Part II: Lattice formulation of non-perfect fluids coupled to gauge fields in a FLRW expanding background and gravitational waves)

Viscosity

$$\partial_\mu T^{\mu\nu}_{\mathrm{pf}} = f^\nu_{viscosity} + f^\nu_{Hubble} + f^\nu_{Lorentz}$$

Viscosity

The stress-energy tensor of an imperfect fluid is described by,

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{pf}} - \Pi^{\mu\nu}$$

viscous stress tensor

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Yielding the following conservation laws:

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \left\{ \begin{aligned} \partial_0 T^{00}_{\text{pf}} + \partial_j T^{0j}_{\text{pf}} &= \partial_j \Pi^{0j} \\ \partial_0 T^{i0}_{\text{pf}} + \partial_j T^{ij}_{\text{pf}} &= \partial_j \Pi^{ij} \end{aligned} \right.$$

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In the sub-relativistic limit we may recast the Navier-Stokes description of viscosity, s.t.:

- Viscous force term: $$f^i_v \equiv \partial_j \Pi^{ij} = 2\nu(1 + c_s^2)\partial_j \left[\rho\left(S^{ij} - \frac{1}{3}S^k_k \delta^{ij}\right)\right]$$
- Viscous energy dissipation: $$f^0_v \equiv \partial_j \Pi^{0j} = \Pi^{ij} S_{ij}$$

with $$S^{ij} = \frac{1}{2}(\partial^i u^j + \partial^j u^i)$$

(rate-of-strain tensor)

Viscosity

Conservation equation with viscosity:

$$\partial_0 T^{00}_{\rm pf} = -\,\partial_j T^{0j}_{\rm pf} + f^0_v$$

$$\partial_0 T^{i0}_{\rm pf} = -\,\partial_j T^{ij}_{\rm pf} + f^i_v$$

with

$$\begin{cases} f^0_{visc} = 2\nu(1+c_s^2)\rho\left(S^{ij}S_{ij} - \frac{1}{3}(\nabla\cdot\mathbf{u})^2\right) \\ f^i_{visc} = \nu(1+c_s^2)\rho\left(\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u}) + 2(\mathbf{\Pi}\cdot\nabla)\ln\rho\right) \end{cases}$$

- <u>conservation form</u>: express ρ and u^i in terms of in the viscosity terms $T^{00}_{\rm pf}$ and $T^{0i}_{\rm pf}$.
- <u>non-conservation form</u>: express $T^{00}_{\rm pf}$ and $T^{0i}_{\rm pf}$ in $T^{0i}_{\rm pf}$ terms of ρ and u^i.

by using: $$\rho = \frac{T^{00}_{\rm pf}}{(1+c_s^2)\gamma^2 - c_s^2}$$ and $$u^i = \frac{T^{0i}_{\rm pf}}{(1+c_s^2)\rho\gamma^2}$$

Viscosity: Discretisation

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- 6th order Laplacian: $\nabla^2 \mathbf{u} = \dfrac{2u_{i,-3i} - 27u_{i,-2i} + 270u_{i,-i} - 490u_i + 270u_{i,+i} - 27u_{i,+2i} + 2u_{i,+3i}}{180\delta x} \rightarrow \nabla^2 \mathbf{u}(x)\,|_{\mathbf{x}=\mathbf{n}\delta x} + \mathcal{O}(\delta x^6)$

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- For cross derivatives…

$$\nabla(\nabla\cdot\mathbf{u}) \to \nabla^0_i\nabla^0_j\mathbf{u} = \nabla^0_j\left[\frac{u_{i,+3i} - 9u_{i,+2i} + 45u_{i,+i} - 45u_{i,-i} + 9u_{i,-2i} - u_{i,-3i}}{60\delta x}\right] = \ldots$$

Viscosity: Discretisation

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- For cross derivatives we use the following Bidiagonal scheme (requires 12 instead of 36 lattice points):

$$\nabla^{\rm Bidiag}_{ij}\mathbf{u} = \frac{-2u_{j,-3j+3i} + 27u_{j,-2j+2i} - 270u_{j,-j+i} + 270u_{j,+j+i} - 27u_{j,+2j+2i} + 2u_{j,+3j+3i}}{720\delta x^2} + \frac{2u_{j,-3j-3i} - 27u_{j,-2i-2j} + 270u_{j,-j-i} - 270u_{j,-j+i} + 27u_{j-2j+2i} - 2u_{j,-3j+3i}}{720\delta x^2}$$

Hydrodynamics is an expanding background

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Fluid dynamics with expansion

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We can then apply a Weyl transformation $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$ and redefine $\tilde{T}_{\mu\nu} = \Omega^{-6} T_{\mu\nu}$, s.t.

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(perfect fluid: $\tilde{T} = a^4 T = \tilde{\rho}(3c_s^2 - 1)$

→ invariance under WT for $c_s^2 = 1/3$)

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For $\Omega = a^{-1}$ and $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ the transformed conservation equations become:

$$\partial_\mu \tilde{T}^{\mu 0} = \tilde{f}^0_H \qquad \text{and} \qquad \partial_\mu \tilde{T}^{\mu i} = 0$$

with

$$\tilde{f}^0_H \equiv -\tilde{T}\mathscr{H}$$

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→ when we include the expansion in CL we are solving for $\tilde{T}^{00}$ and $\tilde{T}^{i0}$

Solving conservation form with expansion

The conservation equations then obtain the following form

- $\partial_0 \tilde{T}^{00} = -\partial_j \tilde{T}^{0j} - \tilde{T}\mathscr{H}$
- $\partial_0 \tilde{T}^{0i} = -\partial_j \tilde{T}^{ij}$

with

$$\tilde{T} = \frac{(3c_s^2 - 1)}{(1 + c_s^2)\gamma^2 - c_s^2}\tilde{T}^{00}$$

(perfect fluid)

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Solve the following set of equations with explicit Runge-Kutta (Williamson) scheme:

$$\partial_\tau \tilde{T}^{00} = \tilde{\mathscr{K}}^i[\tilde{T}^{00}, \tilde{T}^{0i}, a, b; c_s^2] = \mathscr{K}^i[\tilde{T}^{00}, \tilde{T}^{0i}; c_s^2] + \tilde{f}^0_{Hubble}$$

$$\partial_\tau \tilde{T}^{0i} = \mathscr{K}^i[\tilde{T}^{00}, \tilde{T}^{0i}, r^2; c_s^2]$$

$$\partial_\tau b = \mathscr{K}_a[\langle \tilde{T}^{00} \rangle, a] \longrightarrow \mathscr{K}_a[\langle \tilde{T}^{00} \rangle, a] = \frac{1}{6m_p^2 a}(1 - 3c_s^2)\langle \frac{\tilde{T}^{00}}{(1 + c_s^2)\gamma^2 - c_s^2} \rangle$$

$$\partial_\tau a = b$$

$\langle \ldots \rangle$ represents a volume average

Non-conservation form with expansion

The redefined quantities yield the following stress-energy tensor:

$\tilde{T}^{\mu\nu} = a^6 T^{\mu\nu}$ and $\tilde{\rho} = a^4 \rho$ $\tilde{p} = a^4 p$ $\tilde{u}^\mu = u^\mu$ where $u^\mu = \gamma(1, u^i)/a$

$$\Rightarrow \boxed{\tilde{T}^{\mu\nu} = (\tilde{p} + \tilde{\rho}) u^\mu u^\nu + \tilde{p} \eta^{\mu\nu}}$$

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$$\boxed{\tilde{T}^{\mu\nu} = (\tilde{p} + \tilde{\rho}) u^\mu u^\nu + \tilde{p} \eta^{\mu\nu}}$$

In the non-conservation form we then obtain the following conservation equations:

- $$\partial_\tau \ln \tilde{\rho} = -\frac{1 + c_s^2}{1 - c_s^2 u^2} \partial_i u^i - \frac{1 - c_s^2}{1 - c_s^2 u^2} u^i \partial_i \ln \tilde{\rho} + \frac{a'}{a} \frac{(1 + u^2)(1 - 3c_s^2)}{1 - c_s^2 u^2}$$
- $$D_\tau u^i = u^i \frac{1 - u^2}{1 - c_s^2 u^2} \left[c_s^2 \partial_i u^i + c_s^2 \frac{1 - c_s^2}{1 + c_s^2} u^j \partial_j \ln \tilde{\rho} \right] - \frac{c_s^2}{1 + c_s^2} \partial_i \ln \tilde{\rho} - \frac{a'}{a} \frac{(1 - 3c_s^2)(1 - u^2)}{1 - c_s^2 u^2}$$

Solving non-conservation form with expansion

Non-conservation form

$$\partial_\tau \ln \tilde{\rho} = -\frac{1+c_s^2}{1-c_s^2u^2}\partial_i u^i - \frac{1-c_s^2}{1-c_s^2u^2}u^i\partial_i \ln \tilde{\rho} + \frac{a'}{a}\frac{(1+u^2)(1-3c_s^2)}{1-c_s^2u^2}$$

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Solve the following set of equations with explicit Runge-Kutta (Williamson) scheme:

$$\partial_\tau \ln \tilde{\rho} = \tilde{\mathscr{G}}^0[\ln \tilde{\rho}, \tilde{u}^i, a, b; c_s^2] = \mathscr{G}^0[\ln \tilde{\rho}, \tilde{u}^i; c_s^2] + \frac{1+u^2}{(1-c_s^2u^2)\tilde{\rho}}f_H^0$$

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Magnetohydrodynamics

$$\partial_\mu T^{\mu\nu}_{\mathrm{pf}} = f^\nu_{viscosity} + f^\nu_{Hubble} + f^\nu_{Lorentz}$$

Magnetohydrodynamics

The stress-energy tensor of a perfect fluid and electromagnetism is given by

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{pf}} + T^{\mu\nu}_{\mathrm{em}} = T^{\mu\nu}_{\mathrm{pf}} + F^{\lambda}_{\mu} F_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} F^{\sigma\lambda} F_{\sigma\lambda}$$

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Conservation of energy gives:

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Magnetohydrodynamics

The stress-energy tensor of a perfect fluid and electromagnetism is given by

$$T^{\mu\nu} = T^{\mu\nu}_{\mathrm{pf}} + T^{\mu\nu}_{\mathrm{em}} = T^{\mu\nu}_{\mathrm{pf}} + F^{\lambda}_{\mu}F_{\nu\lambda} - \frac{1}{4}g_{\mu\nu}F^{\sigma\lambda}F_{\sigma\lambda}$$

Conservation of energy gives:

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\mu T^{\mu\nu}_{\mathrm{pf}} = -\,\partial_\mu T^{\mu\nu}_{\mathrm{em}}$$

- with the Lorentz force given by:

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The current density J_μ induced by the charged particles of the fluid is described by Ohms law:

$$J_0 = -\,\gamma(\rho_e + \sigma u^i E_i)$$

$$J_i = \gamma(\underbrace{\rho_e}_{\text{charge density}} u_i + \underbrace{\sigma}_{\text{conductivity}}(E_i + \varepsilon_{ijk}u^j B^k))$$

Magnetohydrodynamics: Discretisation and solving equations

The set of Magnetohydrodynamics equation of motions are the following and can be solved again by the explicit Runge-Kutta (Williamson) scheme:

$$\partial_0 T^{00}_{\mathrm{pf}} = -\,\partial_i T^{0i}_{\mathrm{pf}} + f^0_L$$

$$\partial_i T^{i0}_{\mathrm{pf}} = -\,\partial_\mu T^{ij}_{\mathrm{pf}} + f^i_L$$

+

Gauge sector (see Lecture 6)

(with Ohm's law)

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+ Gauge sector (see Lecture 6)

(with Ohm's law)

The discretisation:

- $\nabla_i \to \nabla^{(0)}_i$
- E_i and B_i live at $\mathbf{n} + \frac{\hat{i}}{2}$ and $\mathbf{n} + \frac{\hat{k}}{2} + \frac{\hat{j}}{2}$ respectively and we need to replace in f^μ_L all:

$$E_i \to E^{(2)}_i = \frac{1}{2}\left(E_i + E_{i,-i}\right) \qquad B_i \to B^{(4)}_i = \frac{1}{4}\left(B_i + B_{i,-k} + B_{i,-j} + B_{i,-j-k}\right)$$

Gravitational Waves

Gravitational Waves from fluids

We are solving for the unphysical tensor modes u_{ij} (see Lecture 8 by Nico and Jorge):

$$u''_{ij} - \nabla^2 u_{ij} + 2\mathscr{H} u'_{ij} = \frac{2}{m_p^2 a^2}\Pi^{eff}_{ij}$$

The gravitational waves are sourced by:

- Conservation form: $$\Pi^{eff}_{ij} = (1 + c_s^2 - c_s^2\gamma^{-2})\frac{\tilde{T}^{0i}\tilde{T}^{0j}}{\tilde{T}^{00}}$$
- Non-conservation form: $$\Pi^{eff}_{ij} = (1 + c_s^2)\gamma^2\tilde{\rho}u^i u^j$$

Gravitational Waves from fluids: Discretisation

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Discretising the equation of motions of u_{ij} :

$$\nabla^2 f_i \quad \longrightarrow \quad \Delta^{(6)} f_i = \frac{2f_{i,-3i} - 27f_{i,-2i} + 270f_{i,-i} - 490f_i + 270f_{i,+i} - 27f_{i,+2i} + 2f_{i,+3i}}{180\delta x^2}$$