ICCUB
Institut de Ciències del Cosmos
UNIVERSITAT DE BARCELONA
EXCELENCIA MARÍA DE MAEZTU 04/2025-03/2031
MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES
AGENCIA ESTATAL DE INVESTIGACIÓN

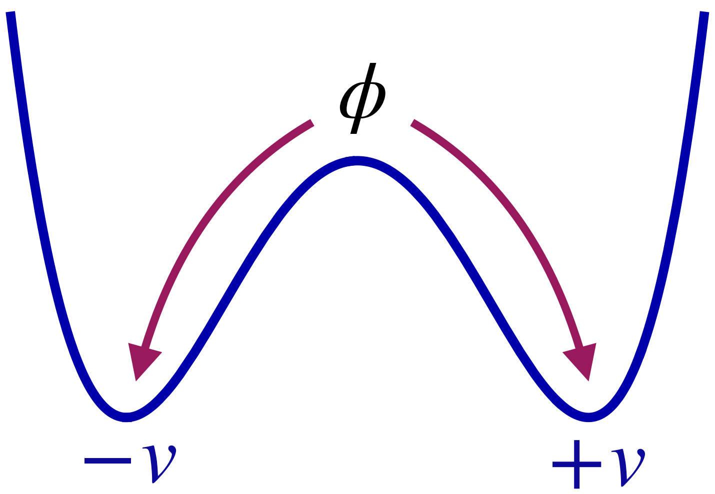
CosmoLattice school:

Lecture 11: Lattice simulations of domain walls

Francisco Torrentí
ICCUB & U. Barcelona

Domain walls

$$\boxed{V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2}$$



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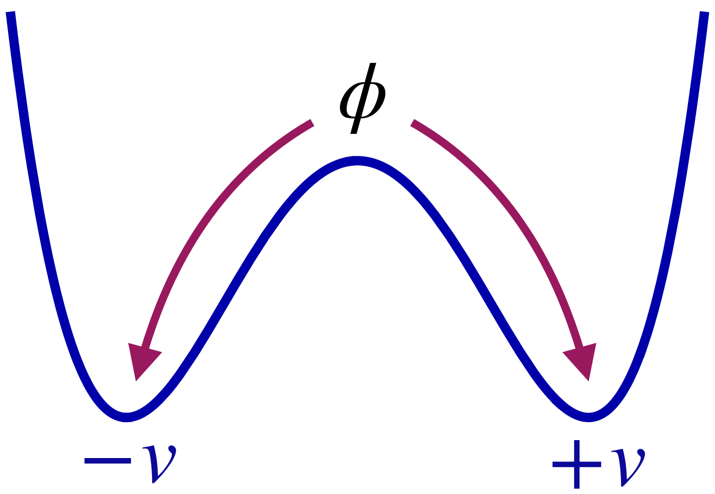

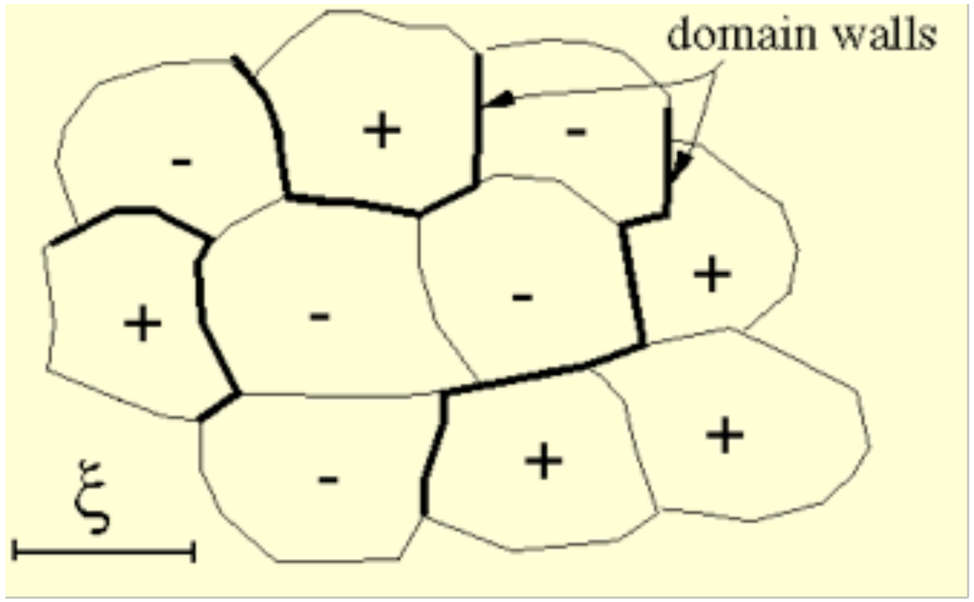

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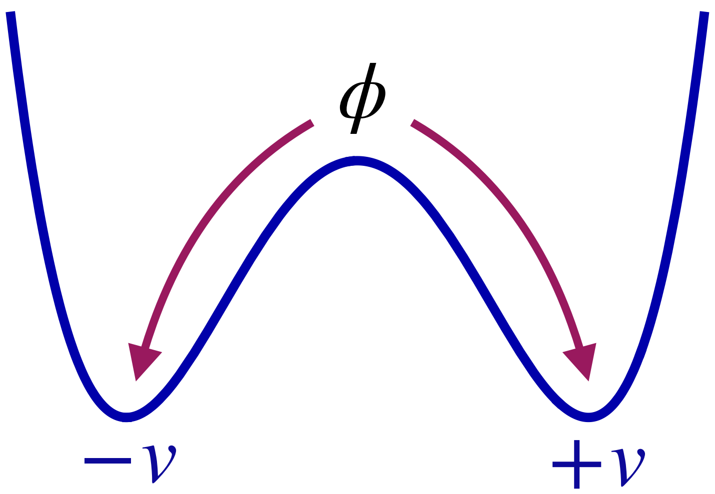

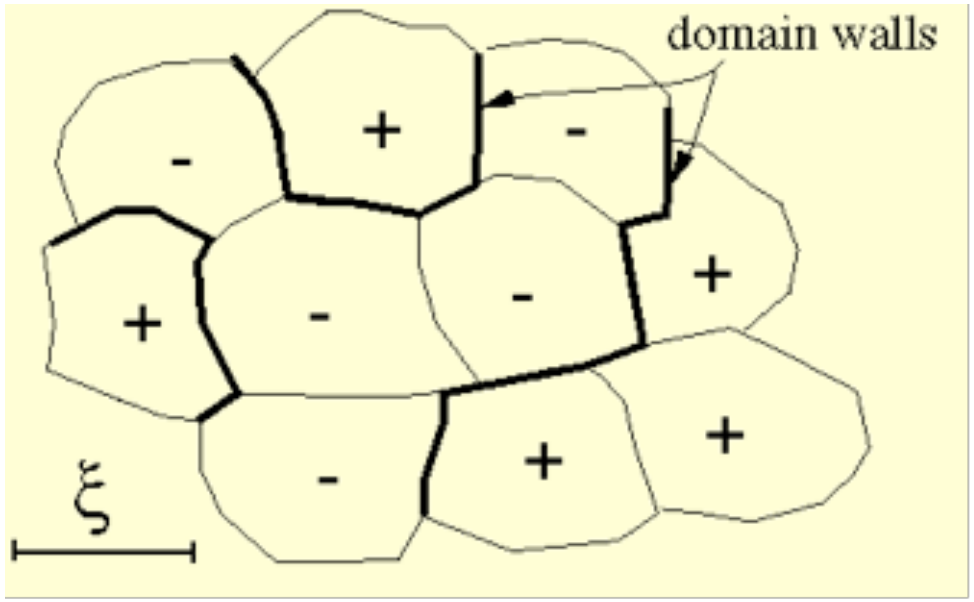

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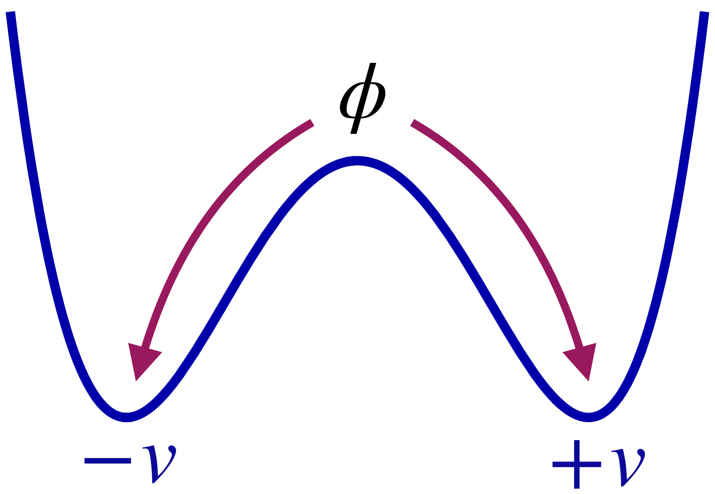
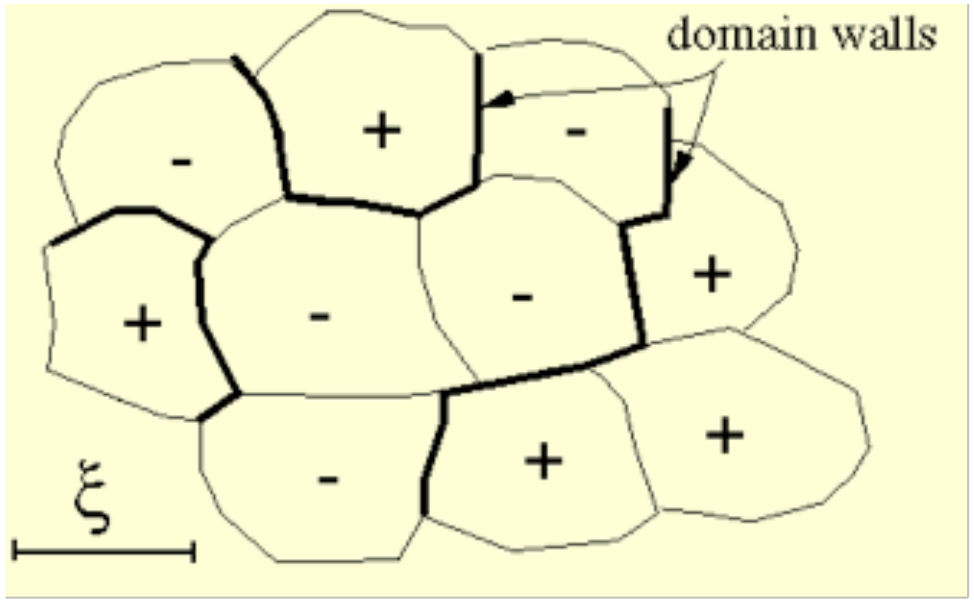
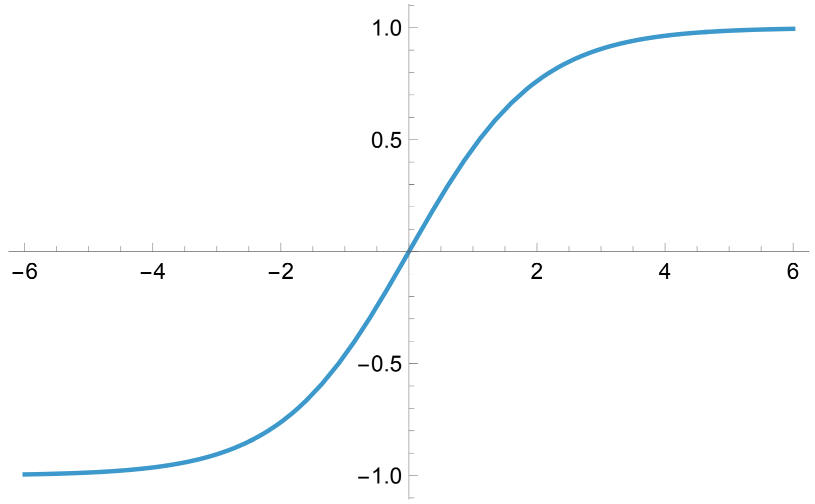




- **1-D solution:**

$$\frac{\partial^2\phi}{\partial z^2} = \frac{dV}{d\phi} \quad \rightarrow \quad \boxed{\phi(z) = v \tanh\left(\frac{m}{2}(z - z_0)\right)}$$

$$m \equiv \sqrt{2\lambda} v$$



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Energy density: $\rho_{\rm dw} = \frac{\sigma A a^2}{V a^3} = 2\mathscr{A}\sigma H$

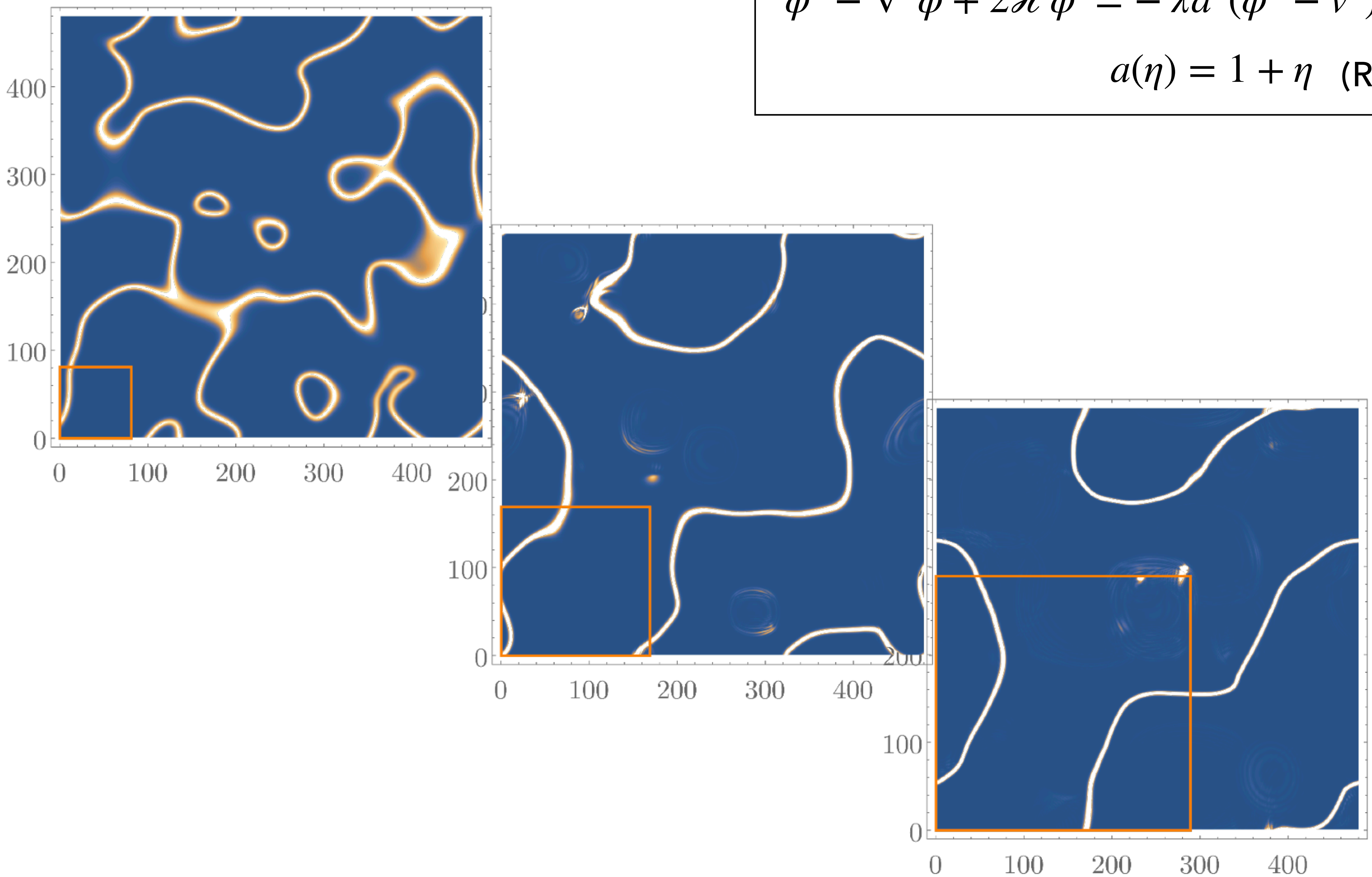
$$\boxed{\mathscr{A} \equiv \frac{A}{V}\frac{1}{2aH} \simeq \text{const}}$$

Area parameter

Domain walls: lattice simulations

$$\phi'' - \nabla^2\phi + 2\mathscr{H}\phi' = -\lambda a^2(\phi^2 - v^2)\phi$$

$$a(\eta) = 1 + \eta \quad \text{(RD)}$$



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Separation of comoving momenta scales:

$k_w \equiv \frac{2\pi a(\eta)}{\delta_w} \propto \eta; \quad k_\mathrm{h} \equiv \frac{2\pi a(\eta)}{H^{-1}(\eta)} \propto \eta^{-1}$ ➡ We require: $k_\mathrm{IR} < k_\mathrm{h} < k_\mathrm{w} < k_\mathrm{UV}$

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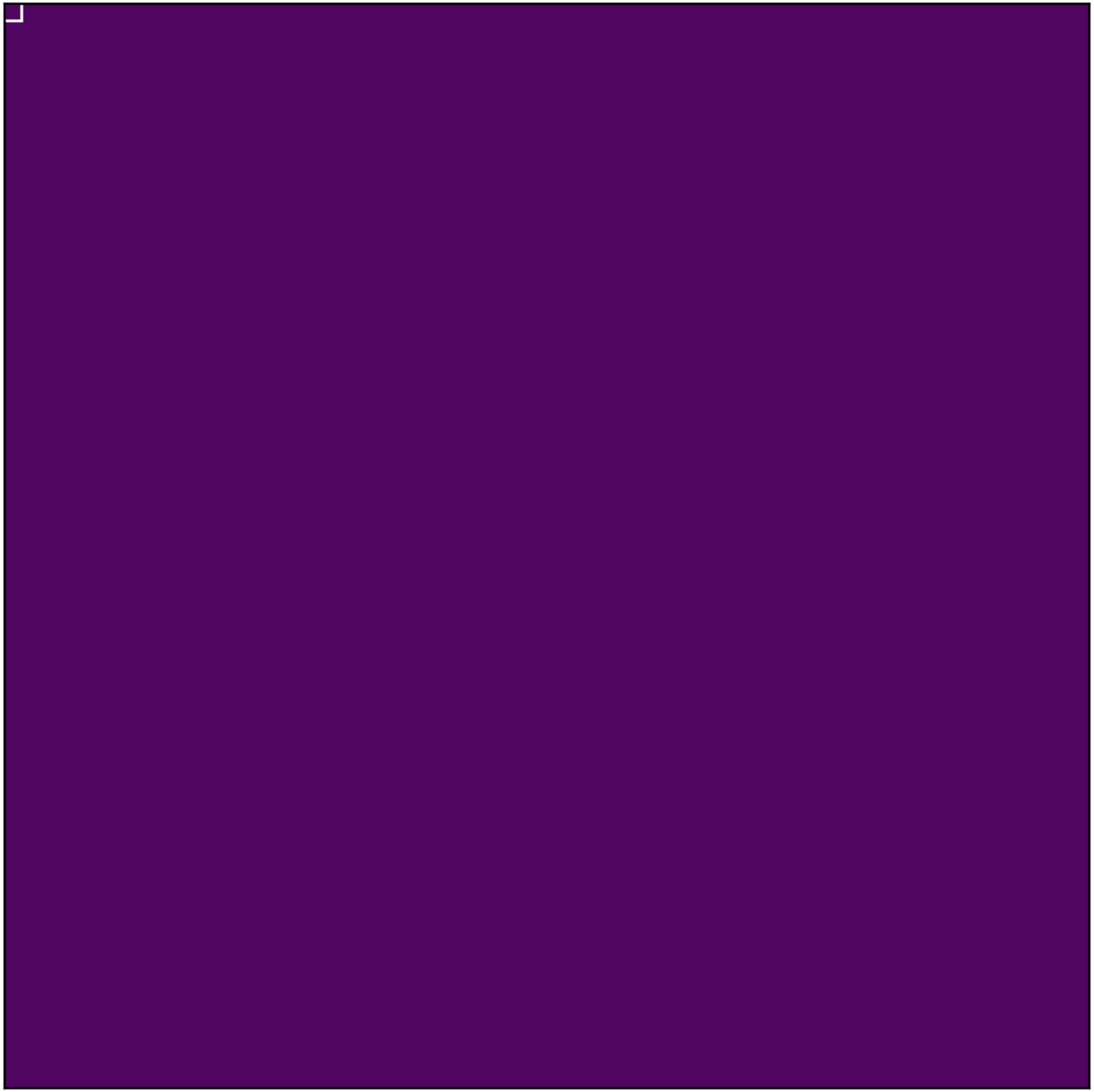
- Given a lattice of N lattice points and comoving length L, the maximum time for which DWs can be resolved are:

$$\eta < \eta_{\rm max} = \left(\frac{\sqrt{3}}{2} \frac{N}{H_i L} - 1 \right) H_i^{-1}$$

$N = 3060$

$L = 70H_i^{-1}$



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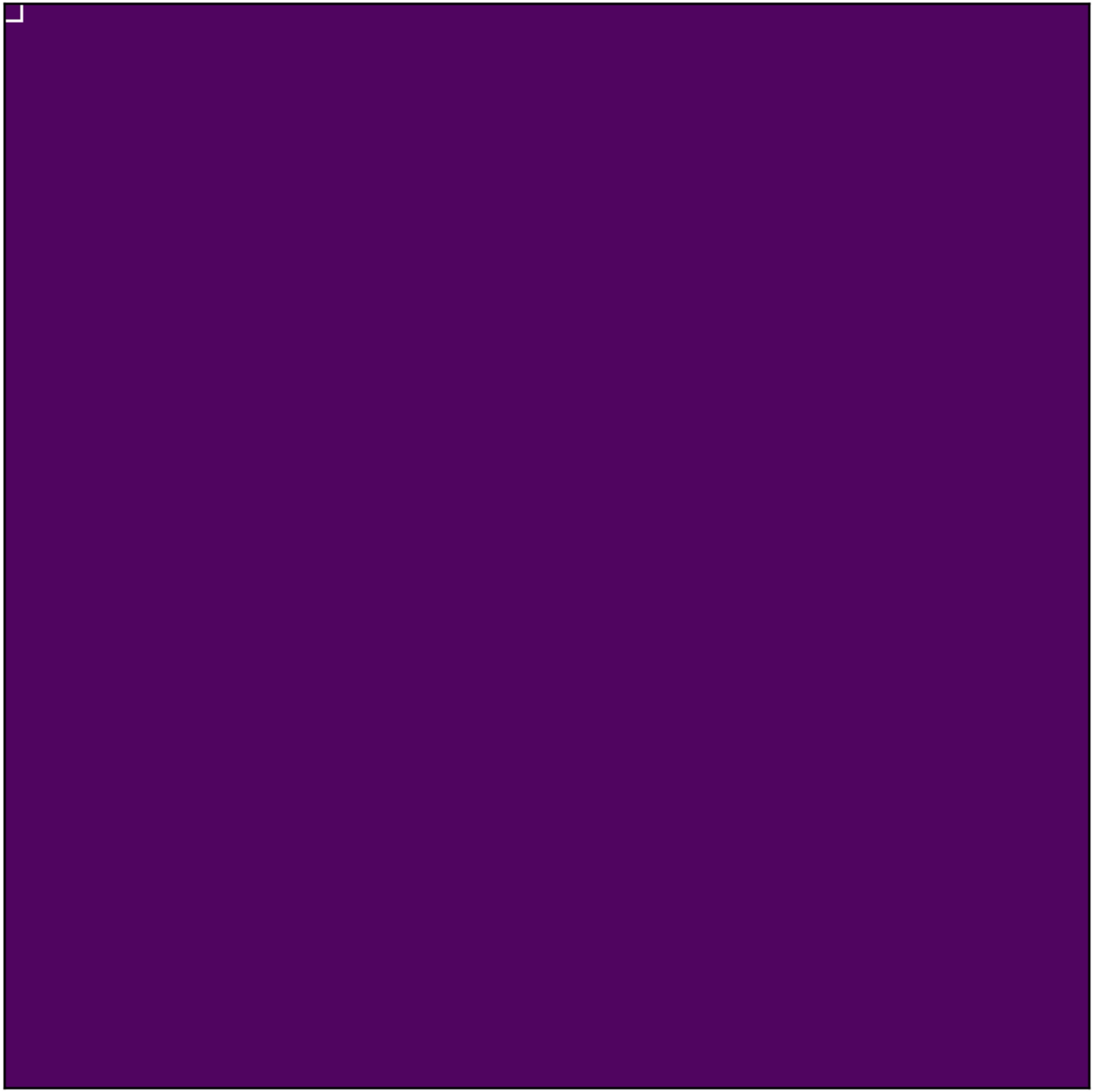

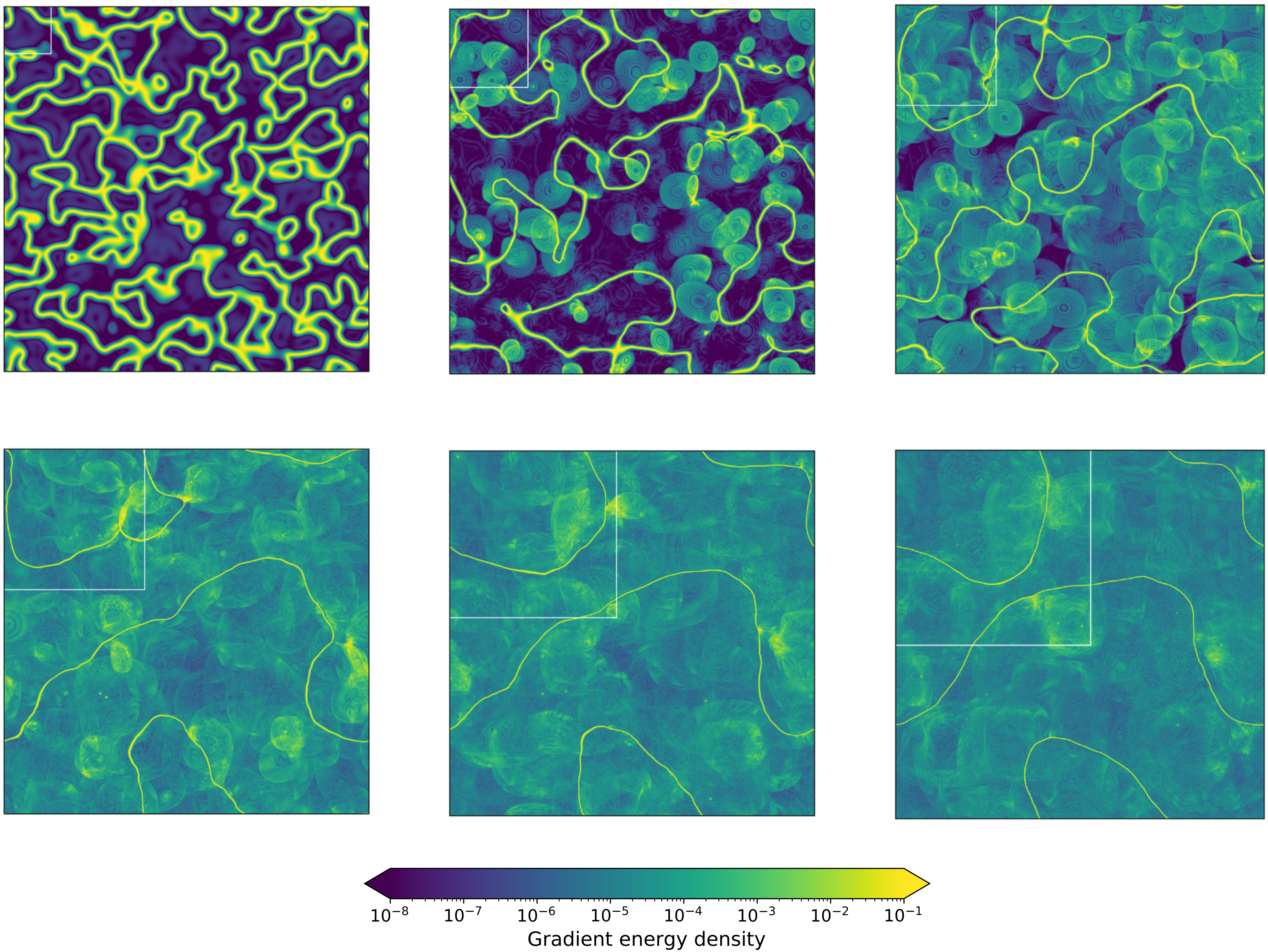

10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
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Gradient energy density

Domain wall evolution: scaling

$[H_i = m = 1]$

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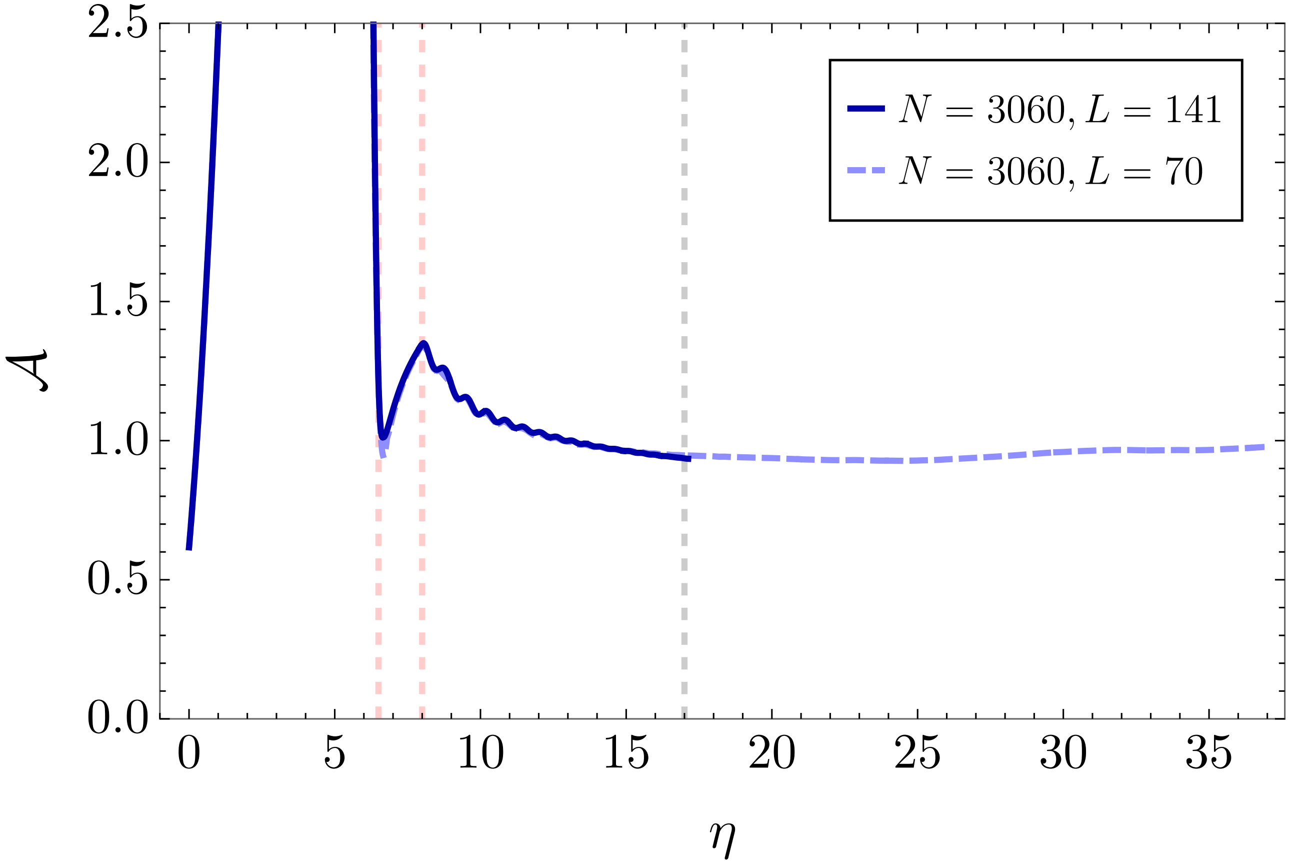
Area parameter



Notari, Rompineve & F.T (JCAP, 2025)

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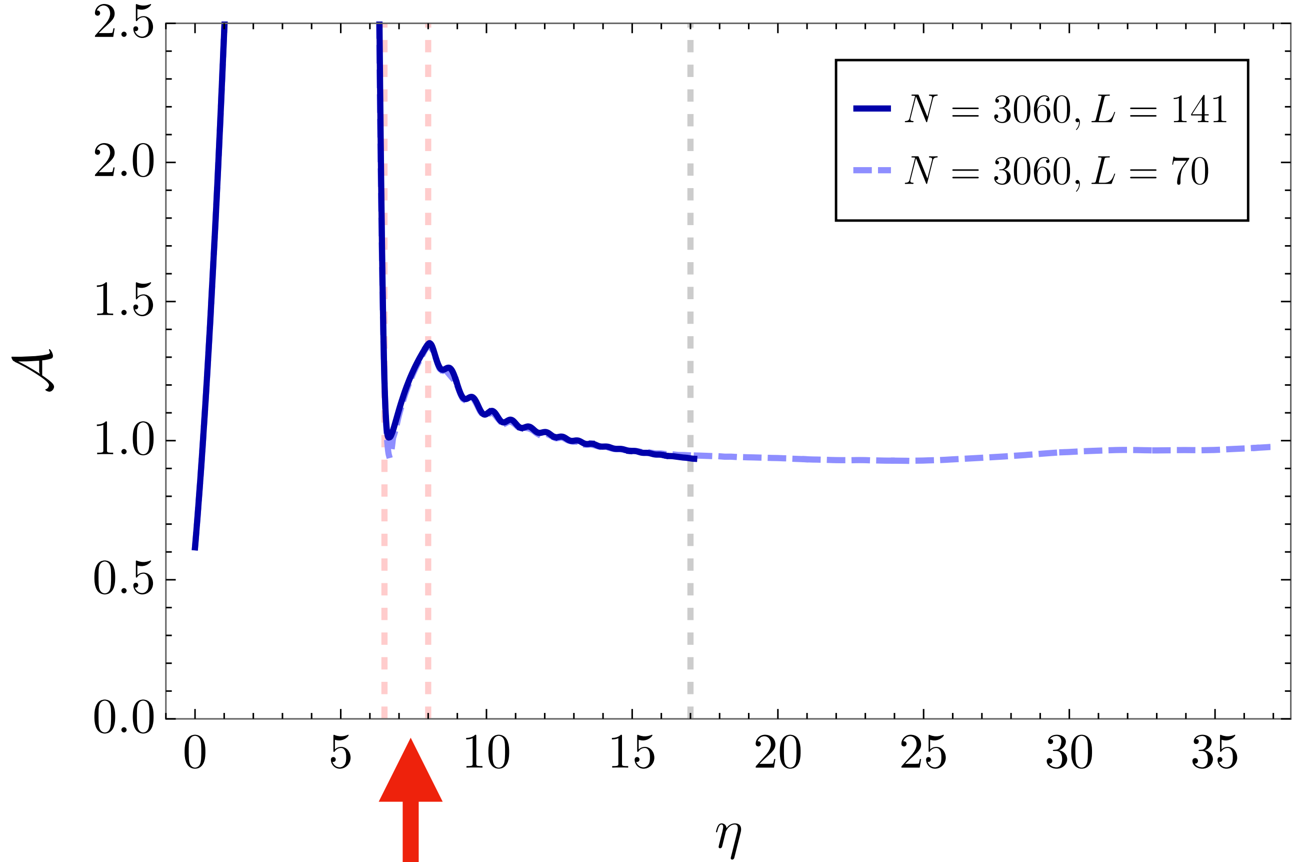

Friction

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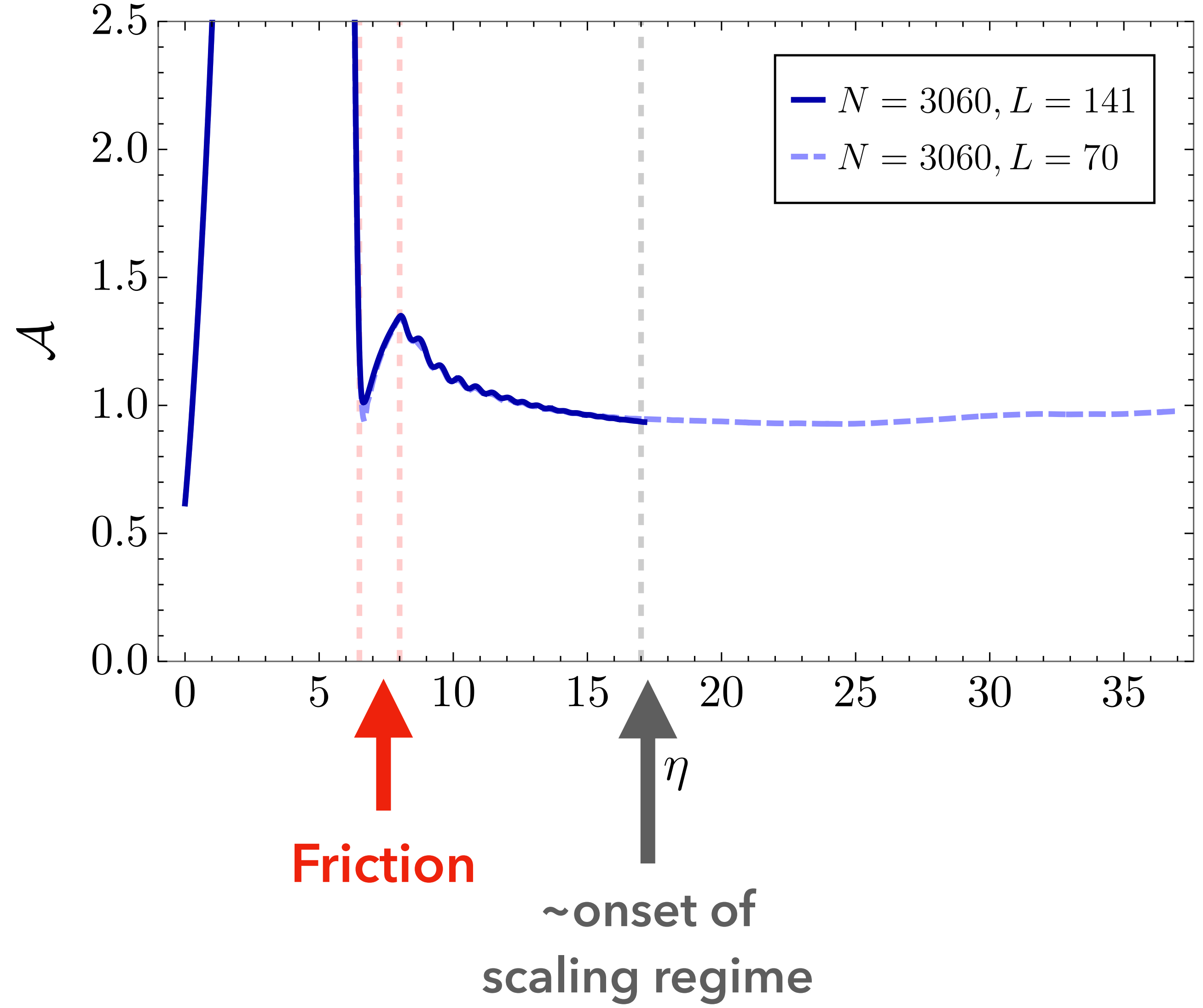

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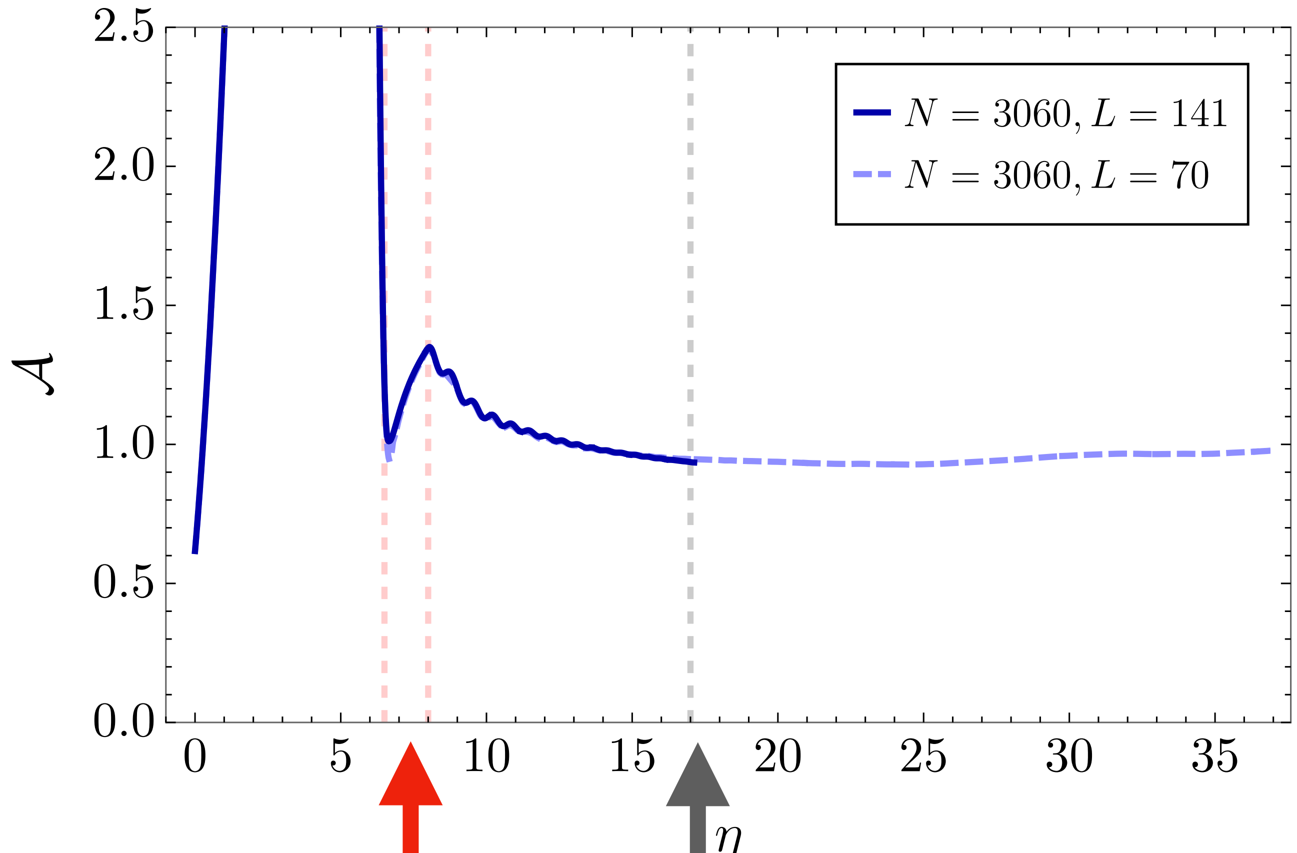
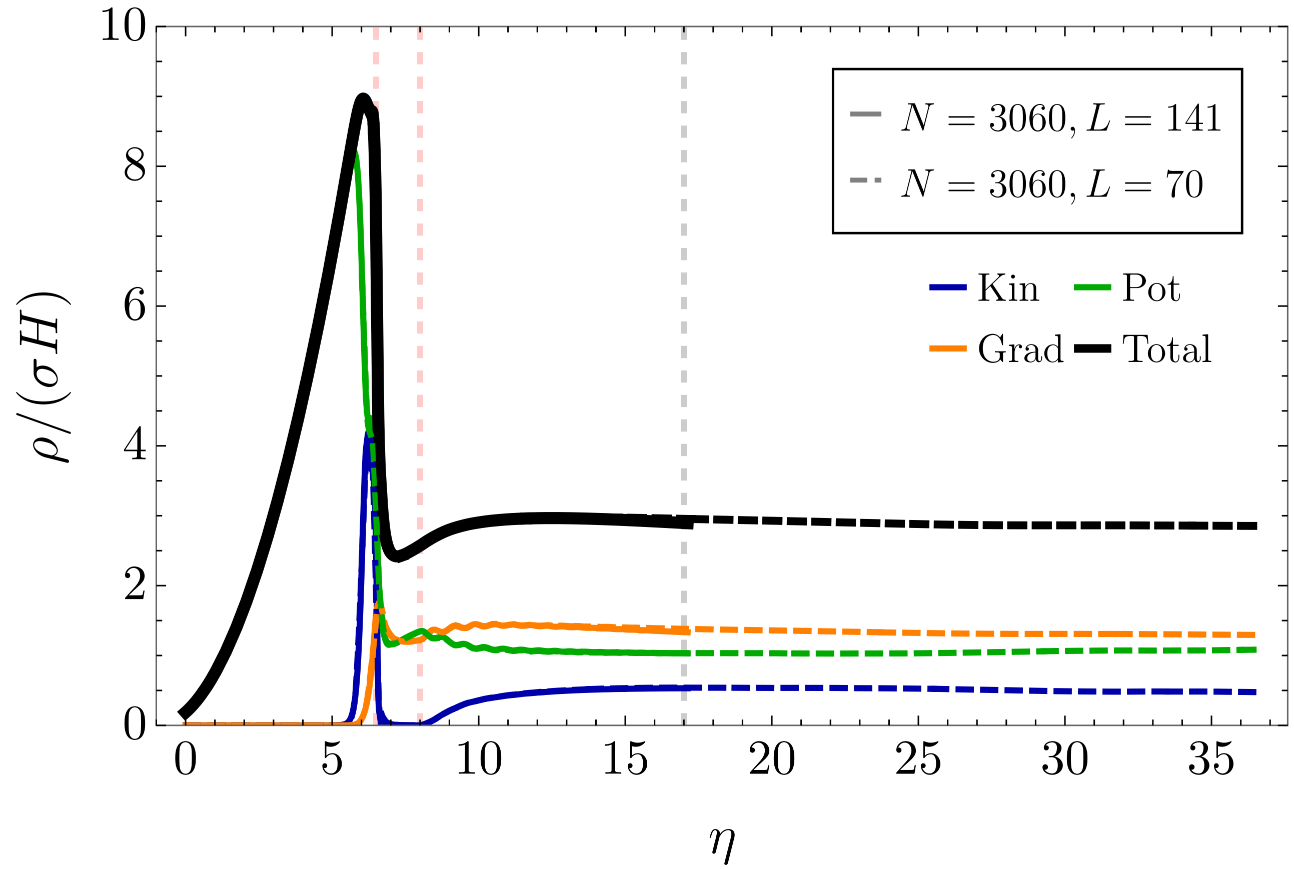


Friction

~onset of scaling regime

Energy density



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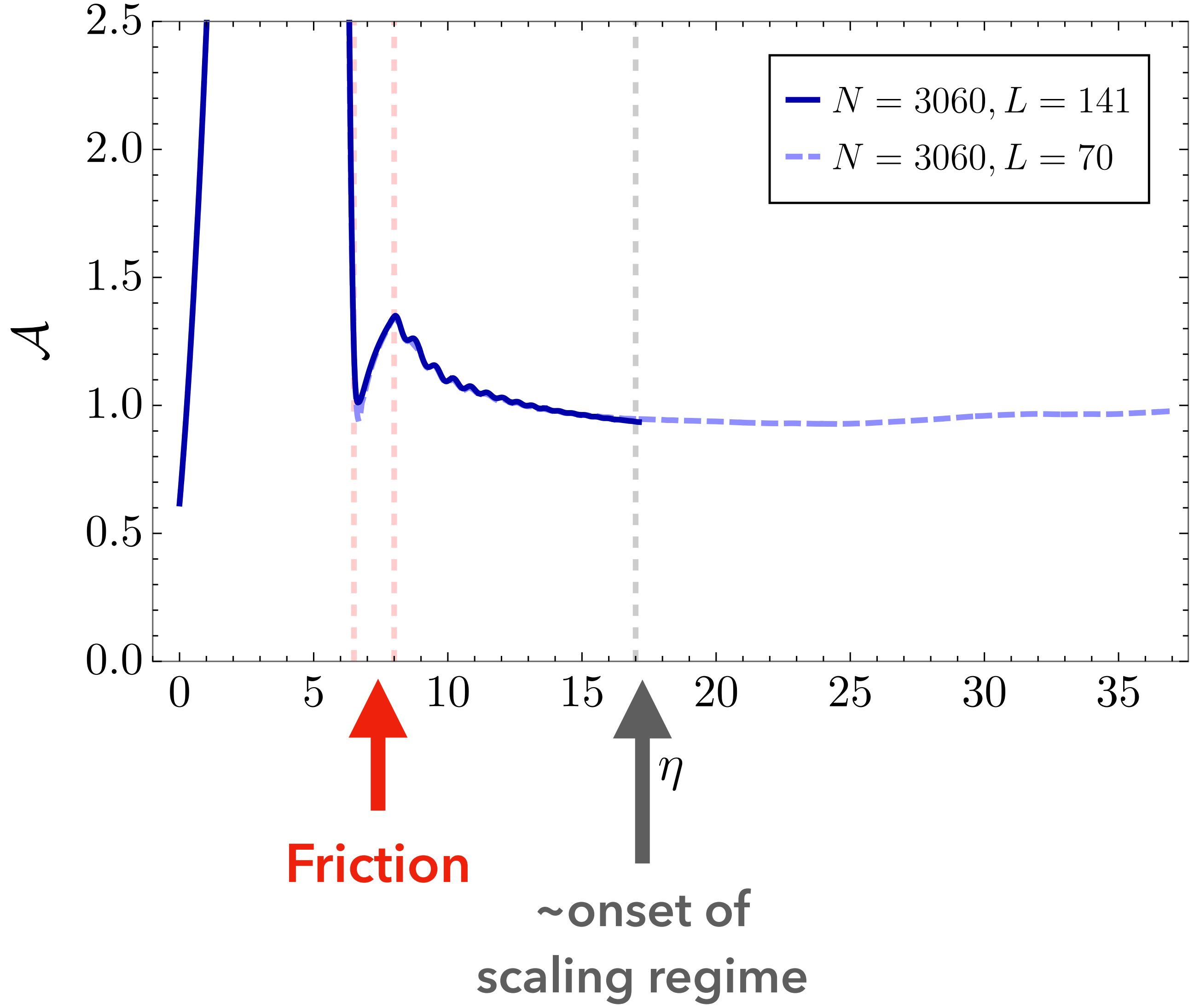

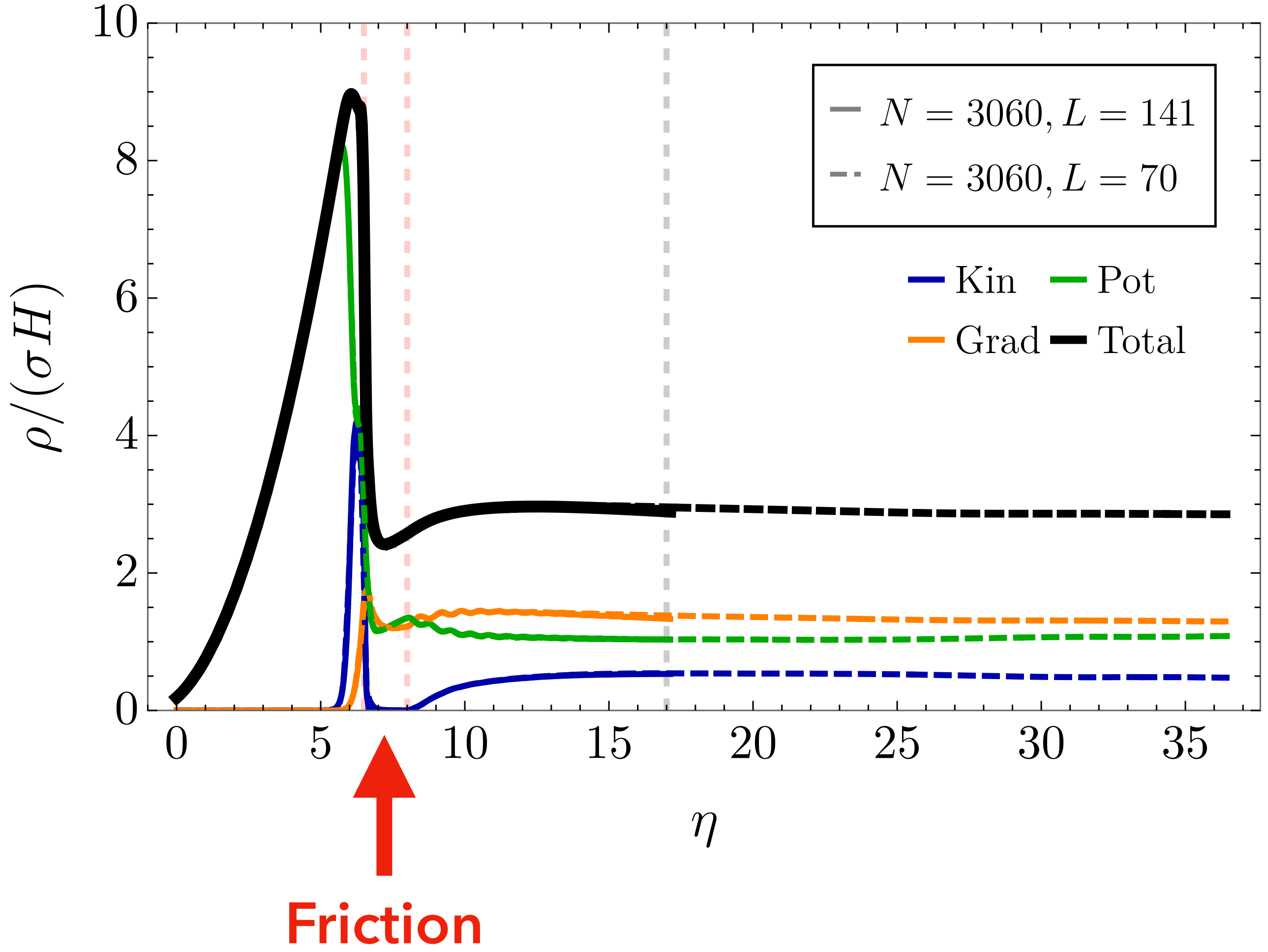

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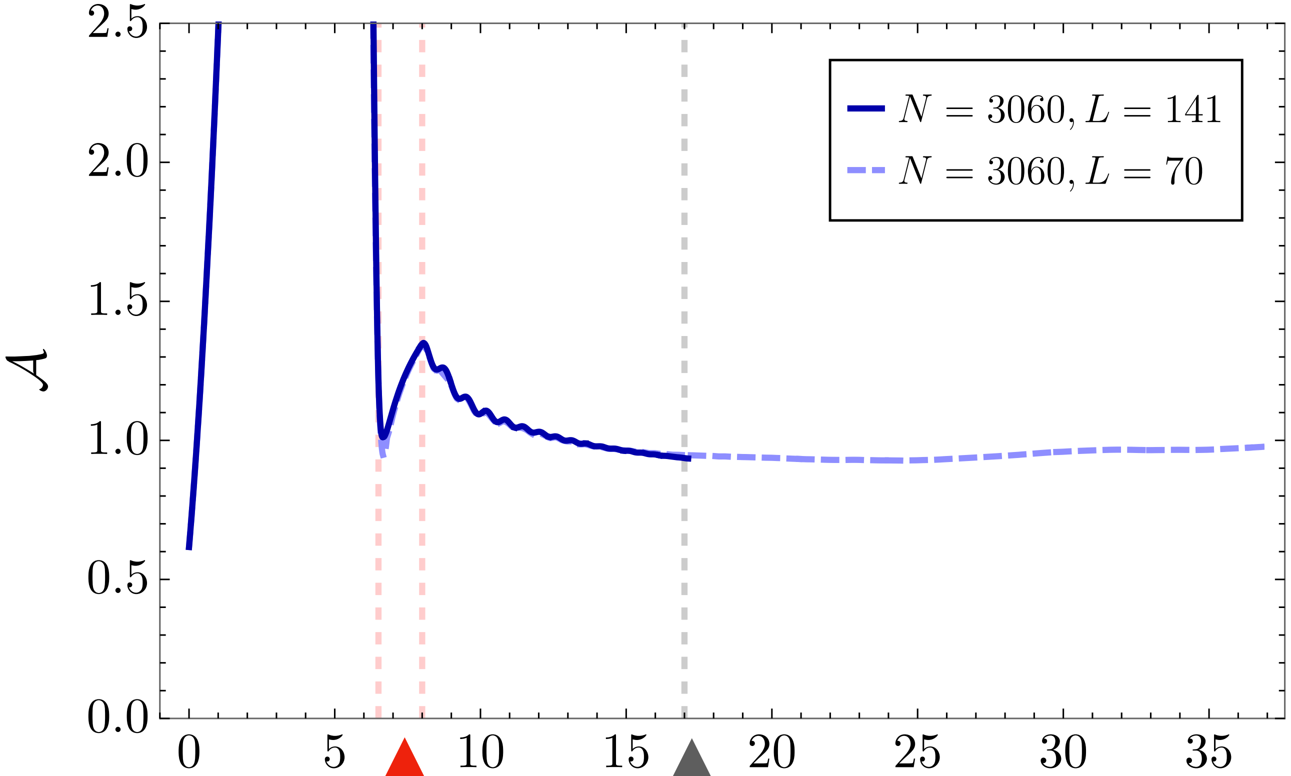

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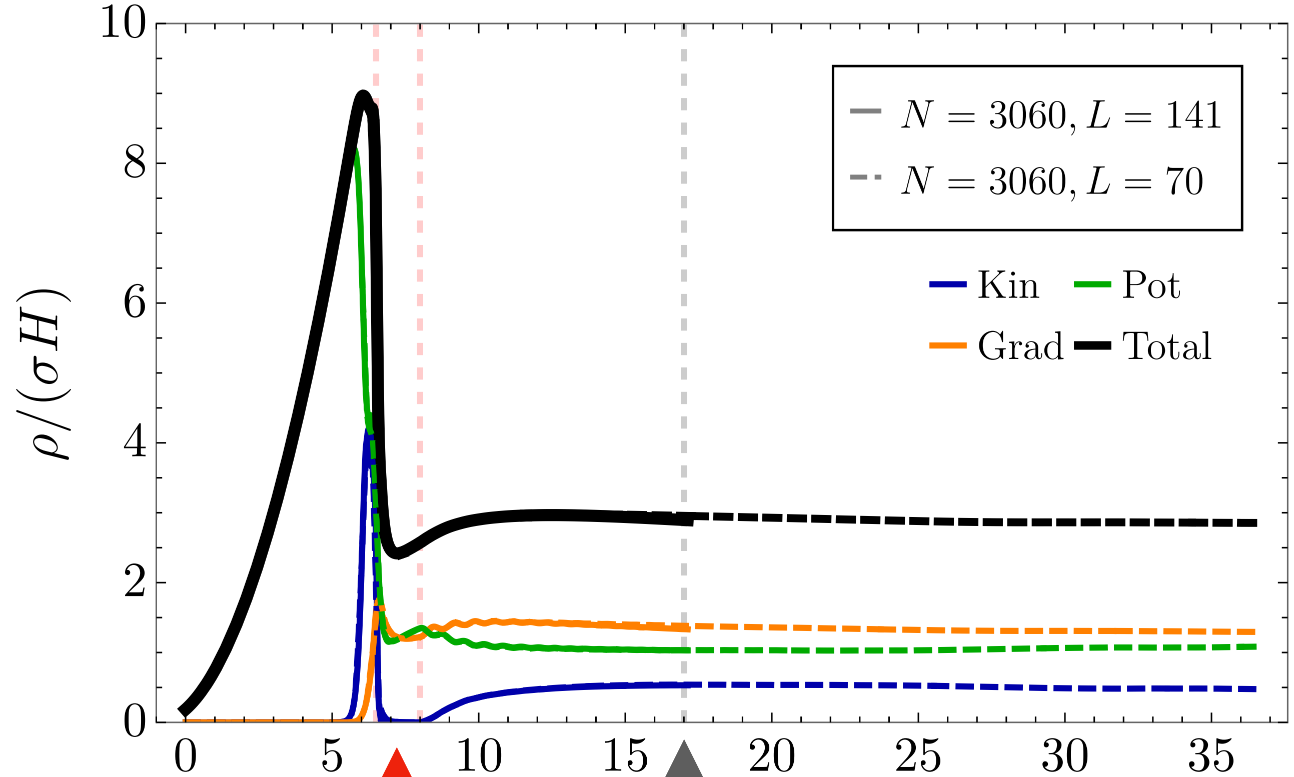

Friction

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Domain wall evolution: scaling

$$\phi'' - \nabla^2\phi + (2\mathscr{H} + \Gamma(a))\phi' = -\lambda a^2(\phi^2 - v^2)\phi$$

$$\boxed{\Gamma(a) \equiv \frac{\Gamma_0/m}{(1 + e^{\gamma(a(\eta_1) - a)})(1 + e^{\gamma(a - a(\eta_2))})}}$$

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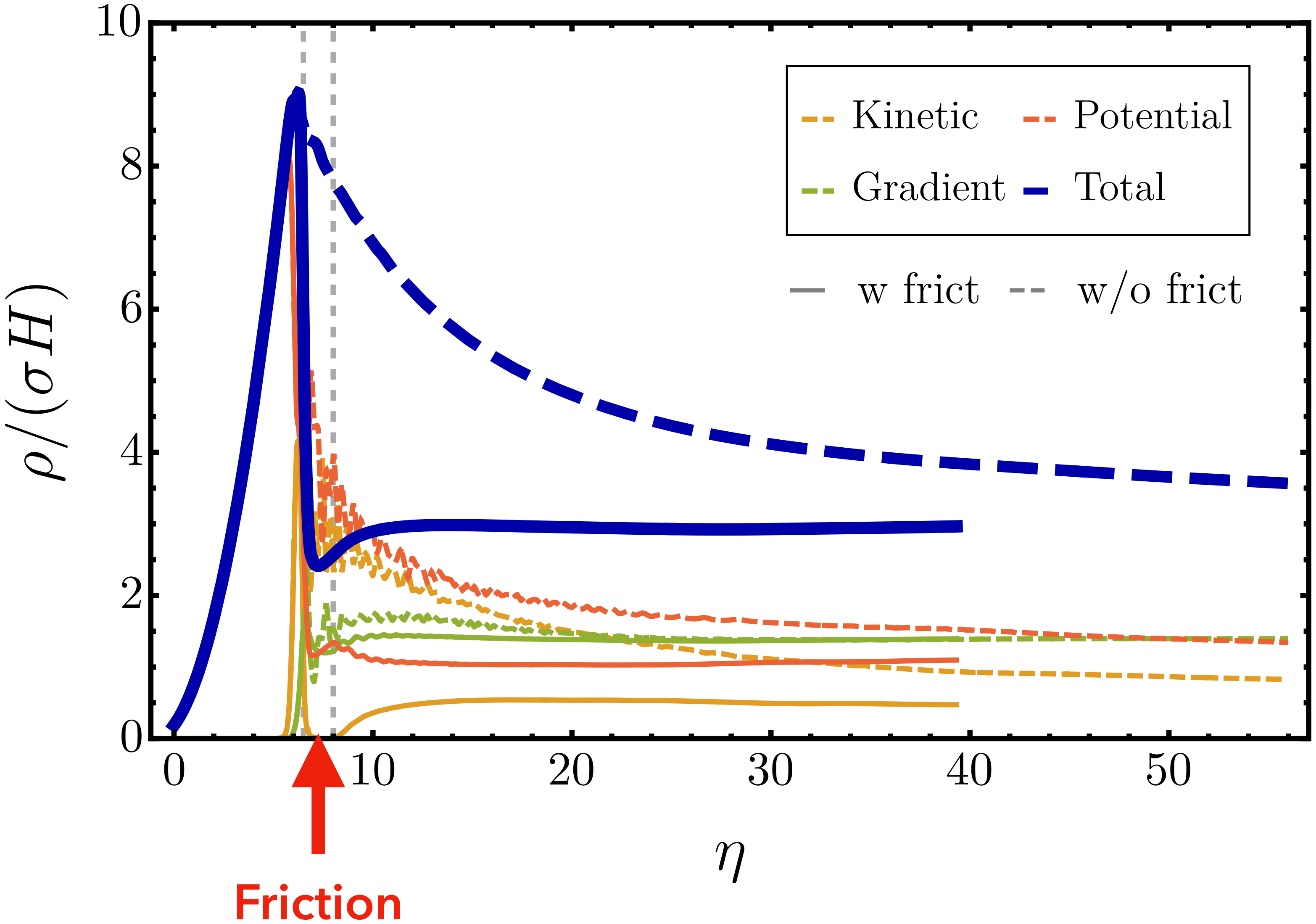

$N = 5000$

GWs from domain walls (scaling)

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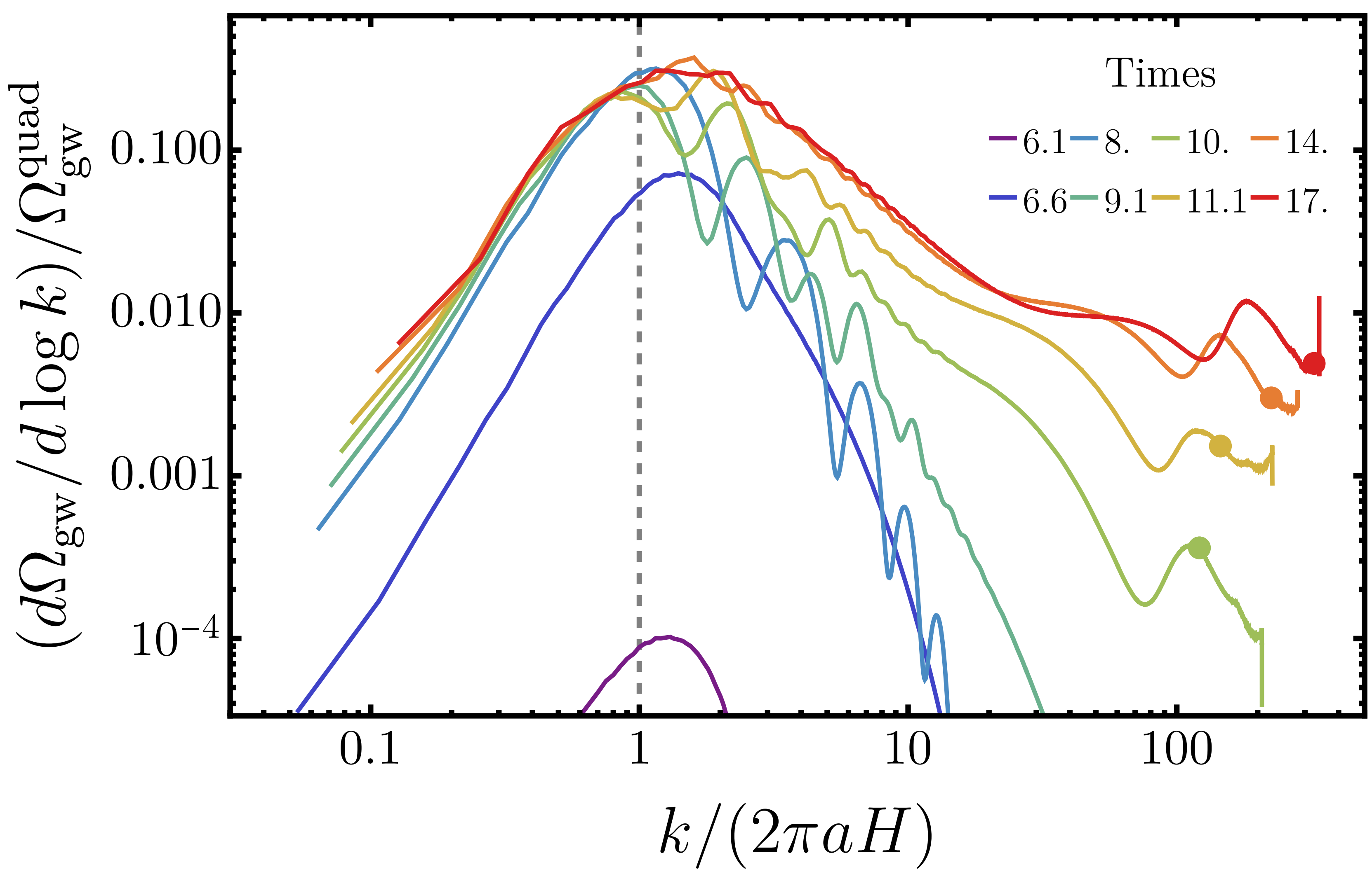

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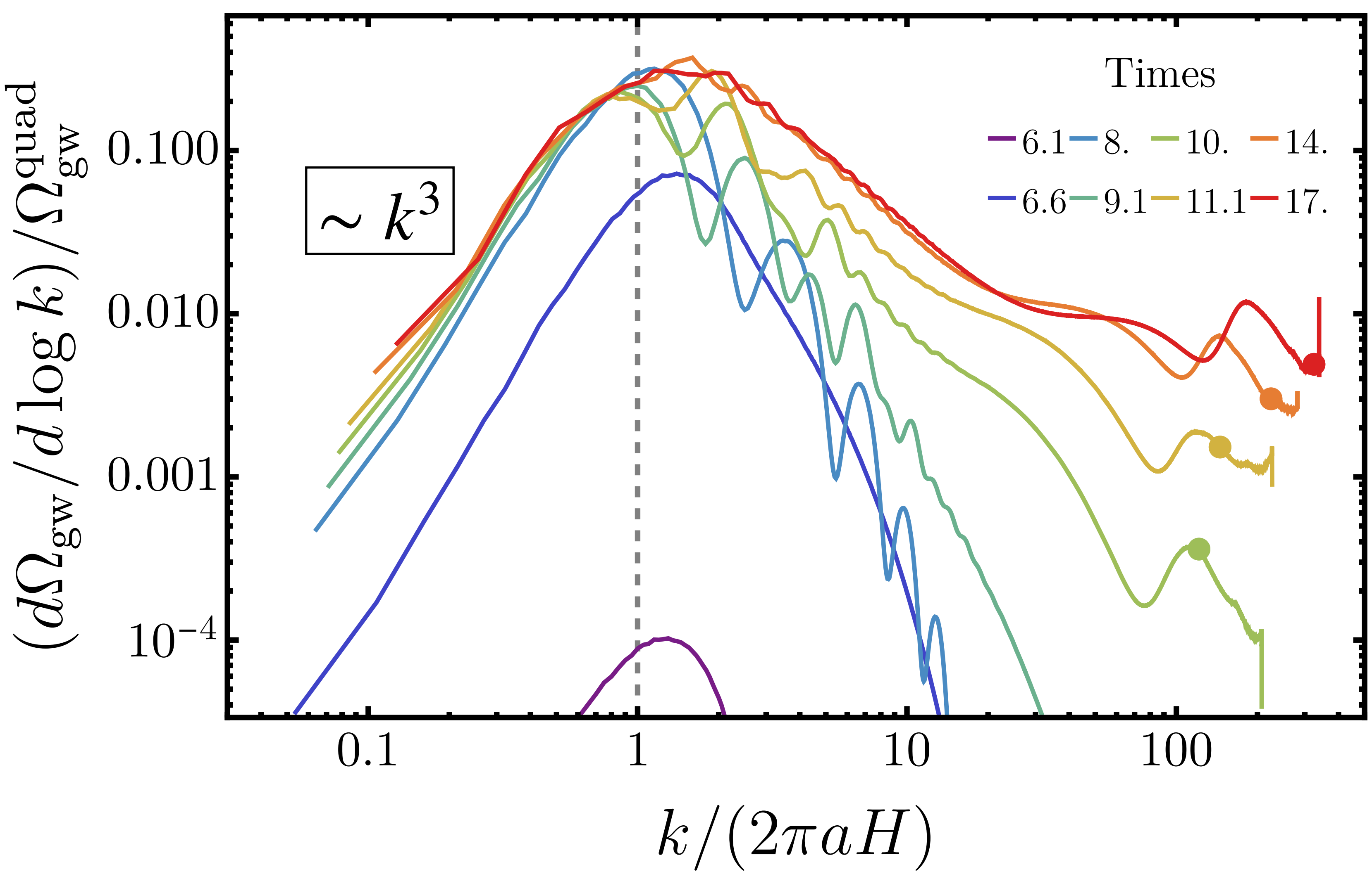

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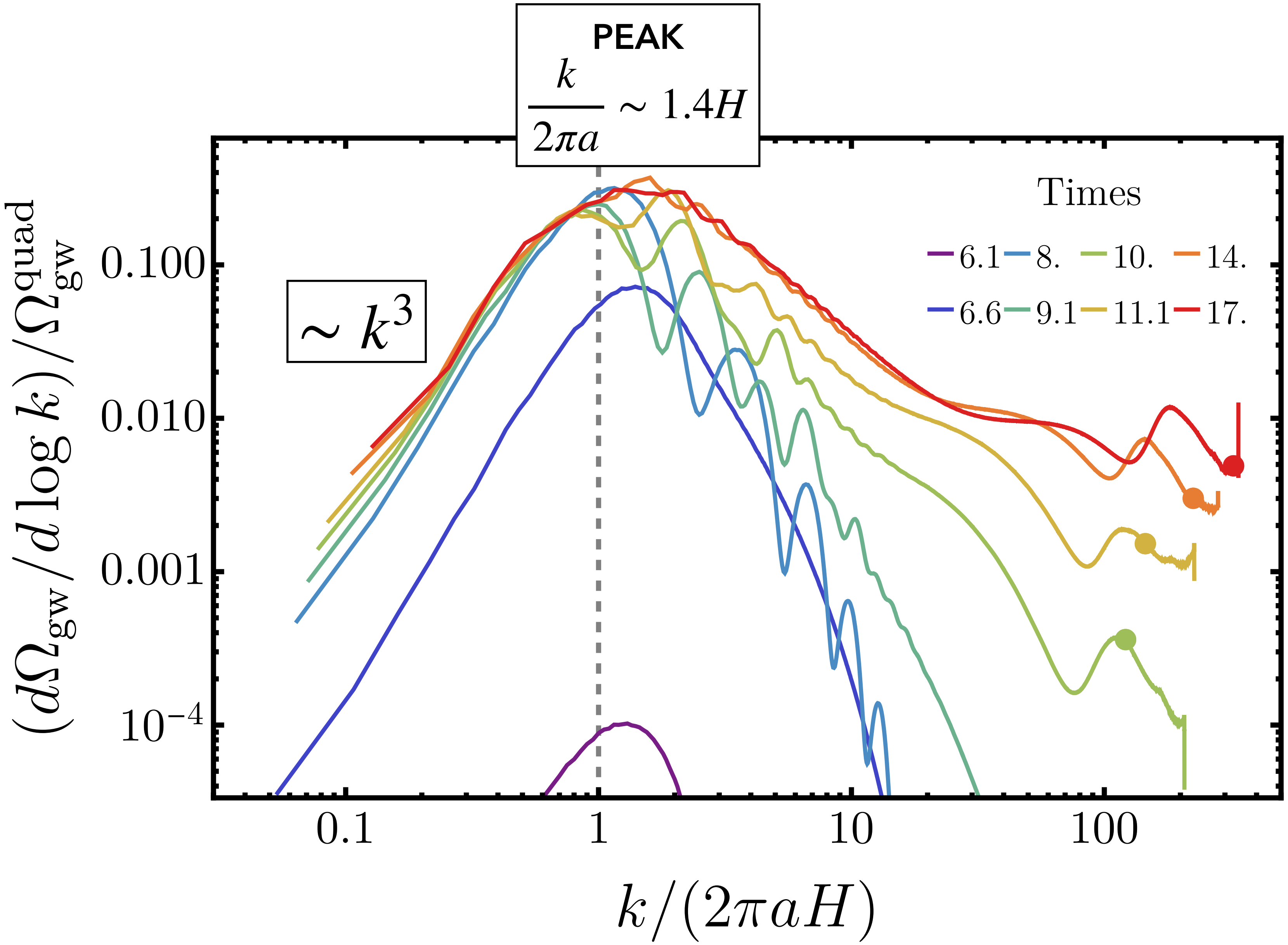

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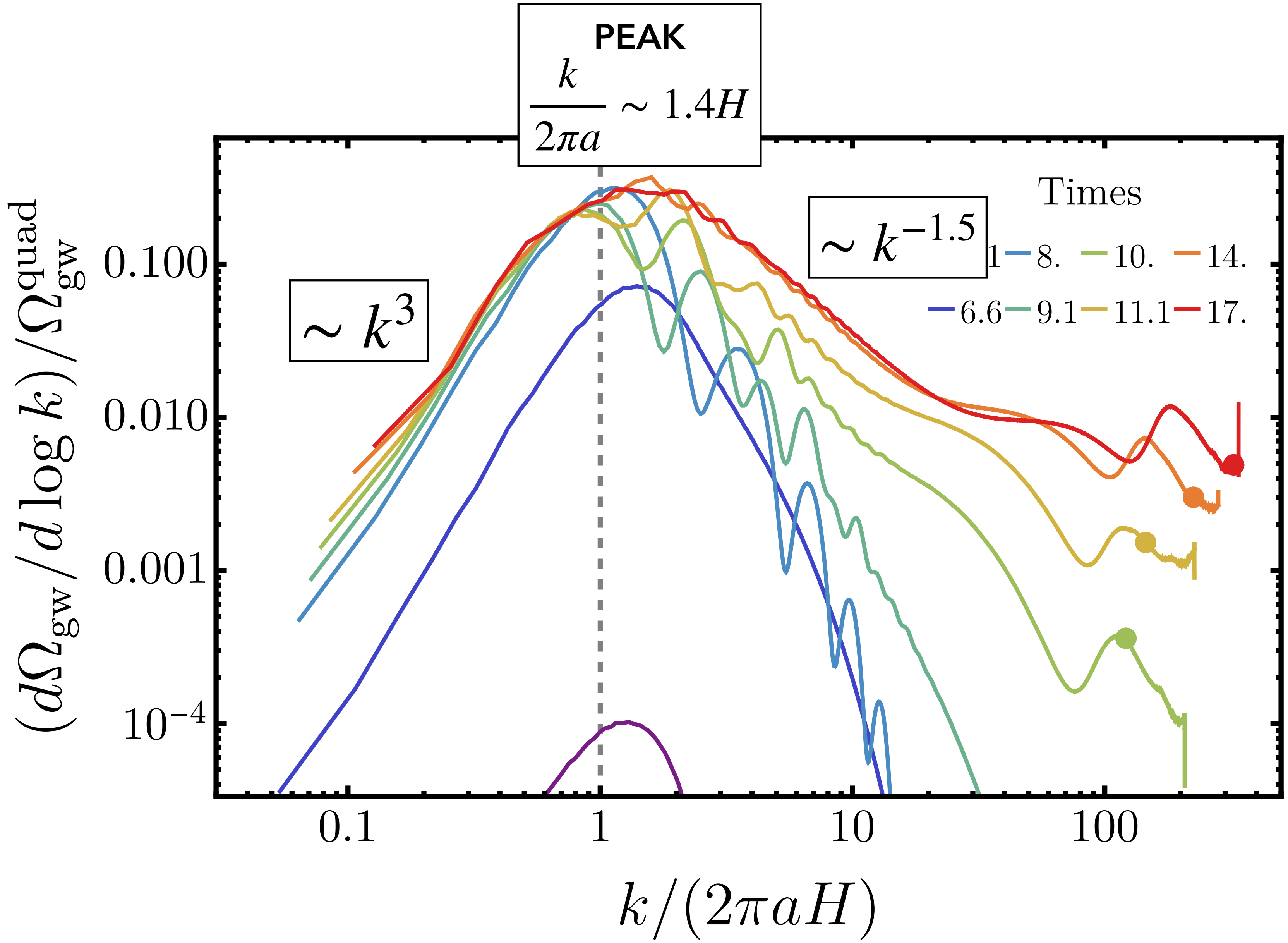

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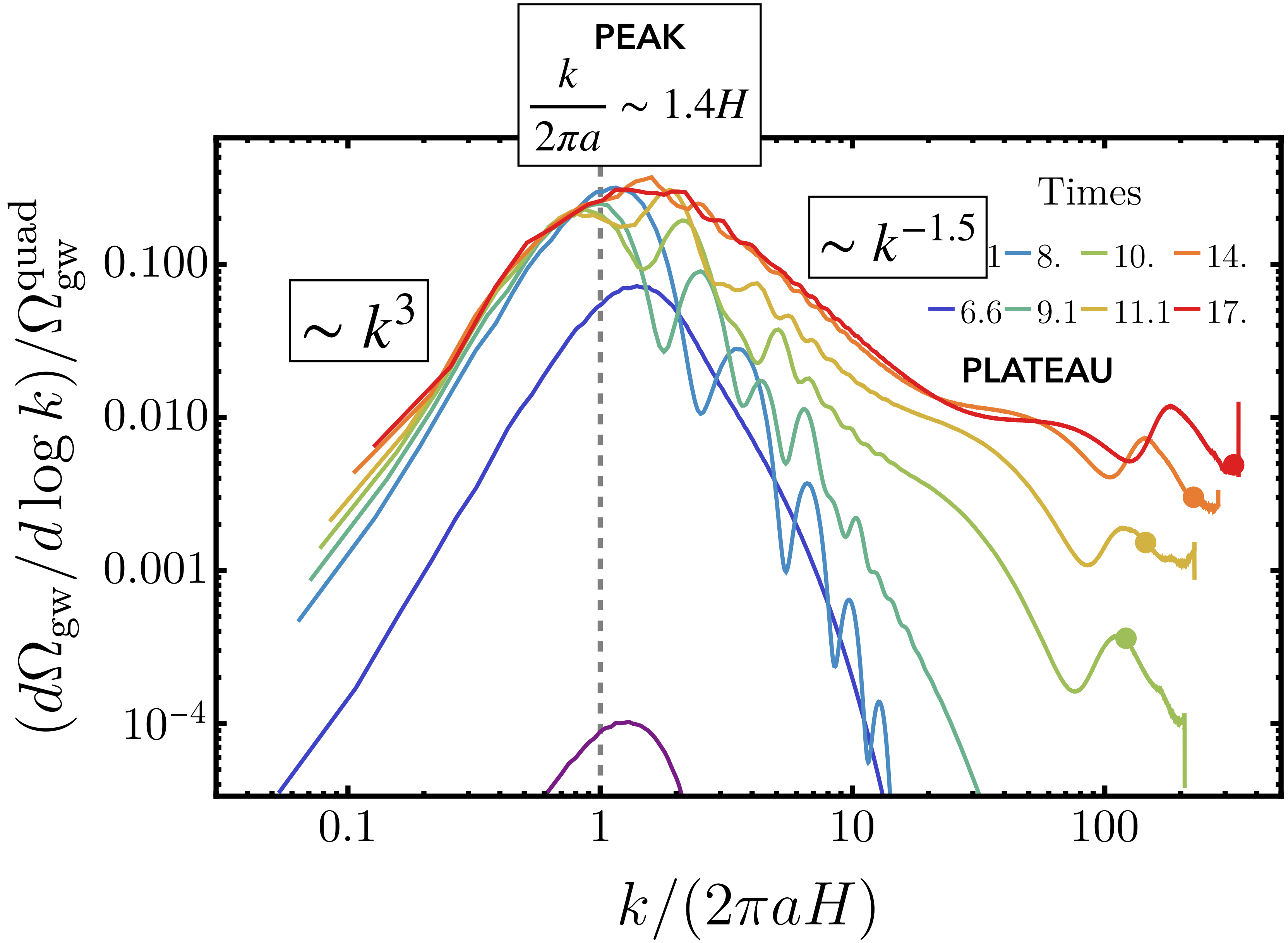

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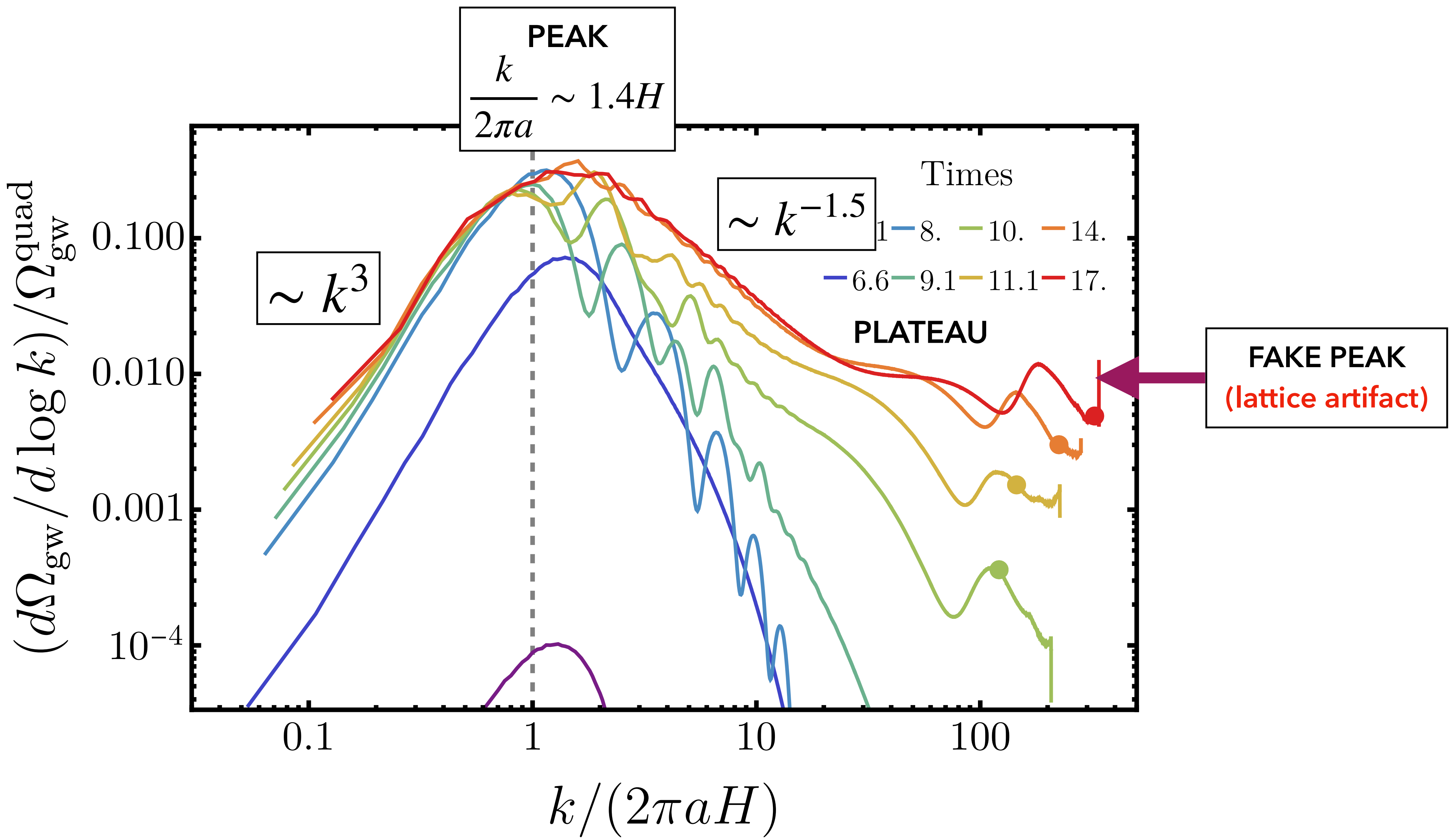

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$$[\nabla_i f]^{(2)} \equiv \frac{f_{\mathbf{n}+\hat{\imath}} - f_{\mathbf{n}-\hat{\imath}}}{2\,\Delta x} \longrightarrow \partial_i \mathtt{f}(\mathbf{x}) + \mathcal{O}(\Delta x^2)$$

$$[\nabla_i^2 f]^{(4)} \equiv \frac{f_{\mathbf{n}+\hat{\imath}} - 2f_{\mathbf{n}} + f_{\mathbf{n}-\hat{\imath}}}{2\,\Delta x^2} \longrightarrow \quad \partial_i^2 \mathtt{f}(\mathbf{x}) + \mathcal{O}(\Delta x^2)$$

ORDER 2

$$[\nabla_i f]^{(4)} \equiv \frac{f_{\mathbf{n}-2\hat{\imath}} - 8f_{\mathbf{n}-\hat{\imath}} + 8f_{\mathbf{n}+\hat{\imath}} - f_{\mathbf{n}+2\hat{\imath}}}{12\,\Delta x} \longrightarrow \partial_i \mathtt{f}(\mathbf{x}) + \mathcal{O}(\Delta x^4)$$

$$[\nabla_i^2 f]^{(4)} \equiv \frac{-f_{\mathbf{n}+2\hat{\imath}} + 16f_{\mathbf{n}+\hat{\imath}} - 30f_{\mathbf{n}} + 16f_{\mathbf{n}-\hat{\imath}} - f_{\mathbf{n}-2\hat{\imath}}}{12\,\Delta x^2} \longrightarrow \quad \partial_i^2 \mathtt{f}(\mathbf{x}) + \mathcal{O}(\Delta x^4)$$

ORDER 4

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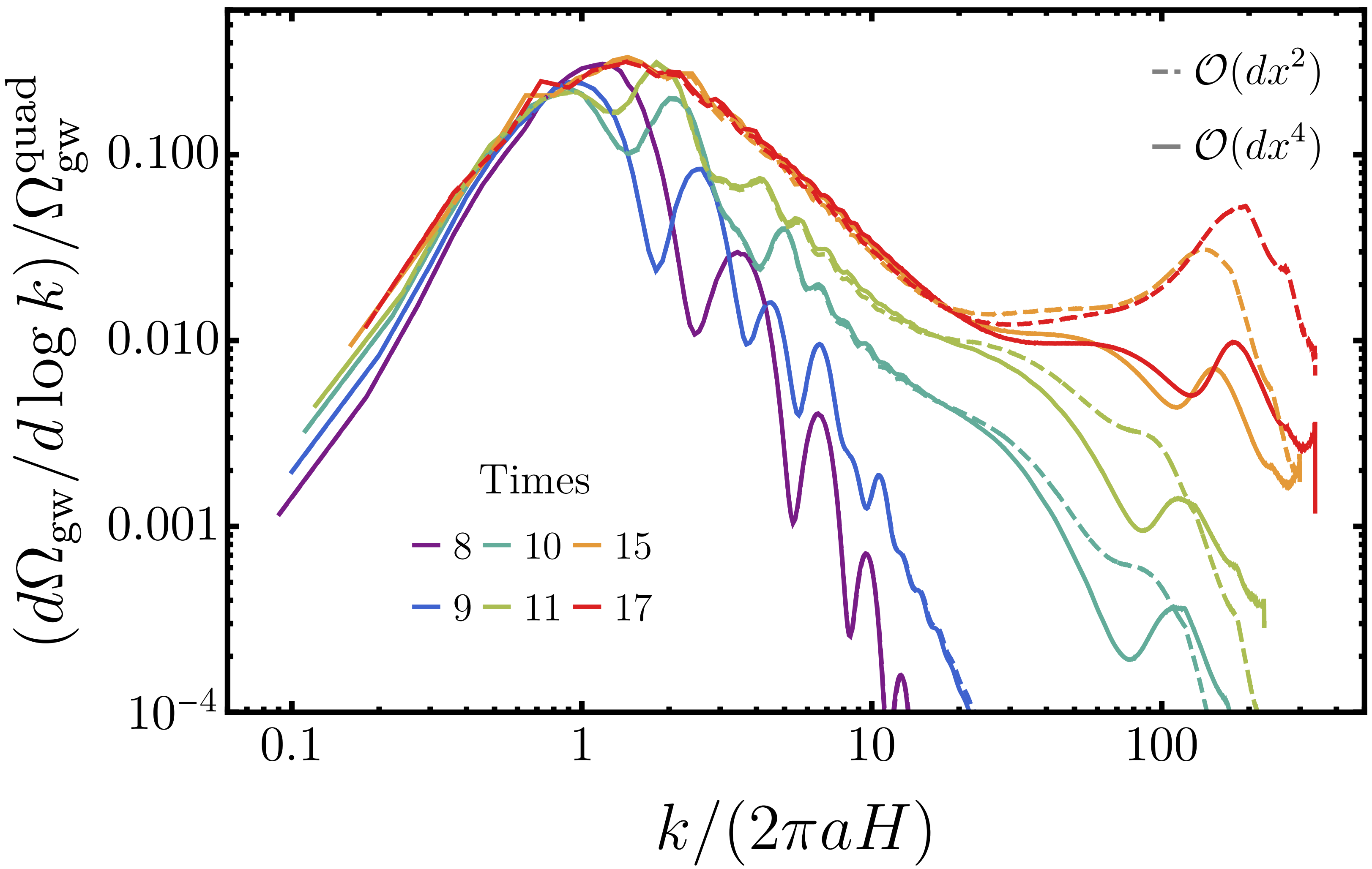

Annihilating domain walls

- DWs eventually dominate the energy density of the universe if $\sigma \gtrsim (\mathrm{MeV})^3$:

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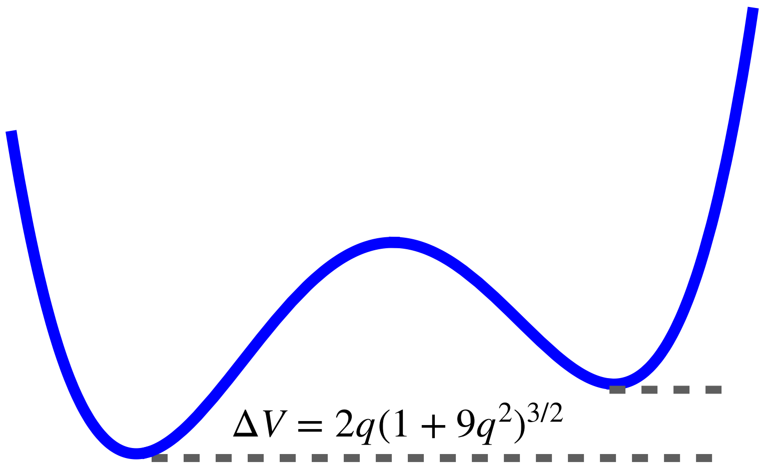

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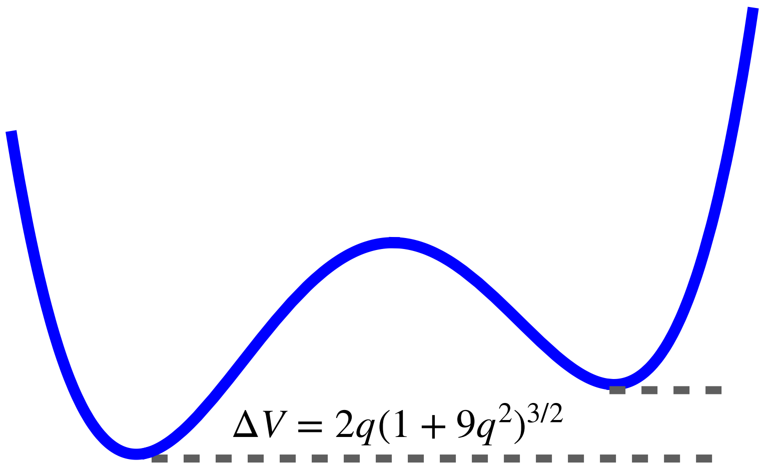

→ Annihilation starts when: $\sigma H(\eta_{\Delta V}) = \Delta V$

Annihilating domain walls

- DWs eventually dominate the energy density of the universe if $\sigma \gtrsim (\mathrm{MeV})^3$:

$$\rho_{\rm dw} \sim \sigma H \sim a^{-2} \longrightarrow$$ **DW problem!**

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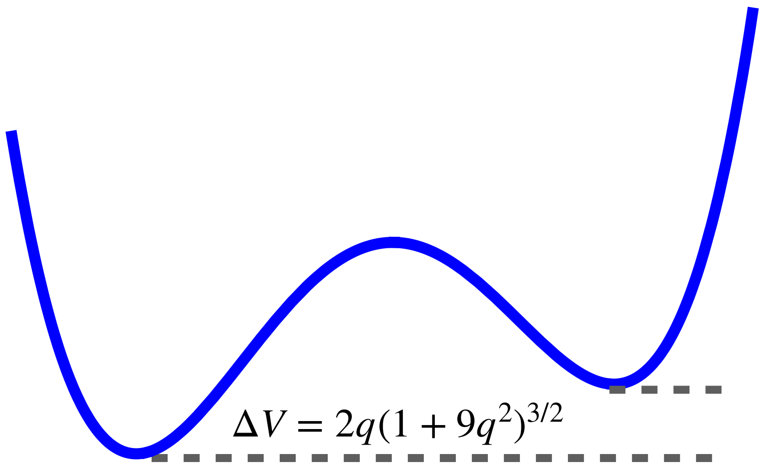

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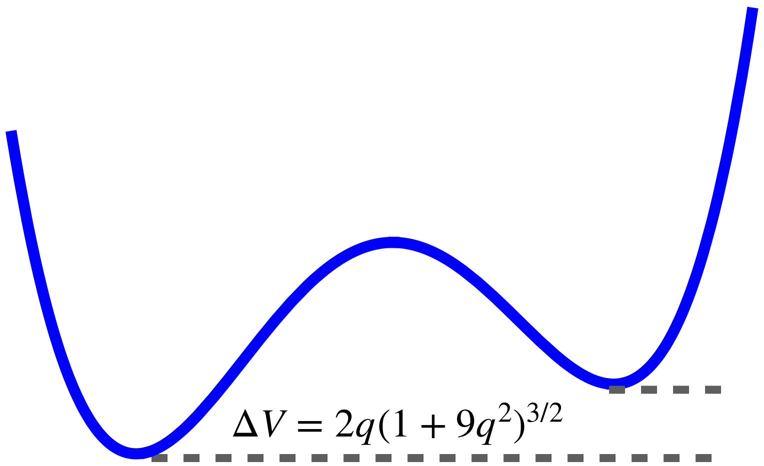

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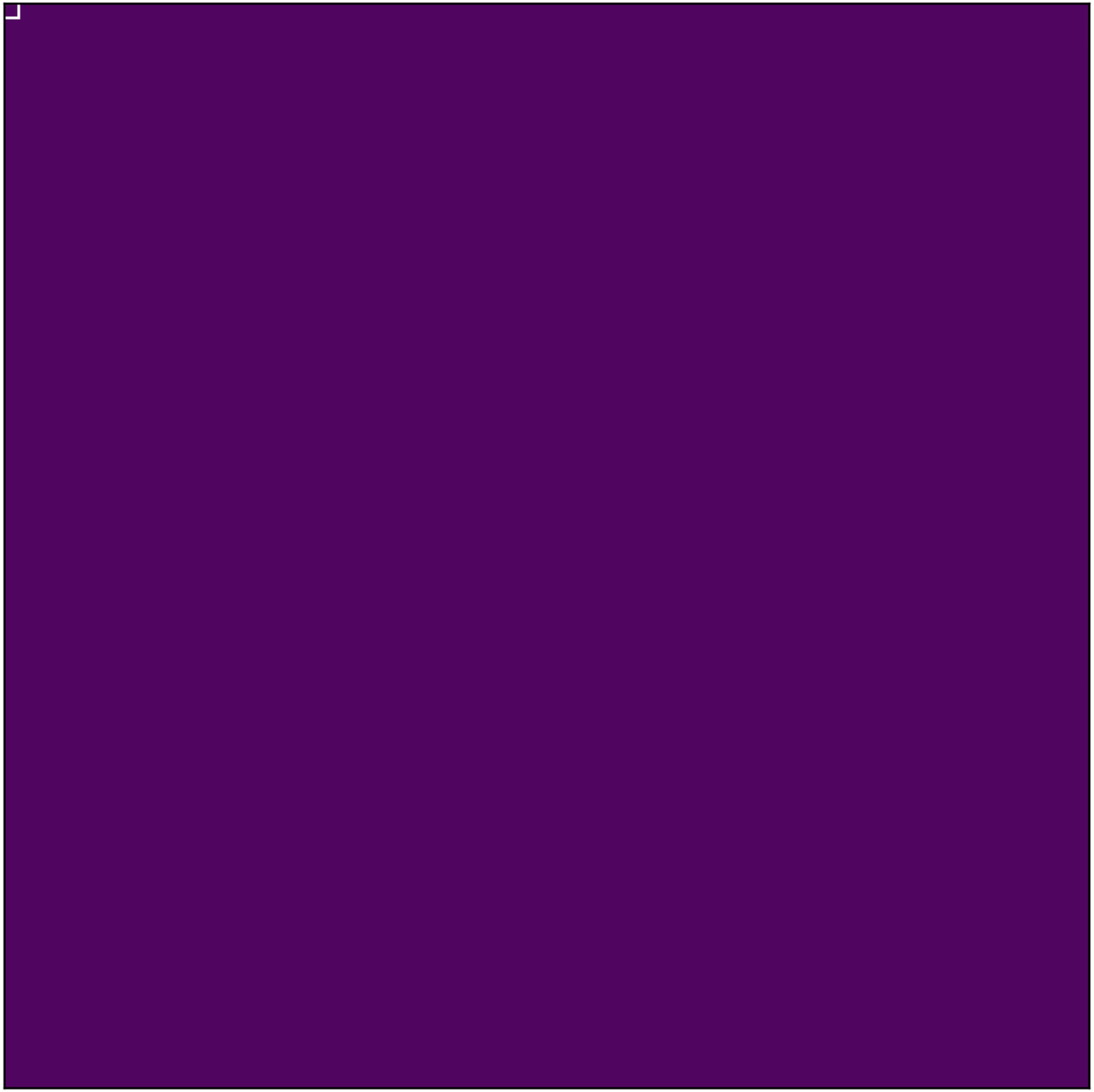
$L = 80H_i^{-1}$



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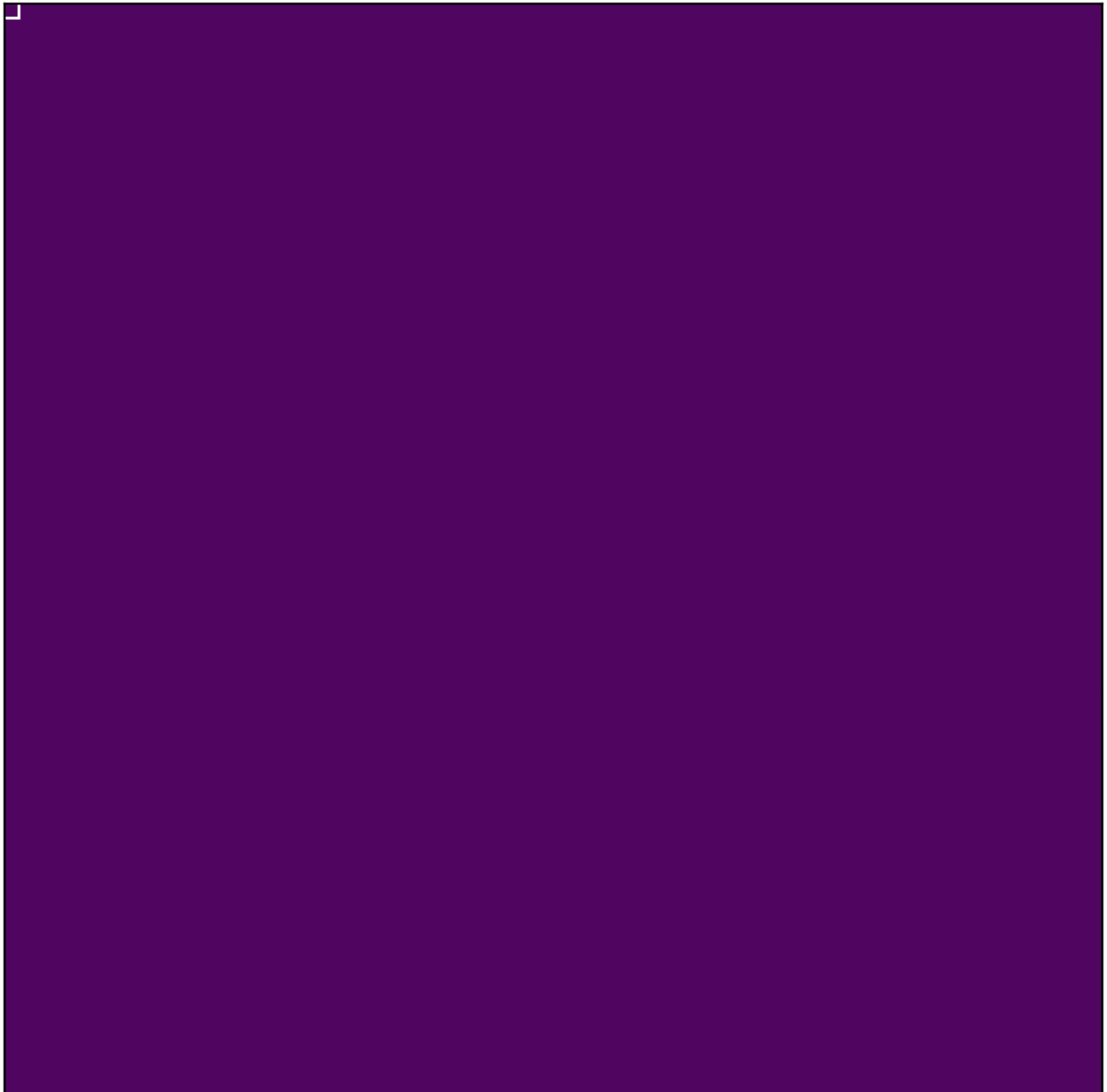

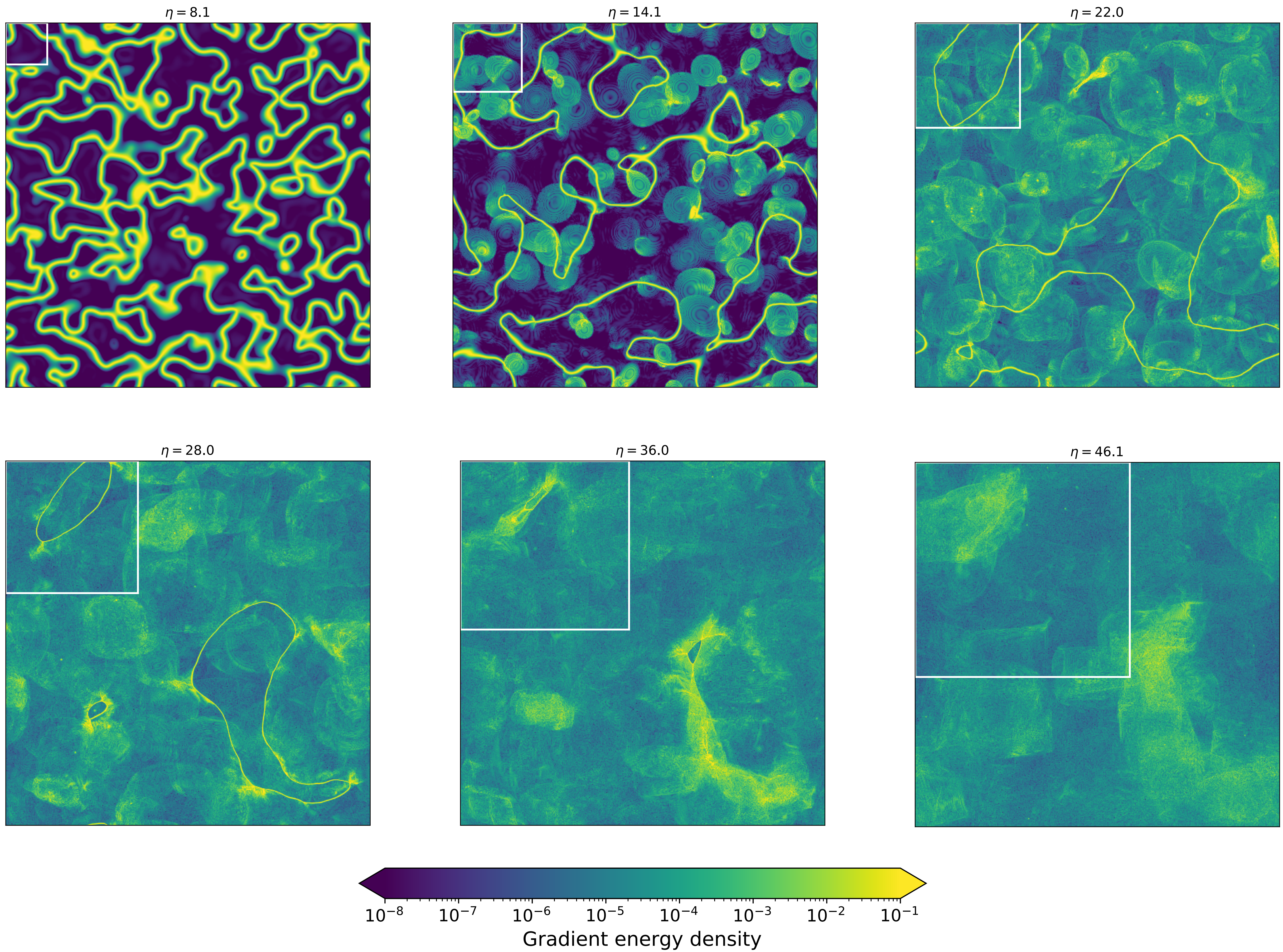

$\eta = 8.1$
$\eta = 14.1$
$\eta = 22.0$
$\eta = 28.0$
$\eta = 36.0$
$\eta = 46.1$
10^{-8} 10^{-7} 10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1}
Gradient energy density

Domain wall evolution (biased case)

$\eta_{\Delta V} = 14$

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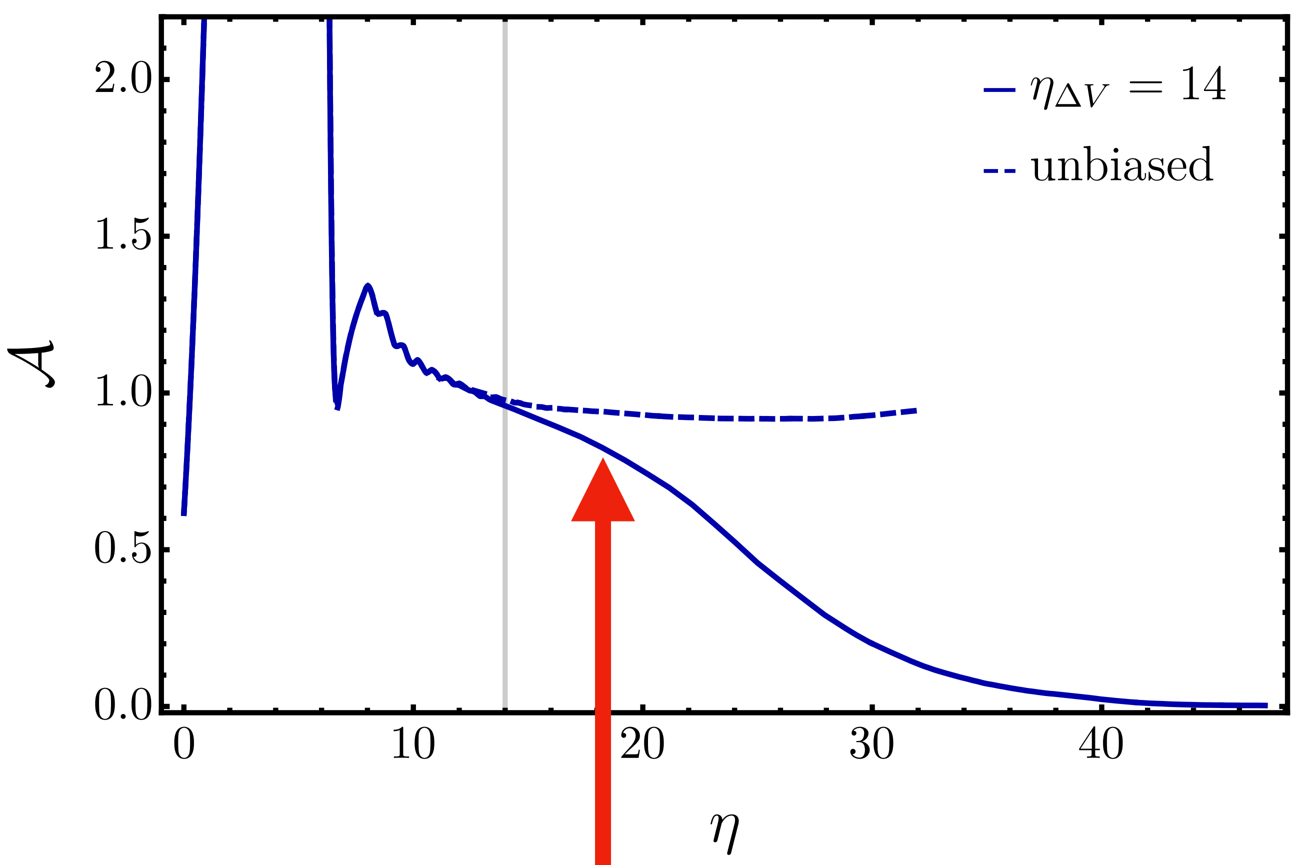
Area parameter



Annihilation

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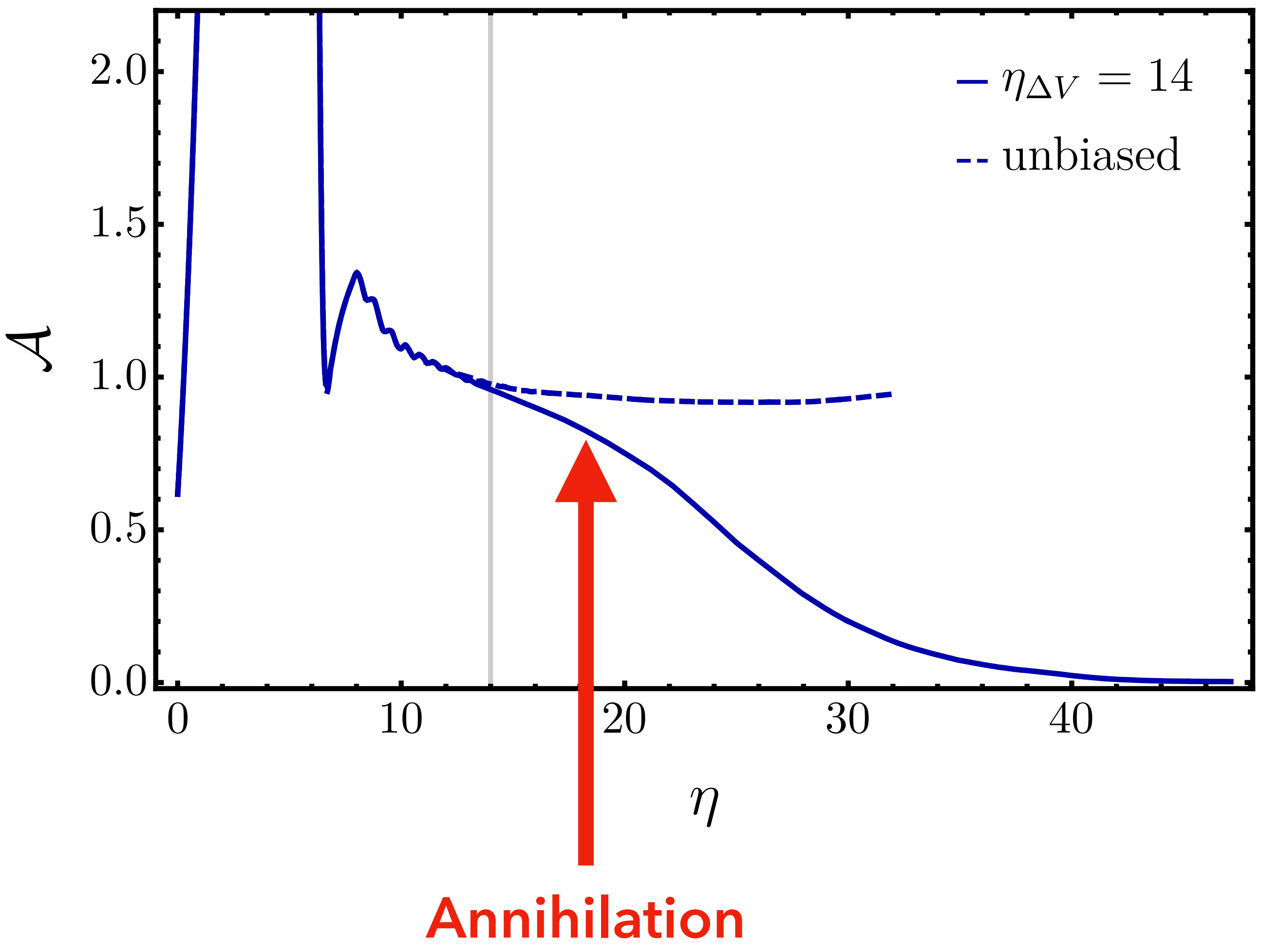
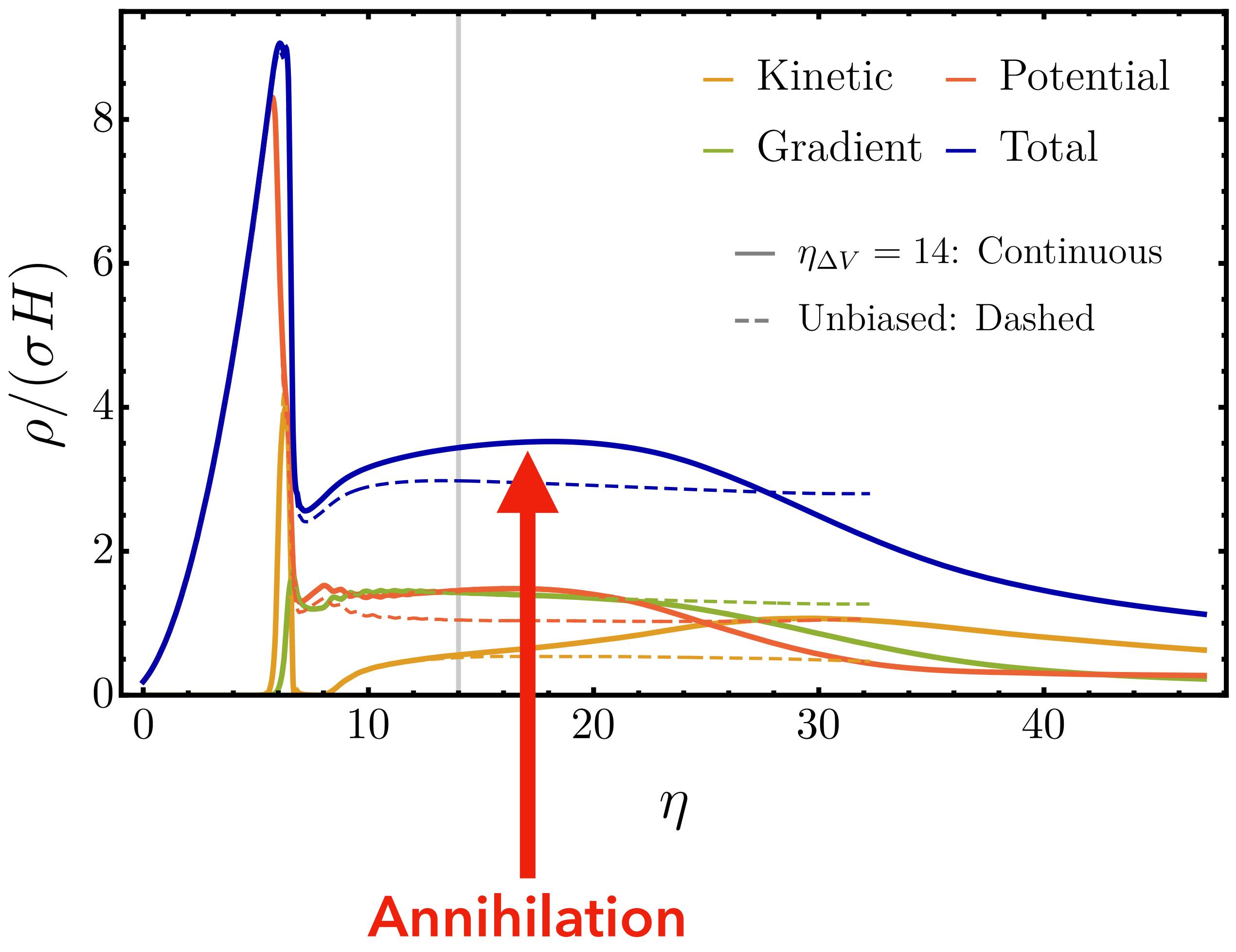
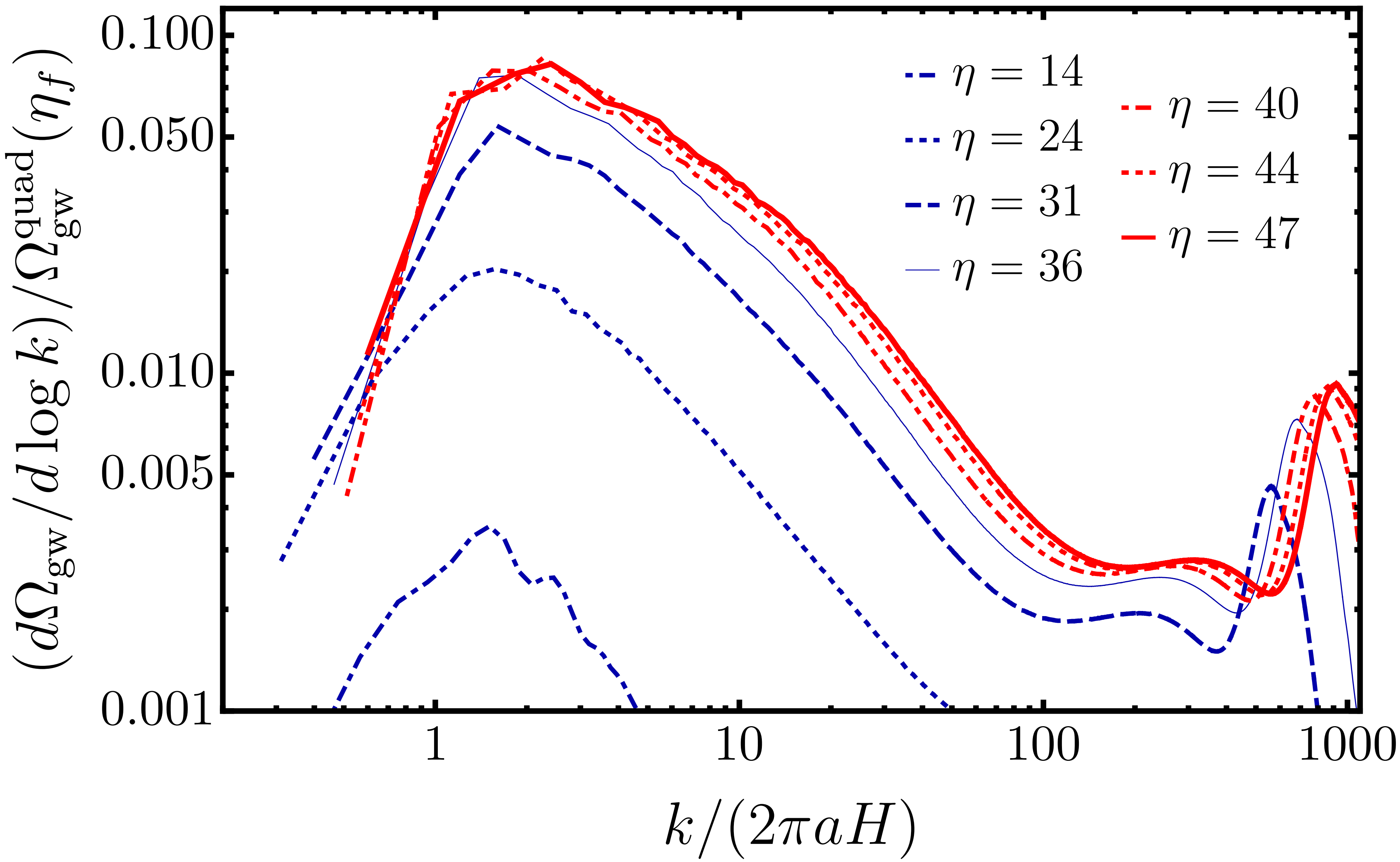


Energy density



GWs from annihilating DW networks



$\Omega_{\rm gw}^{\rm quad} = 3/(32\pi)[2\sigma H/(3H^2 M_p)^2]$

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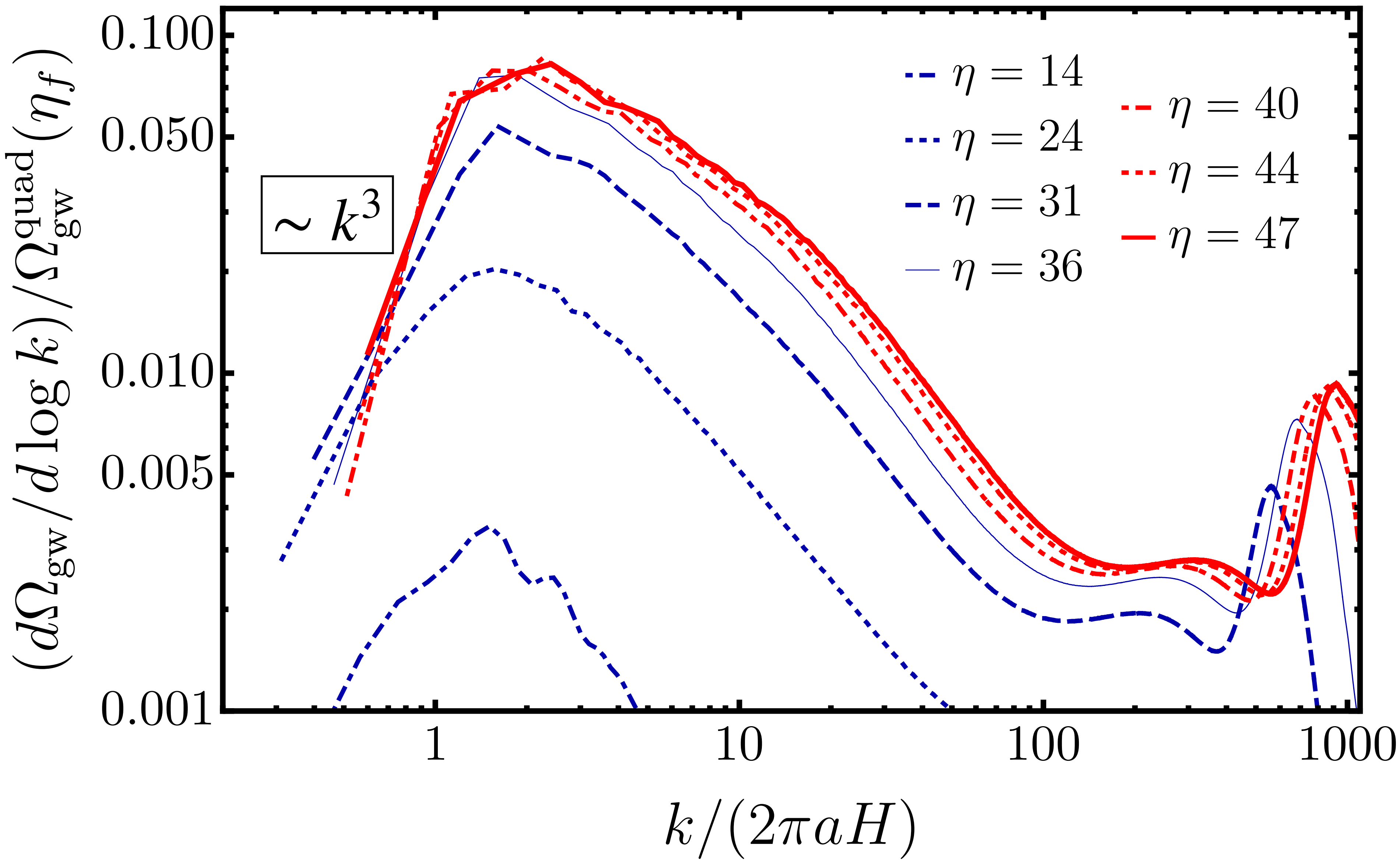

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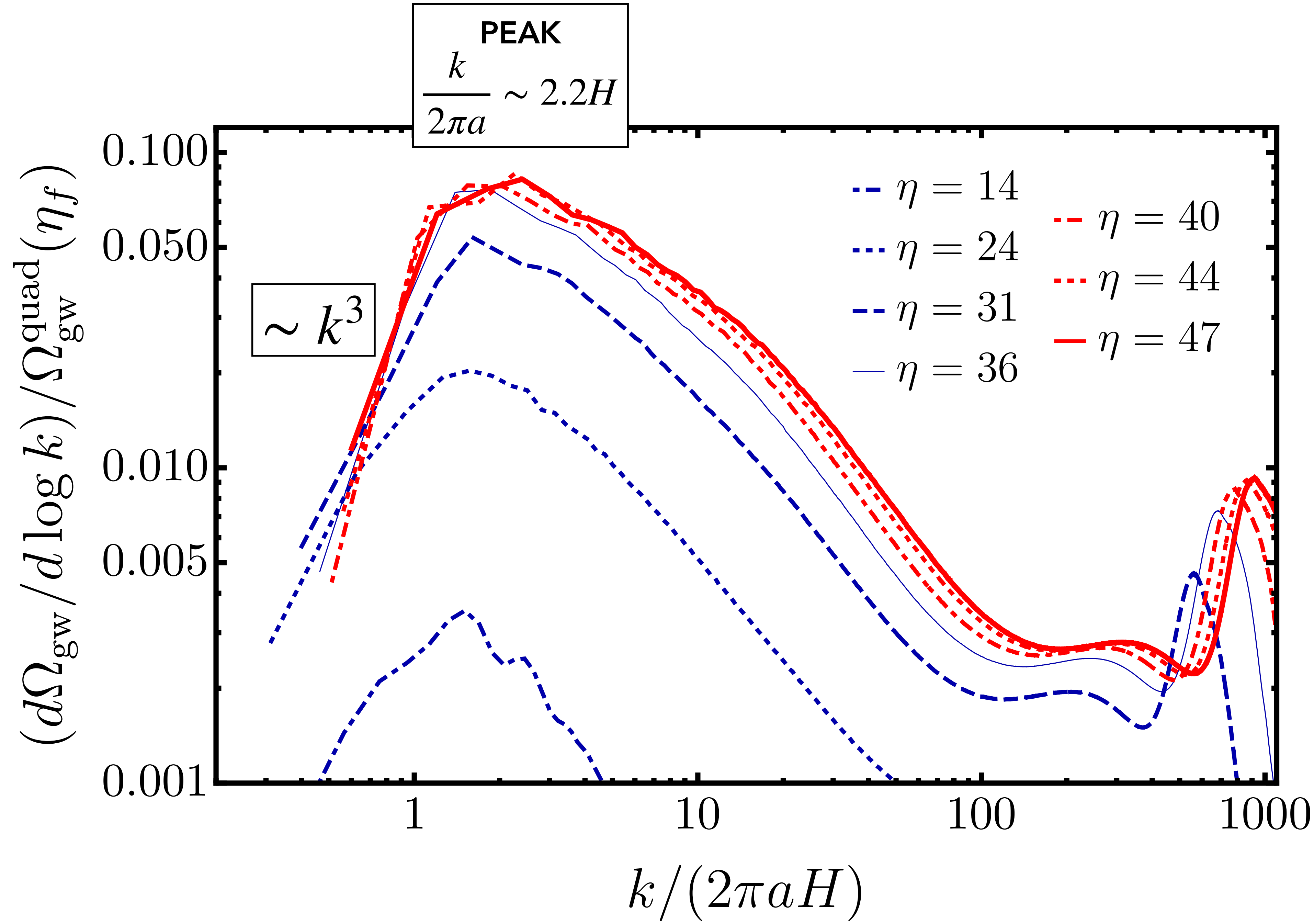

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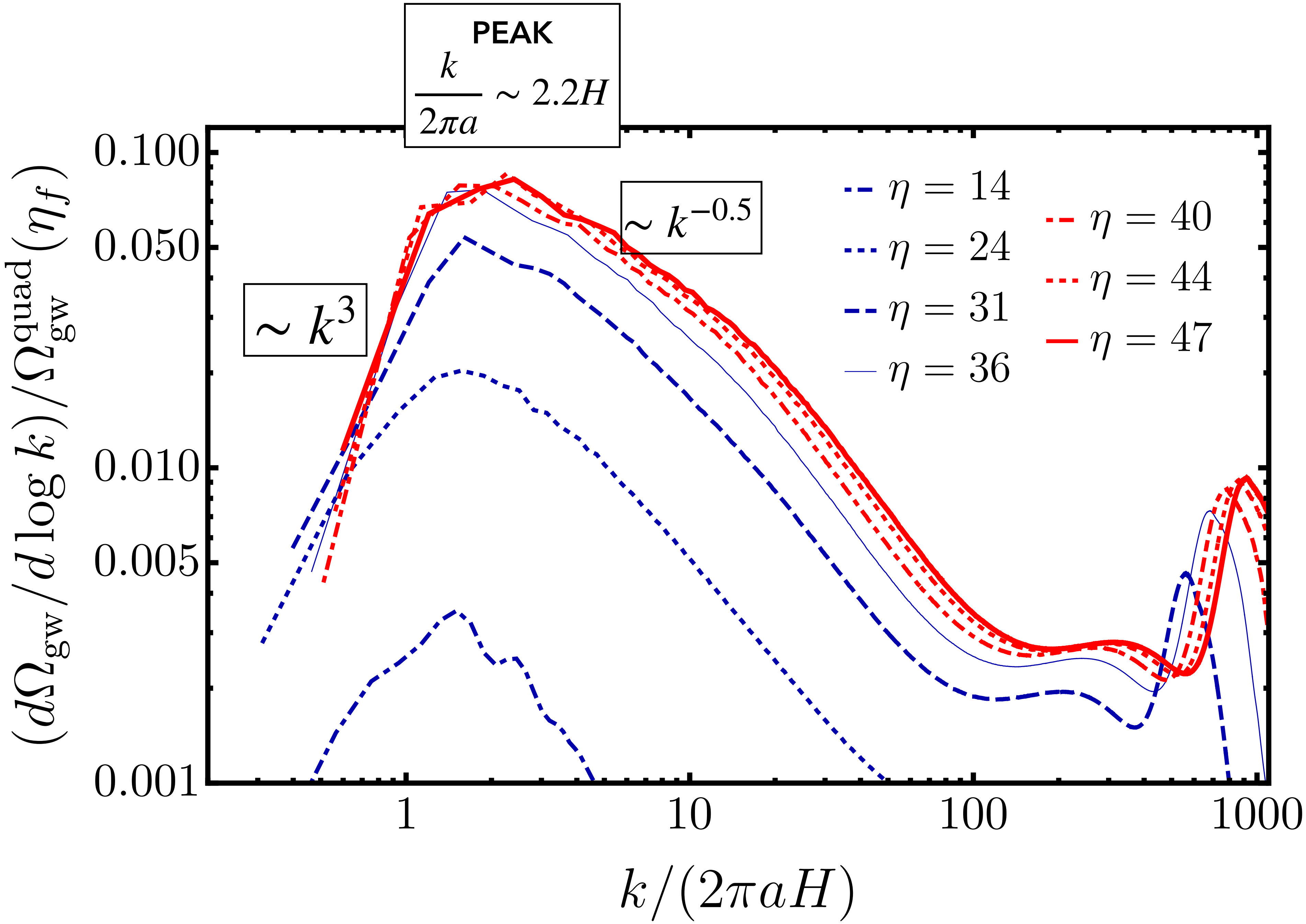

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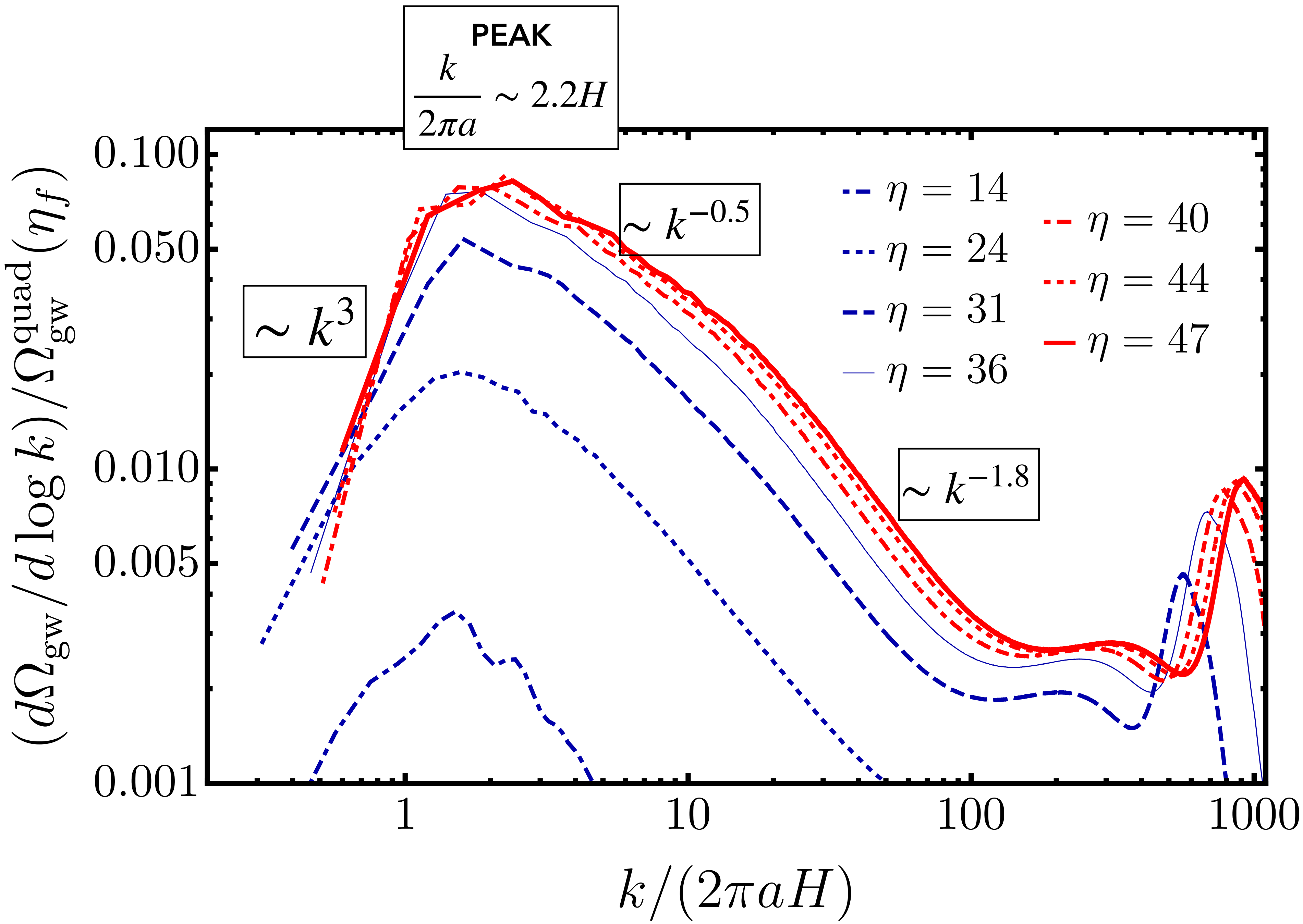

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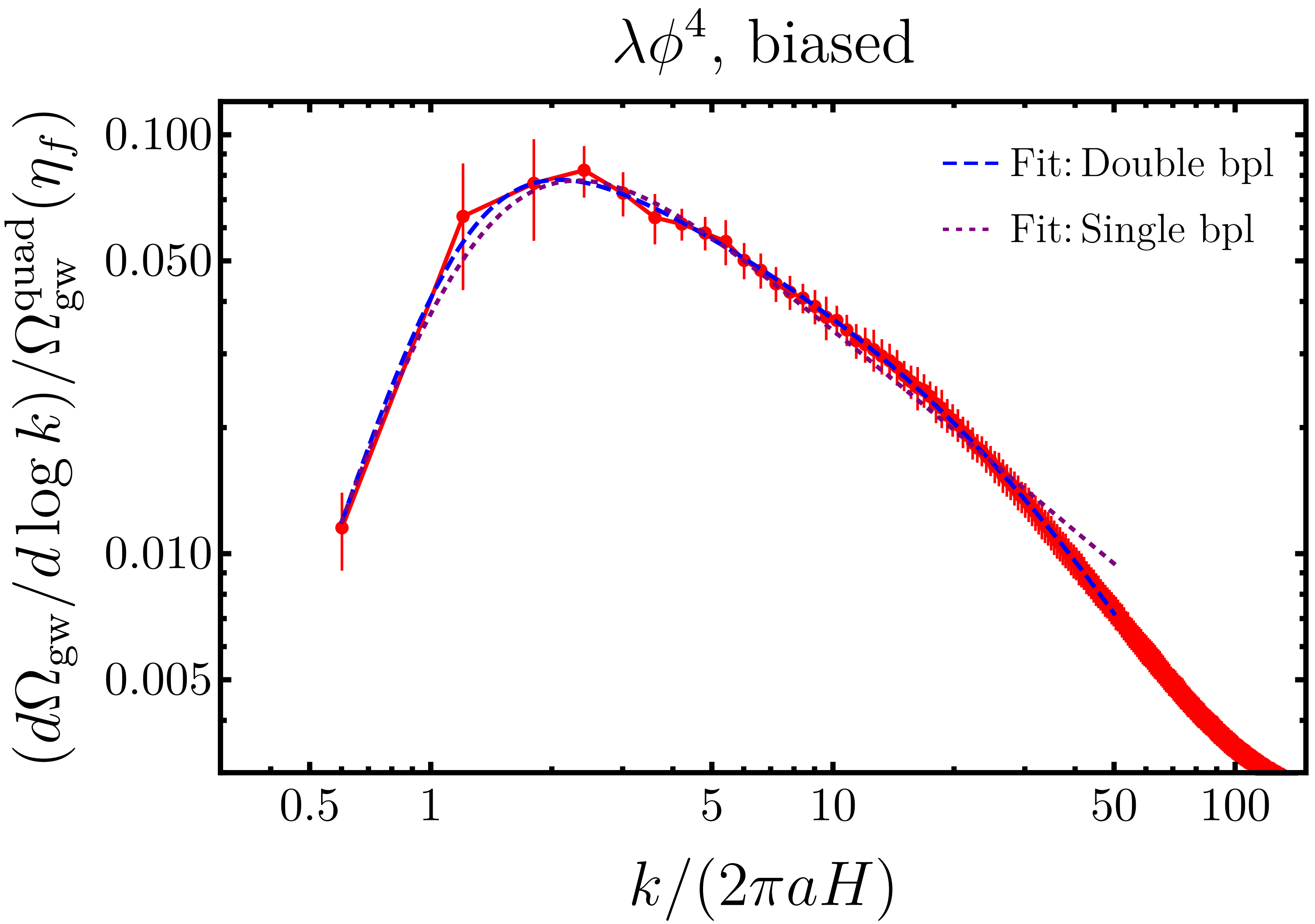
THANK YOU!

GWs from annihilating DW networks



Notari, Rompineve & F.T (JCAP, 2025)