# CosmoLattice

2025-Daejeon

Based on Adrien Florio's lecture

# Lecture 3:
# Evolution algorithms for ordinary differential equations

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- Part I: Symplectic integrators

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- Part II: Non-symplectic integrators

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From discretized
partial differential equation (PDE)

- Problem at hand?

**Solve coupled** ordinary differential equation's **(ODE's)**

$$\dot{\vec{f}} = F(\vec{f}) \text{ where } \vec{f} = (f_1\ f_2\ \dots\ f_N)$$

**Aim:** given $f(t=0)$ $\rightarrow$ find $f(t)$

- Method of choice $\longrightarrow$ **Discrete timesteping**

$$t_0 \rightarrow \vec{f}(t_0) \qquad \vec{f}(t_1) = \mathscr{M}(\vec{f}(t_0))$$

$$t_1 \rightarrow \vec{f}(t_1) \qquad \vec{f}(t_2) = \mathscr{M}(\vec{f}(t_1))$$

$$\vdots$$

$$t_{N-1} \rightarrow \vec{f}(t_{N-1}) \qquad \vec{f}(t_N) = \mathscr{M}(\vec{f}(t_{N-1}))$$

$\mathscr{M}$ : some map

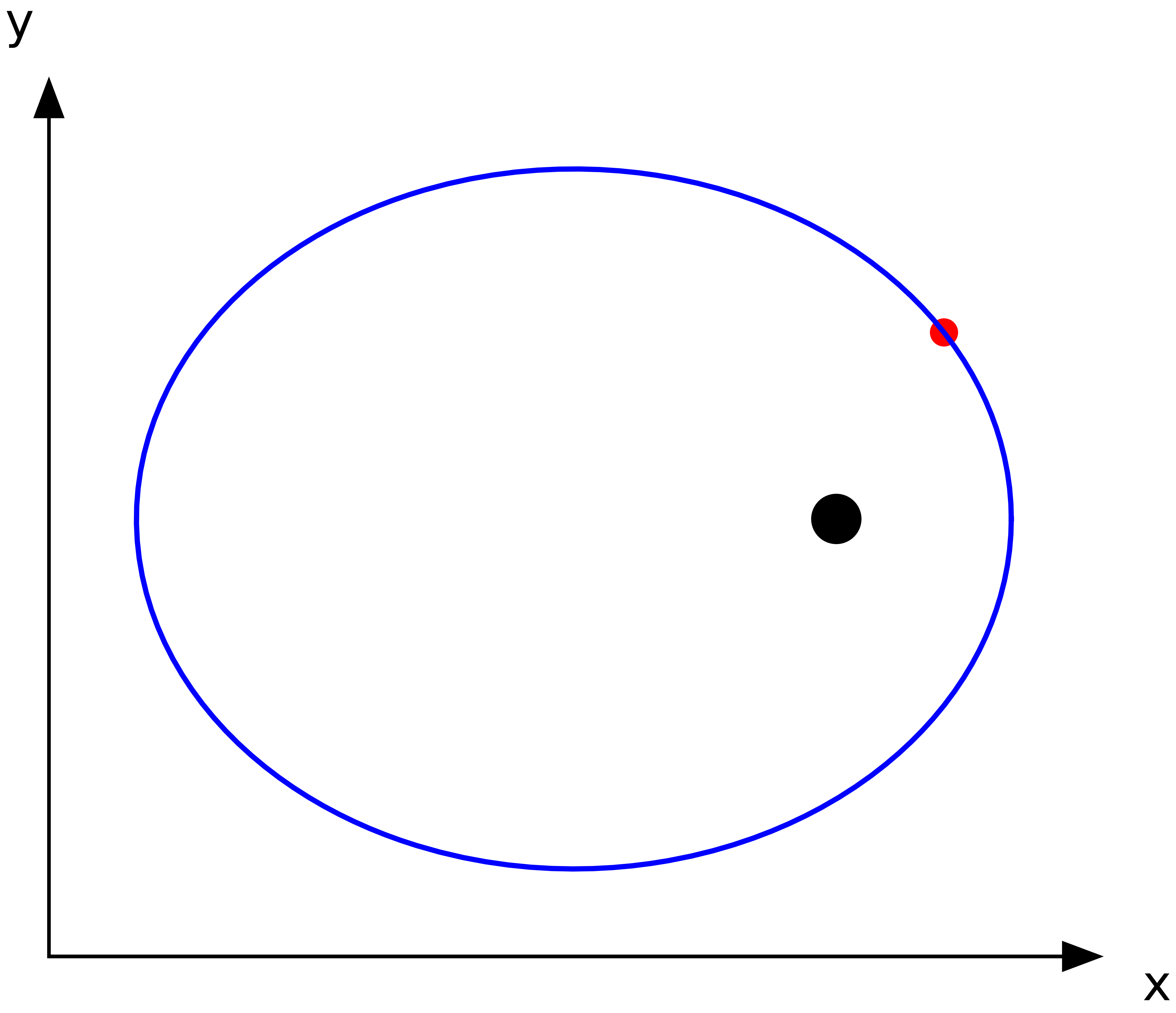
$\downarrow$

$$\dot{\vec{f}} = F[\vec{f}(t)]$$

$\downarrow$

# Algorithm!

- Learn with examples → Orbit in a radial potential




**Kepler problem**

https://en.wikipedia.org/wiki/Kepler_problem

$$V(x, y) = -\frac{\lambda}{\sqrt{x^2 + y^2}}$$

Conservative force

Does not depend on velocities

**Kepler problem** $V(x,y) = -\dfrac{\lambda}{\sqrt{x^2+y^2}}$

Lagrangian

$$\mathscr{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - V(x,y)$$

Conjugate momenta

$$p_x = \frac{\partial \mathscr{L}}{\partial \dot{x}} = \dot{x}\,,\; p_y = \frac{\partial \mathscr{L}}{\partial \dot{y}} = \dot{y}$$

$\xrightarrow[\text{Transf}]{\text{Legendre}}$

Hamiltonian

$$\mathscr{H} = \underbrace{\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2}_{\text{Kin}} + \underbrace{V(x,y)}_{\text{Pot}}$$

\+

EOM

$$\dot{p}_x = -\frac{\partial \mathscr{H}}{\partial x} = -\frac{\partial V}{\partial x} = -x\frac{\lambda}{(x^2+y^2)^{3/2}} = \mathscr{K}_x(x,y)$$

$$\dot{p}_y = -\frac{\partial \mathscr{H}}{\partial y} = -\frac{\partial V}{\partial y} = -y\frac{\lambda}{(x^2+y^2)^{3/2}} = \mathscr{K}_y(x,y)$$

**Kepler problem** $V(x, y) = -\dfrac{\lambda}{\sqrt{x^2 + y^2}}$

Poisson Bracket $\{A, B\} = \dfrac{\partial A}{\partial x}\dfrac{\partial B}{\partial p_x} + \dfrac{\partial A}{\partial y}\dfrac{\partial B}{\partial p_y} - \dfrac{\partial A}{\partial p_x}\dfrac{\partial B}{\partial x} - \dfrac{\partial A}{\partial p_y}\dfrac{\partial B}{\partial y}$

Exercise

Angular momentum $L_z = xp_y - yp_x \longrightarrow \{L_z, \mathscr{H}\} = 0$

$+$ Hamiltonian does not depend on time

$\longrightarrow$ Conservation of **Energy & Angular momentum**

Analytical solution

$$r(\theta) = -\frac{L_z^2}{1 + e\cos\theta}$$ where $e \equiv \sqrt{1 + 2EL^2} \equiv$ eccentrity

Polar coordinates $\rightarrow r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$

Or Runge-Kutta1
(Antonino's part)

# #1 Forward-Euler (FE)

Based on: $$\frac{df(t)}{dt} \simeq \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$\begin{cases} \vec{p}(t+\Delta t) = \vec{p}(t) + \Delta t \; \overrightarrow{\mathscr{K}}(x(t), y(t)) \\ \vec{x}(t+\Delta t) = \vec{x}(t) + \Delta t \; \vec{p}(t) \end{cases}$$

**Kepler problem** $$V(x,y) = -\frac{\lambda}{\sqrt{x^2+y^2}}$$

$$\mathscr{K}_x = -x\frac{\lambda}{(x^2+y^2)^{3/2}}$$

$$\mathscr{K}_y = -y\frac{\lambda}{(x^2+y^2)^{3/2}}$$

Let's see Python Notebook!

Open
**AlgorithmPlots.ipynb** in [https://colab.research.google.com/](https://colab.research.google.com/)

**Notation:** $f(t+\Delta t) \equiv f^{+1}$ , $f(t) \equiv f$, $\mathscr{K}(x(t+\Delta t), y(t+\Delta t)) \equiv \mathscr{K}^{+1}$

# #1 Forward-Euler (FE)

$\longrightarrow$ Does not preserve angular momentum

$$L_z^{+1} \neq L_z$$

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**Check**

$$L_z^{+1} = x^{+1}p_y^{+1} - y^{+1}p_x^{+1}$$

**FE** $$\begin{cases} x^{+1}p_y^{+1} = xp_y + (p_xp_y + x\mathscr{K}_y)\Delta t + p_x\mathscr{K}_y\Delta t^2 \\ y^{+1}p_x^{+1} = yp_x + (p_xp_y + y\mathscr{K}_x)\Delta t + p_y\mathscr{K}_x\Delta t^2 \end{cases}$$

$$\longrightarrow \quad L_z^{+1} = L_z + (p_x\mathscr{K}_y - p_y\mathscr{K}_x)\Delta t^2$$

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**Suggestion**

# #2 Semi-Implicit Euler (SE)

$\longrightarrow$ Does not preserve angular momentum

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**Check**

$$L_z^{+1} = x^{+1} p_y^{+1} - y^{+1} p_x^{+1}$$

$$\text{FE} \begin{cases} x^{+1} p_y^{+1} = x p_y + (p_x p_y + x \mathscr{K}_y)\Delta t + p_x \mathscr{K}_y \Delta t^2 \\ y^{+1} p_x^{+1} = y p_x + (p_x p_y + y \mathscr{K}_x)\Delta t + p_y \mathscr{K}_x \Delta t^2 \end{cases}$$

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$$L_z^{+1} = x^{+1}p_y^{+1} - y^{+1}p_x^{+1}$$

**SE**

$$\begin{cases} x^{+1}p_y^{+1} = xp_y + (p_xp_y + x\mathcal{K}_y)\Delta t + p_x\mathcal{K}_y\Delta t^2 \\ \qquad + p_y\mathcal{K}_x\Delta t^2 + \mathcal{K}_x\mathcal{K}_y\Delta t^3 \\ y^{+1}p_x^{+1} = yp_x + (p_xp_y + y\mathcal{K}_x)\Delta t + p_y\mathcal{K}_x\Delta t^2 \end{cases}$$

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→ Does not preserve angular momentum

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$$\rightarrow \quad L_z^{+1} = L_z + \cancel{(p_x\mathcal{K}_y - p_y\mathcal{K}_x)\Delta t^2}$$

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**Suggestion**

# #2 Semi-Implicit Euler (SE)

$\longrightarrow$ Preserves angular momentum

$$L_z^{+1} = L_z$$

$$\begin{cases} \vec{p}(t+\Delta t) = \vec{p}(t) + \Delta t \; \overrightarrow{\mathscr{K}}(x(t), y(t)) \\ \vec{x}(t+\Delta t) = \vec{x}(t) + \Delta t \; \vec{p}(t+\Delta t) \end{cases}$$

Let's see Python Notebook!

**Check**

$$L_z^{+1} = x^{+1} p_y^{+1} - y^{+1} p_x^{+1}$$

$$\textbf{SE} \begin{cases} x^{+1} p_y^{+1} = x p_y + (p_x p_y + x \mathscr{K}_y)\Delta t + p_x \mathscr{K}_y \Delta t^2 \\ \quad + p_y \mathscr{K}_x \Delta t^2 + \mathscr{K}_x \mathscr{K}_y \Delta t^3 \\ y^{+1} p_x^{+1} = y p_x + (p_x p_y + y \mathscr{K}_x)\Delta t + p_y \mathscr{K}_x \Delta t^2 \\ \quad + p_x \mathscr{K}_y \Delta t^2 + \mathscr{K}_x \mathscr{K}_y \Delta t^3 \end{cases}$$

$$\longrightarrow \quad L_z^{+1} = L_z + \cancel{(p_x \mathscr{K}_y - p_y \mathscr{K}_x)\Delta t^2}$$

- Take home message: Physics $\leftarrow$ Algorithm $\rightarrow$

Built an algorithm that preserves the conservation laws:
**"Symplectic"**

$\longrightarrow$

Symplectic
$=$
preserves $\{\,\cdot\,,\,\cdot\,\}$
$\Updownarrow$
Preserves conservation laws

**Key for Hamiltonian systems**

$$\{x(t), p_x(t)\} = 1$$

Lecture 7:
Axion-Coupling on the Lattice

# Important algorithm properties:

- $\boxed{\text{Explicit}}$ **vs** $\boxed{\text{Implicit}}$

  ✓ Cheaper — ✗ More expensive

  ✗ Less stable — ✓ More stable

- Order of convergence: $x^{exact}(t) - x(t) = \mathcal{O}(\Delta t^l)$

- Depending on if Hamiltonian: Symplectic **vs** Non-symplectic

Antonino's part

Let's see Python Notebook!

## Higher order integrators

Combination of SI Euler for half-steps

# Verlet

**Notation:** $f(t + \Delta t/2) \equiv f^{+1/2}$

## #3 Velocity-Verlet (VV)

$$\vec{p}^{\,+1/2} = \vec{p} + \frac{1}{2}\Delta t\ \overrightarrow{\mathscr{K}}$$

$$\vec{x}^{\,+1} = \vec{x} + \Delta t\ \vec{p}^{\,+1/2}$$

$$\vec{p}^{\,+1} = \vec{p}^{\,1/2} + \frac{1}{2}\Delta t\ \overrightarrow{\mathscr{K}}^{+1}$$

## #4 Position-Verlet (PV)

$$\vec{x}^{\,+1/2} = \vec{x} + \frac{1}{2}\Delta t\ \vec{p}$$

$$\vec{p}^{\,+1} = \vec{p} + \Delta t\ \overrightarrow{\mathscr{K}}^{+1/2}$$

$$\vec{x}^{\,+1} = \vec{x}^{\,1/2} + \frac{1}{2}\Delta t\ \vec{p}^{\,+1}$$

Exercise:
Check approximate the ODE to $\mathcal{O}(\Delta t^2)$

**Hint**: use Taylor

Let's see Python Notebook!

## Higher order integrators

Non synchronized
PV

# #5 Staggered Leapfrog (LF)

$$\vec{x}^{+1/2} = \vec{x} + \frac{1}{2}\Delta t\ \vec{p}$$

$$\vec{p}^{+1} = \vec{p} + \Delta t\ \overrightarrow{\mathscr{K}}^{+1/2}$$

✓ One less operation

↓

Synchronize
only for measurements

Antonino’s turn now!