frac{1}{\delta x_\mu} \left[U_\mu(n) \Phi(n + \hat{\mu}) - \Phi(n) \right]$$

Recall Φ and U_μ are SU(2) matrices!

➤ We use **plaquettes to discretize the EOM of the gauge field**

$$\mathcal{D}_i G_{ij} \approx \frac{1}{\delta x} \left[U_{ij}(n) - U_i^\dagger(n - \hat{i}) U_{ij}(n - \hat{i}) U_i(n - \hat{i}) \right]$$

- Used to discretize spacial derivatives in EOM
- For time derivatives, we keep gauge field
- Plaquettes are also used for **magnetic field**

Discrete equations of motion

Using this discretization, the **equations of motion** become

$$\begin{aligned}
\boldsymbol{\Phi}: &\quad \partial_0(a^{3-\alpha}\partial_0\Phi) - a^{1+\alpha} D_i^- D_i^+ \Phi = -\frac{a^{\alpha+3}}{2}\frac{\partial V}{\partial|\Phi|}\frac{\Phi}{|\Phi|} \\
\boldsymbol{U_i}: &\quad \partial_0(a^{1-\alpha}\partial_0 B_i) - a^{\alpha-1}\mathcal{D}_j^- G_{ji} = a^{1+\alpha} J_i
\end{aligned}$$

Discrete equations of motion

Using this discretization, the **equations of motion** become

$$\mathbf{\Phi}: \quad \partial_0(a^{3-\alpha}\partial_0\Phi) - a^{1+\alpha}D_i^- D_i^+\Phi = -\frac{a^{\alpha+3}}{2}\frac{\partial V}{\partial|\Phi|}\frac{\Phi}{|\Phi|}$$

$$\boldsymbol{U_i}: \quad \partial_0(a^{1-\alpha}\partial_0 B_i) - a^{\alpha-1}\mathcal{D}_j^- G_{ji} = a^{1+\alpha}J_i$$

Introduce **conjugate momenta**:

$$\pi_\Phi \equiv a^{3-\alpha}\partial_0\Phi \qquad\qquad (\pi_B)_i \equiv a^{1-\alpha}G_{0i} = a^{1-\alpha}\partial_0^+ B_i$$

Dynamical variables: $U_i \in \mathrm{SU}(2)$, $\pi_B \in \mathfrak{su}(2)$, Φ, π_Φ

- Evolve using some numerical algorithm (LF, VV...)!
- Initial conditions analogous to U(1) case (see Art)

Evolving the equations of motion

How do we evolve U_i from $(\pi_B)_i$?

Evolving the equations of motion

How do we evolve U_i from $(\pi_B)_i$?

- **Option 1**: Use an **approximation**

$$\partial_0 U_i = \partial_0 \left[\mathrm{e}^{-ig_B Q_B \delta x B_i} \right] \approx -ig_B Q_b \delta x (\partial_0 B_i) U_i + \mathcal{O}(\delta x^2)$$
$$= -i \frac{g_B Q_B \delta x}{a^{1-\alpha}} (\pi_B)_i U_i + \mathcal{O}(\delta x^2)$$

and invert the relation for $U_{i,+0}$

$$U_{i,+0} = \left[\mathbb{1} - i \frac{g_B Q_B \delta\eta \delta x}{2 a_{+1/2}^{1-\alpha}} (\pi_B)_{i,+1/2} \right]^2 U_i + \mathcal{O}(\delta\eta^2, \delta x^2)$$

- Approach used in $\mathcal{C}$osmo$\mathcal{L}$attice
- **Caution**: Introduces errors at $\mathcal{O}(\delta x)$

Evolving the equations of motion

How do we evolve U_i from $(\pi_B)_i$?

- **Option 2**: Use a time-like plaquette

$$U_{0i} = \exp\left[-ig_B Q_B \delta\eta a^{\alpha-1} (\pi_B)_i\right]$$

to determine the evolved link

$$U_{0i} = U_{i,+0} U_i^\dagger \longrightarrow U_{i,+0} = U_{0i} U_i$$

Require to exponentiate a matrix $\longrightarrow$ **Complicated for SU(N)**

Summary of non-Abelian gauge theories

- Lattice formulation of non-Abelian gauge theories requires of the **compact formulation**
  - Gauge-covariant derivatives using **links**
  - Field-strength tensor using **plaquettes**
- Numerical simulations of non-Abelian gauge theories are similar to Abelian gauge theories
  - Analogous evolution algorithms and initial conditions
  - Approximation to evolve link variables in $\mathcal{C}$osmo$\mathcal{L}$attice

Any questions?