CosmoLattice – School 2025

— Lecture 4 —

Lattice Formulation of Interactive Scalar Fields

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Lattice Formulation of Interactive Scalar Fields (@ Expanding Backgrounds)

Continuum Formulation

Interacting real scalar fields: $\{\phi_a\}_{a=1,2,\ldots,N_s}$ **(w/ canonically normalized kinetic terms)**

Continuum Formulation

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Action: $$S = -\int d^4x \sqrt{-g} \left(\frac{1}{2}\partial_\mu \phi_b \partial^\mu \phi_b + V(\{\phi_c\}) \right) = \left(\frac{f_*}{\omega_*} \right)^2 \tilde{S}$$

Continuum Formulation

Interacting real scalar fields: $\{\phi_a\}_{a=1,2,\ldots,N_s}$ **(w/ canonically normalized kinetic terms)**

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$$\boxed{\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b\tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}}$$

$$\left[\begin{array}{c} \tilde{\phi}_a \equiv \dfrac{\phi_a}{f_*}\,, \qquad d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt\,, \qquad d\tilde{x}^i \equiv \omega_* dx^i \\ \widetilde{V}(\{\tilde{\phi}_c\}) \equiv \dfrac{1}{f_*^2\omega_*^2}V(\{\phi_c\})\Big|_{\phi_c = f_*\tilde{\phi}_c} \end{array}\right]$$

Continuum Formulation

$$\boxed{\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b^{\prime 2} - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}}$$

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$$\delta\tilde{S} = 0 \quad \Rightarrow \quad \boxed{\tilde{\phi}_a'' - a^{-2(1-\alpha)}\tilde{\nabla}^2\tilde{\phi}_a + (3-\alpha)\frac{a'}{a}\tilde{\phi}_a' + a^{2\alpha}\widetilde{V}_{,\tilde{\phi}_a} = 0}$$

$$(a = 1,2,\ldots,N_s)$$

EOM (dimensionless variables)

Continuum Formulation

$$\tilde{S} = \int d^3\tilde{x} d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,k} (\tilde{\nabla}_k \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

$$\delta\tilde{S} = 0 \Rightarrow \boxed{\tilde{\phi}_a'' - a^{-2(1-\alpha)} \tilde{\nabla}^2 \tilde{\phi}_a + (3-\alpha)\frac{a'}{a}\tilde{\phi}_a' + a^{2\alpha} \widetilde{V}_{,\tilde{\phi}_a} = 0}$$

$(a = 1,2,\ldots,N_s)$

EOM (dimensionless variables)

$$\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2 \left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right],$$

$$\frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2 \left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right],$$

$$\left(\begin{array}{l} \widetilde{E}_K \equiv \dfrac{1}{2a^{2\alpha}} \displaystyle\sum_i \left\langle (\tilde{\phi}_i')^2 \right\rangle \\ \widetilde{E}_G \equiv \dfrac{1}{2a^2} \displaystyle\sum_{i,k} \left\langle (\widetilde{\nabla}_k \tilde{\phi}_i)^2 \right\rangle \\ \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_j\}) \right\rangle \end{array} \right)$$

EOM of Expanding Background

Continuum Formulation

Simple Prescription:

Continuum		Lattice
∂_μ	$\longrightarrow$	$\nabla_\mu^{(\mathrm{L})}$

Continuum Formulation

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Valid for Scalar Fields ✓

Continuum Formulation

Simple Prescription:

Continuum		Lattice
∂_μ	$\longrightarrow$	$\nabla_\mu^{(\mathrm{L})}$

Valid for Scalar Fields ✓

↓

Not enough for Gauge theories !

See Lecture 6 (by Jorge B.B)

Lattice Formulation

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

Lattice Formulation

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

i) ACTION Approach: Discretize the continuum action

$$\tilde{S} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b^{\prime 2} - \frac{1}{2}a^{1+\alpha}\sum_{b,k}(\tilde{\nabla}_k\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice Formulation

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

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ii) EOM approach: Discretize the continuum EOM

$$\tilde{\phi}_a'' - a^{-2(1-\alpha)} \tilde{\nabla}^2 \tilde{\phi}_a + (3-\alpha) \frac{a'}{a} \tilde{\phi}_a' = -a^{2\alpha} \widetilde{V}_{,\tilde{\phi}_a}$$

Lattice Formulation

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^{(\mathrm{L})} \end{pmatrix}$$

i) ACTION Approach: Discretize the continuum action

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?

ii) EOM approach: Discretize the continuum EOM

$$\tilde{\phi}_a'' - a^{-2(1-\alpha)} \tilde{\nabla}^2 \tilde{\phi}_a + (3-\alpha)\frac{a'}{a}\tilde{\phi}_a' = -\, a^{2\alpha} \widetilde{V}_{,\tilde{\phi}_a}$$

Lattice Formulation

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla^{(\mathrm{L})}_\mu \end{pmatrix}$$

i) ACTION Approach: Discretize the continuum action

$$\tilde{S} = \int d^3\tilde{x}\, d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,k} (\tilde{\nabla}_k \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

ii) EOM approach: Discretize the continuum EOM

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Lattice Action Approach

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice Action Approach (Full)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}\, d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

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Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\qquad\qquad\qquad\qquad\right\}$$

Lattice Action Approach (Full)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}\, d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\,\delta\tilde{x}^3 \underbrace{\sum_{n_0} \sum_{\mathbf{n}}} \left\{ \qquad \qquad \qquad \qquad \qquad \qquad \right\}$$

time step

lattice spacing

integration $\simeq$ Sum

Lattice Action Approach (Full)

Continuum:
$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice:
$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{ \qquad \right\}$$

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

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Implicitly
fields live at
integer times
$\tilde{\phi}_b(n_0\delta\tilde{\eta})\,,\quad n_0 = 1,2,3,...$

Lattice Action Approach (Full)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}\, d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\,\delta\tilde{x}^3 \sum_{n_0} \sum_{\mathbf{n}} \left\{ \frac{1}{2} a^{3-\alpha}_{+0/2} \sum_b \underbrace{(\widetilde{\Delta}_0^+ \tilde{\phi}^b)^2} - \frac{1}{2} a^{1+\alpha} \sum_{b,i} \underbrace{(\widetilde{\Delta}_i^+ \tilde{\phi}^b)^2} - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Forward
Derivatives
(it's a choice)

Lattice Action Approach (Full)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}\,d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\,\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

$$\underbrace{\frac{\tilde{\phi}^b(n_0+1)-\tilde{\phi}^b(n_0)}{\delta\tilde{\eta}}}_{\equiv\,(\widetilde{\Delta}_0^+\tilde{\phi}^b)\big|_{\mathbf{n},n_0+1/2}}\qquad\qquad \underbrace{\frac{\tilde{\phi}^b(\mathbf{n}+\hat{i})-\tilde{\phi}^b(\mathbf{n})}{\delta\tilde{x}^i}}_{\equiv\,(\widetilde{\Delta}_i^+\tilde{\phi}^b)\big|_{\mathbf{n}+\hat{i}/2,n_0}}$$

(@ semi-integer times) (@ integer times)

Lattice Action Approach (Full)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\frac{1}{2}\underbrace{a^{3-\alpha}_{+0/2}}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2 - \frac{1}{2}\underbrace{a^{1+\alpha}}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

scale factor at $(n_0 + 1/2)\delta\tilde{\eta}$

(**semi-integer times**)

scale factor at $n_0\delta\tilde{\eta}$

(**integer times**)

Lattice Action Approach (Full)

Continuum:
$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}\,d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice:
$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\,\delta\tilde{x}^3 \sum_{n_0} \sum_{\mathbf{n}} \left\{ \frac{1}{2} \underbrace{a^{3-\alpha}_{+0/2}} \sum_b (\widetilde{\Delta_0^+}\tilde{\phi}^b)^2 - \frac{1}{2} \underbrace{a^{1+\alpha}} \sum_{b,i} (\widetilde{\Delta_i^+}\tilde{\phi}^b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

scale factor
at $(n_0 + 1/2)\delta\tilde{\eta}$
(**semi-integer times**)

scale factor
at $n_0\delta\tilde{\eta}$
(**integer times**)

Notation:
$f_{\pm 0/2} \equiv f((n_0 \pm 1/2)\delta\tilde{\eta})$

Notation:
$f \equiv f(n_0\delta\tilde{\eta})$

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x} d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3 \sum_{n_0} \sum_{\mathbf{n}} \left\{ \underbrace{\frac{1}{2} a^{3-\alpha}_{+0/2} \sum_b (\widetilde{\Delta}_0^+ \tilde{\phi}^b)^2}_{@\ (n_0 + 1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2} a^{1+\alpha} \sum_{b,i} (\widetilde{\Delta}_i^+ \tilde{\phi}^b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}} \right\}$$

(semi-integer times) **(integer times)**

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

(semi-integer times) **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0$$

(Variational Principle)

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

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(semi-integer times) **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \ \Rightarrow\ \widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b} \quad , b = 1,2,\ldots,N_s$$

Lattice Action Approach (Full)

Continuum:
$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

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(semi-integer times) **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \quad \Rightarrow \quad \underbrace{\widetilde{\Delta}_0^-}_{\text{Backward time deriv.}}[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \underbrace{\widetilde{\Delta}_i^-}_{\text{Backward spat. deriv.}}\times\underbrace{\widetilde{\Delta}_i^+}_{\text{Forward spat. deriv.}}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b} \quad , b = 1,2,\ldots,N_s$$

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

(semi-integer times) (integer times)

$$\delta_{\phi_a}S_{\rm L} = 0 \ \Rightarrow\ \widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b} \quad , b = 1,2,\ldots,N_s$$

scale factor at $(n_0+1/2)\delta\tilde{\eta}$ (semi-integer times)

scale factor at $n_0\delta\tilde{\eta}$ (integer times)

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

(semi-integer times) **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \;\Rightarrow\; \widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = \underbrace{a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}_{@\ n_0\delta\tilde{\eta}}\quad,\ b = 1,2,\ldots,N_s$$

(integer times)

scale factor
at $(n_0+1/2)\delta\tilde{\eta}$
(semi-integer times)

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

(semi-integer times) **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \Rightarrow \widetilde{\Delta}_0^-[\underbrace{a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b}_{@\ (n_0+1/2)\delta\tilde{\eta}}] = \underbrace{a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}_{@\ n_0\delta\tilde{\eta}}\quad ,\ b = 1,2,\ldots,N_s$$

(integer times)

Lattice Action Approach (Full)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

(semi-integer times) **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \Rightarrow \underbrace{\widetilde{\Delta}_0^-[\underbrace{a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b}_{@\,(n_0+1/2)\delta\tilde{\eta}}]}_{@\,n_0\delta\tilde{\eta}} = \underbrace{a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}_{@\,n_0\delta\tilde{\eta}}\quad,\ b=1,2,\ldots,N_s$$

(integer times) **(integer times)**

Lattice Action Approach (Full)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a_{+0/2}^{3-\alpha}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

(semi-integer times) **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \;\Rightarrow\; \underbrace{\widetilde{\Delta}_0^-[\underbrace{a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b}_{@\,(n_0+1/2)\delta\tilde{\eta}}]}_{@\,n_0\delta\tilde{\eta}} = \underbrace{a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}_{@\,n_0\delta\tilde{\eta}}\quad,\ b=1,2,\ldots,N_s$$

(integer times) $\Leftrightarrow$ consistency ✔ **(integer times)**

Lattice Action Approach (Full)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\underbrace{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2}_{@\ (n_0+1/2)\delta\tilde{\eta}} - \underbrace{\frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})}_{@\ n_0\delta\tilde{\eta}}\right\}$$

(semi-integer times) **(integer times)**

$$\delta_{\phi_a}S_{\rm L} = 0 \Rightarrow \boxed{\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}\ ,\ b = 1,2,\ldots,N_s$$

Lattice Action Approach (Full)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\delta\tilde{x}^3\sum_{n_0}\sum_{\mathbf{n}}\left\{\frac{1}{2}a^{3-\alpha}_{+0/2}\sum_b(\widetilde{\Delta}_0^+\tilde{\phi}^b)^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\widetilde{\Delta}_i^+\tilde{\phi}^b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

EOM

$$\boxed{\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}} \quad , \; b = 1,2,\ldots,N_s$$

Lattice Action Approach (Hybrid)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \delta\tilde{\eta}\sum_{n_0}\delta\tilde{x}^3\sum_{\mathbf{n}}\left\{ \qquad ? \qquad \right\}$$

Lattice Action Approach (Hybrid)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}\, d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \underbrace{\delta\tilde{\eta} \sum_{n_0}}_{\int d\tilde{\eta}} \delta\tilde{x}^3 \sum_{\mathbf{n}} \left\{ \qquad ? \qquad \right\}$$

Lattice Action Approach (Hybrid)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}d\tilde{\eta}\left\{\frac{1}{2}a^{3-\alpha}\sum_b \tilde{\phi}_b'^2 - \frac{1}{2}a^{1+\alpha}\sum_{b,i}(\tilde{\nabla}_i\tilde{\phi}_b)^2 - a^{3+\alpha}\widetilde{V}(\{\tilde{\phi}_c\})\right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \int d\tilde{\eta}\sum_{\mathbf{n}}\delta\tilde{x}^3\left\{\qquad ? \qquad\right\}$$

Lattice Action Approach (Hybrid)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x} d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \int d\tilde{\eta} \sum_{\mathbf{n}} \delta \tilde{x}^3 \left\{ \frac{1}{2} a^{3-\alpha} \sum_b (\tilde{\partial}_0 \tilde{\phi}^b)^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\widetilde{\Delta_i^+} \tilde{\phi}^b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice Action Approach (Hybrid)

Continuum:

$$\tilde{S}_{\rm cont} = \int d^3\tilde{x}\, d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice:

$$\widetilde{S}_{\rm L} = \int d\tilde{\eta} \sum_{\mathbf{n}} \delta\tilde{x}^3 \left\{ \underbrace{\frac{1}{2} a^{3-\alpha} \sum_b (\tilde{\partial}_0 \tilde{\phi}^b)^2} - \underbrace{\frac{1}{2} a^{1+\alpha} \sum_{b,i} (\widetilde{\Delta_i^+} \tilde{\phi}^b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\})} \right\}$$

All at the same time

Lattice Action Approach (Hybrid)

Continuum: $$\tilde{S}_{\rm cont} = \int d^3\tilde{x} d\tilde{\eta} \left\{ \frac{1}{2} a^{3-\alpha} \sum_b \tilde{\phi}_b'^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\tilde{\nabla}_i \tilde{\phi}_b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

Lattice: $$\widetilde{S}_{\rm L} = \int d\tilde{\eta} \sum_{\mathbf{n}} \delta\tilde{x}^3 \left\{ \frac{1}{2} a^{3-\alpha} \sum_b (\tilde{\partial}_0 \tilde{\phi}^b)^2 - \frac{1}{2} a^{1+\alpha} \sum_{b,i} (\widetilde{\Delta}_i^+ \tilde{\phi}^b)^2 - a^{3+\alpha} \widetilde{V}(\{\tilde{\phi}_c\}) \right\}$$

EOM

$$\delta_{\phi_a} S_{\rm L} = 0 \;\Rightarrow\; \boxed{\tilde{\partial}_0 \left(a^{3-\alpha} \tilde{\partial}_0 \tilde{\phi}_b \right) = a^{1+\alpha} \sum_i \widetilde{\Delta}_i^- \widetilde{\Delta}_i^+ \tilde{\phi}_b - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}_b}} \;, \; b = 1,2,\ldots,N_s$$

Lattice Formulation

i) ACTION Approach: Discretize the continuum action

ii) EOM Approach: Discretize the continuum EOM

Lattice Formulation

i) ACTION Approach: Discretize the continuum action $\longrightarrow \begin{cases} \text{- Full} \\ \text{- Hybrid} \end{cases}$

$$\boxed{\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^+ \end{pmatrix}$$

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_i & \longrightarrow & \nabla_i^+ \end{pmatrix}$$

ii) EOM Approach: Discretize the continuum EOM

Lattice Formulation

i) ACTION Approach: Discretize the continuum action $\longrightarrow$
- Full
- Hybrid

$$\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^+ \end{pmatrix}$$

$$\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_i & \longrightarrow & \nabla_i^+ \end{pmatrix}$$

ii) EOM Approach: Discretize the continuum EOM

$$\tilde{\phi}_b'' - a^{-2(1-\alpha)}\widetilde{\nabla}^2\tilde{\phi}_b + (3-\alpha)\frac{a'}{a}\tilde{\phi}_b' = -a^{2\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

Lattice Formulation

i) ACTION Approach: Discretize the continuum action $\longrightarrow \begin{cases} \text{- Full} \\ \text{- Hybrid} \end{cases}$

$$\boxed{\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^+ \end{pmatrix}$$

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_i & \longrightarrow & \nabla_i^+ \end{pmatrix}$$

ii) EOM Approach: Discretize the continuum EOM

$$\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\widetilde{\nabla}^2\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

Lattice Formulation

i) ACTION Approach: Discretize the continuum action $\longrightarrow$ $\left\{\begin{array}{l}\textbf{- Full}\\ \textbf{- Hybrid}\end{array}\right.$

$$\boxed{\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix}\text{Continuum} & & \text{Lattice}\\ \partial_\mu & \longrightarrow & \nabla^+_\mu\end{pmatrix}$$

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix}\text{Continuum} & & \text{Lattice}\\ \partial_i & \longrightarrow & \nabla^+_i\end{pmatrix}$$

ii) EOM Approach: Discretize the continuum EOM

$$\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\cancel{\nabla}^2\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

$$\longrightarrow \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\,,$$

Lattice Formulation

i) ACTION Approach: Discretize the continuum action $\longrightarrow$ $\begin{cases} \text{- Full} \\ \text{- Hybrid} \end{cases}$

$$\boxed{\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}} \qquad \begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla_\mu^+ \end{pmatrix}$$

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}} \qquad \begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_i & \longrightarrow & \nabla_i^+ \end{pmatrix}$$

ii) EOM Approach: Discretize the continuum EOM

$$\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\cancel{\widetilde{\nabla}}^2\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

$$\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+,\ \widetilde{\Delta}_i^+\widetilde{\Delta}_i^-,\ \widetilde{\Delta}_i^-\widetilde{\Delta}_i^-,\ \widetilde{\Delta}_i^+\widetilde{\Delta}_i^+,\quad \ldots$$

Lattice Formulation

i) ACTION Approach: Discretize the continuum action $\longrightarrow$

- **Full**
- **Hybrid**

$$\boxed{\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix}\text{Continuum} & & \text{Lattice}\\ \partial_\mu & \longrightarrow & \nabla^+_\mu\end{pmatrix}$$

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix}\text{Continuum} & & \text{Lattice}\\ \partial_i & \longrightarrow & \nabla^+_i\end{pmatrix}$$

ii) EOM Approach: Discretize the continuum EOM

$$\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\widetilde{\cancel{\nabla}}^2\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

$$\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+,\ \widetilde{\Delta}_i^+\widetilde{\Delta}_i^-,\ \widetilde{\Delta}_i^-\widetilde{\Delta}_i^-,\ \widetilde{\Delta}_i^+\widetilde{\Delta}_i^+,\ \ \ldots\ \widetilde{\nabla}^2_{(\mathrm{L})}$$

Lattice Formulation

i) ACTION Approach: Discretize the continuum action $\longrightarrow$ $\begin{cases} \text{- Full} \\ \text{- Hybrid} \end{cases}$

$$\boxed{\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_\mu & \longrightarrow & \nabla^+_\mu \end{pmatrix}$$

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\sum_i \widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_i & \longrightarrow & \nabla^+_i \end{pmatrix}$$

ii) EOM Approach: Discretize the continuum EOM

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\widetilde{\nabla}^2_{(\mathrm{L})}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\begin{pmatrix} \text{Continuum} & & \text{Lattice} \\ \partial_i\partial_i & \longrightarrow & \nabla^2_{(\mathrm{L})} \end{pmatrix}$$

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}^+_0\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^+_k\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b \left\langle (\widetilde{\Delta}^+_0\tilde{\phi}_b)^2 \right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k} \left\langle (\widetilde{\Delta}^+_k\tilde{\phi}_b)^2 \right\rangle, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\}) \right\rangle \quad\right]$$

Where does the scale factor live?

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b \left\langle(\widetilde{\Delta}^+_0\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^+_k\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

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i) IF $a(n_0 + 1/2)$

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\, ; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b \left\langle (\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\, , \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle (\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\, , \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

Cont. **Lattice**

i) IF $a(n_0 + 1/2)$ $\Rightarrow$
$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b\ , \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$$

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b\left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\ \right]$$

Cont. **Lattice**

i) IF $a(n_0 + 1/2)$ $\Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b & \text{, [integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$$

[semi-integer times]

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b \left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\right]$$

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i) IF $a(n_0 + 1/2)$ $\Longrightarrow$

	Cont.		**Lattice**
	a'	$\longrightarrow$	$\widetilde{\Delta}_0^+ a_{-0/2} \equiv b$, **[integer times]**
	a''	$\longrightarrow$	$\widetilde{\Delta}_0^+ b = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right]$

[semi-integer times]

$\overline{\widetilde{E}}_G \to \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2}$ $\quad$ $\overline{\widetilde{E}}_V \to \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$

consistency ✓

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b \left\langle (\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle (\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\})\right\rangle\ \right]$$

Cont. **Lattice**

i) IF $a(n_0 + 1/2)$ $\Rightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b \ , \text{ [integer times]} \\[2ex] a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}_G} + (\alpha+1)\overline{\widetilde{E}_V}\right] \end{cases}$$

[semi-integer times]

$$\overline{\widetilde{E}_G} \to \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \overline{\widetilde{E}_V} \to \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,,\quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\right]$$

Cont. **Lattice**

i) IF $a(n_0+1/2)$ $\Longrightarrow$

$a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b$, **[integer times]**

$a'' \longrightarrow \widetilde{\Delta}_0^+ b = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}_{+0/2}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}_G} + (\alpha+1)\overline{\widetilde{E}_V}\right]$

[semi-integer times]

$\overline{\widetilde{E}_G} = \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2}$, $\overline{\widetilde{E}_V} = \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$

ii) IF $a(n_0)$ $\Longrightarrow$

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b \left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle , \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle , \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\right]$$

Cont. **Lattice**

i) IF $a(n_0 + 1/2) \Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b \text{ , [integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}_G} + (\alpha+1)\overline{\widetilde{E}_V}\right] \end{cases}$$

[semi-integer times]

$$\overline{\widetilde{E}_G} \rightarrow \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2}, \qquad \overline{\widetilde{E}_V} \rightarrow \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

ii) IF $a(n_0) \Longrightarrow$

$$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a \equiv b_{+0/2} , \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}$$

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

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Cont. **Lattice**

i) IF $a(n_0 + 1/2)$ $\Longrightarrow$ $$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b \text{ , [integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] \end{cases}$$

[semi-integer times]

$$\overline{\widetilde{E}}_G \to \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2}, \qquad \overline{\widetilde{E}}_V \to \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

ii) IF $a(n_0)$ $\Longrightarrow$ $$\begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a \equiv b_{+0/2} \text{ , [semi-integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}$$

[integer times]

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \quad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b\left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\ \right]$$

Cont. **Lattice**

i) IF $a(n_0 + 1/2) \quad \Longrightarrow \quad \begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a_{-0/2} \equiv b\,, & \textbf{[integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right] & \end{cases}$

[semi-integer times]

$$\overline{\widetilde{E}}_G = \frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \overline{\widetilde{E}}_V = \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

ii) IF $a(n_0) \quad \Longrightarrow \quad \begin{cases} a' \longrightarrow \widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,, & \textbf{[semi-integer times]} \\ a'' \longrightarrow \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] & \end{cases}$

[integer times]

$\Longleftarrow$ consistency ✓ $\Longrightarrow$

$$\overline{\widetilde{E}}_K = \frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}$$

Expansion of the Universe

Cont: $$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \qquad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}_0^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}_k^+\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\quad\right]$$

In summary, depending where a(t) lives:

Expansion of the Universe

Cont:
$$\boxed{\left(\frac{a'}{a}\right)^2 = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right]\,; \qquad \frac{a''}{a} = \frac{a^{2\alpha}}{3}\left(\frac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]}$$

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In summary, depending where a(t) lives:

Expansion of the Universe

i) IF $a(n_0 + 1/2)$ $\Longrightarrow$

$$\boxed{\begin{cases} \widetilde{\Delta}_0^+ a_{-0/2} \equiv b\,, \\ \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}_G} + (\alpha+1)\overline{\widetilde{E}_V}\right] \end{cases}}$$

$$\boxed{\frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2}} \qquad \boxed{\frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}}$$

ii) IF $a(n_0)$ $\Longrightarrow$

$$\boxed{\begin{cases} \widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,, \\ \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\,\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}}$$

$$\boxed{\frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}}$$

Putting all together …

Expansion of the Universe

i) IF $a(n_0 + 1/2)$ $\Longrightarrow$

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$$\frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}$$

Putting all together ...

Scalar Fld. Dynamics

(Full)

$$\widetilde{\Delta}_0^-[a^{3-\alpha}_{+0/2}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

$$\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

(Hybrid)

ACTION Approach

$$\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\widetilde{\nabla}^2_{(\mathrm{L})}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

EOM Approach

Expansion of the Universe

i) IF $a(n_0+1/2)$ $\Longrightarrow$

$$\widetilde{\Delta}_0^+ a_{-0/2} \equiv b\,,$$

$$\widetilde{\Delta}_0^+ b = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}_{+0/2}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}}_G + (\alpha+1)\overline{\widetilde{E}}_V\right]$$

$$\frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

ii) IF $a(n_0)$ $\Longrightarrow$

$$\widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,,$$

$$\widetilde{\Delta}_0^+ b_{-0/2} = \frac{1}{6}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}}_K + \alpha\,\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]$$

$$\frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}$$

Putting all together …

Scalar Fld. Dynamics

(Full)

$$\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

ACTION Approach

Expansion of the Universe

i) IF $a(n_0 + 1/2)$ $\Longrightarrow$

$$\begin{cases} \widetilde{\Delta}_0^+ a_{-0/2} \equiv b\ , \\ \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}_G} + (\alpha+1)\overline{\widetilde{E}_V}\right] \end{cases}$$

$$\frac{(\widetilde{E}_G + \widetilde{E}_{G,+\hat{0}})}{2} \qquad \frac{(\widetilde{E}_V + \widetilde{E}_{V,+\hat{0}})}{2}$$

Putting all together ...

Scalar Fld. Dynamics

(Full)

$$\boxed{\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

$$\boxed{\frac{(a_{-0/2} + a_{+0/2})}{2}}$$

ACTION Approach

Expansion of the Universe

i) IF $a(n_0 + 1/2)$ $\Longrightarrow$

$$\boxed{\begin{cases} \widetilde{\Delta}_0^+ a_{-0/2} \equiv b\ , \\ \widetilde{\Delta}_0^+ b = \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0/2}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\overline{\widetilde{E}_G} + (\alpha+1)\overline{\widetilde{E}_V}\right] \end{cases}}$$

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Putting all together ...

Scalar Fld. Dynamics

(Full)

$$\boxed{\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

ACTION Approach

Expansion of the Universe

ii) IF $a(n_0)$ $\Longrightarrow$

$$\boxed{\begin{cases} \widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,, \\ \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}}_K + \alpha\,\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}}$$

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Putting all together ...

Scalar Fld. Dynamics

(Full)

$$\widetilde{\Delta}_0^-[a_{+0/2}^{3-\alpha}\widetilde{\Delta}_0^+\tilde{\phi}_b] = a^{1+\alpha}\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}$$

$$\frac{(a+a_{+0})}{2}$$

ACTION Approach

Expansion of the Universe

ii) IF $a(n_0) \Rightarrow$

$$\begin{cases} \widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,, \\ \widetilde{\Delta}_0^+ b_{-0/2} = \frac{1}{6}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}}_K + \alpha\,\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}$$

$$\frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}$$

Putting all together ...

Scalar Fld. Dynamics

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\widetilde{\Delta}_i^-\widetilde{\Delta}_i^+\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

(Hybrid)

ACTION Approach

Expansion of the Universe

ii) IF $a(n_0)$ $\Rightarrow$

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Putting all together …

Scalar Fld. Dynamics

$$\boxed{\tilde{\partial}_0\left(a^{3-\alpha}\tilde{\partial}_0\tilde{\phi}_b\right) = a^{1+\alpha}\widetilde{\nabla}^2_{(\mathrm{L})}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}$$

EOM Approach

Expansion of the Universe

ii) IF $a(n_0)$ $\Longrightarrow$

$$\boxed{\begin{cases} \widetilde{\Delta}_0^+ a \equiv b_{+0/2}\,, \\ \widetilde{\Delta}_0^+ b_{-0/2} = \dfrac{1}{6}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\,\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right] \end{cases}}$$

$$\boxed{\frac{(\widetilde{E}_{K,-\hat{0}/2} + \widetilde{E}_{K,+\hat{0}/2})}{2}}$$

How do we solve these EOM ??

… We apply previous evolution algorithms !

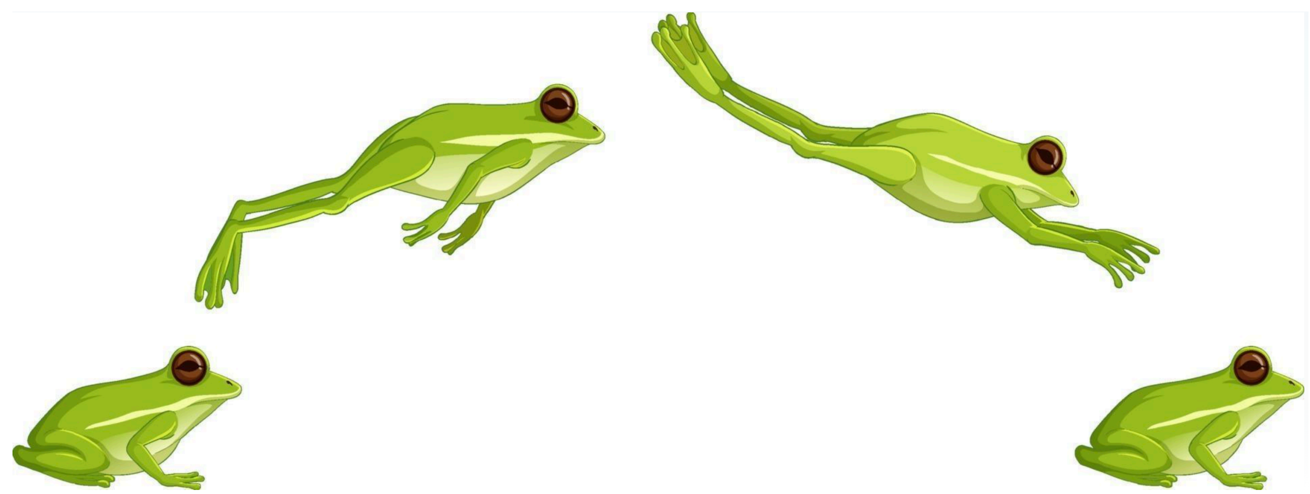
* LeapFrog
* Verlet methods
* Runge-Kutta methods
* Higher Order integrators

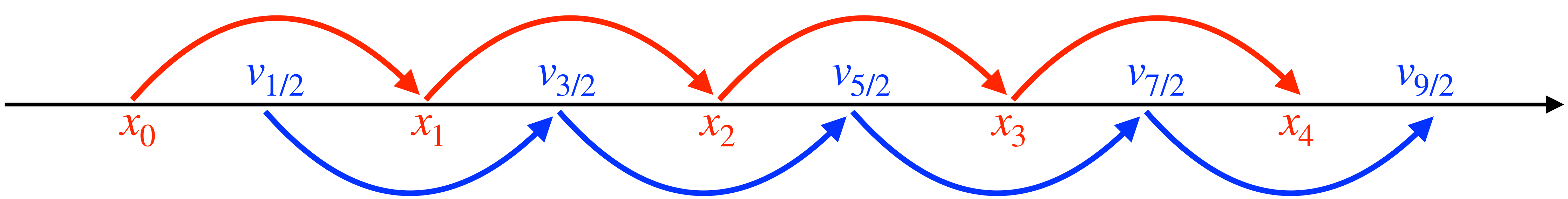
Notation

$$\phi_{+0} \equiv \phi(\mathbf{n}, n_0 + 1)$$

$$\pi_{+0/2} \equiv \pi(\mathbf{n}, n_0 + 1/2)$$

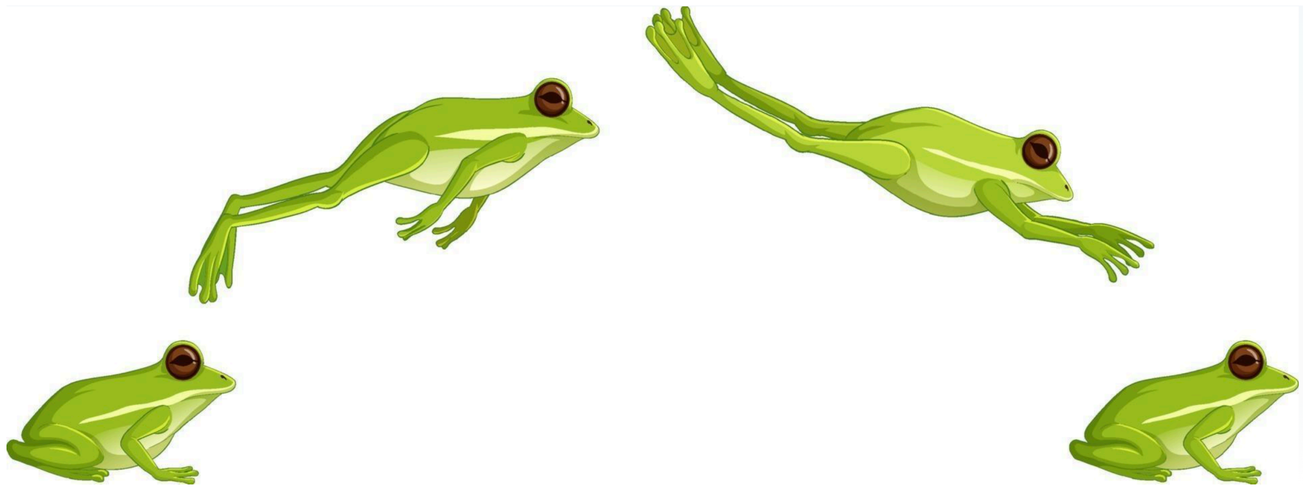
(Staggered) LeapFrog

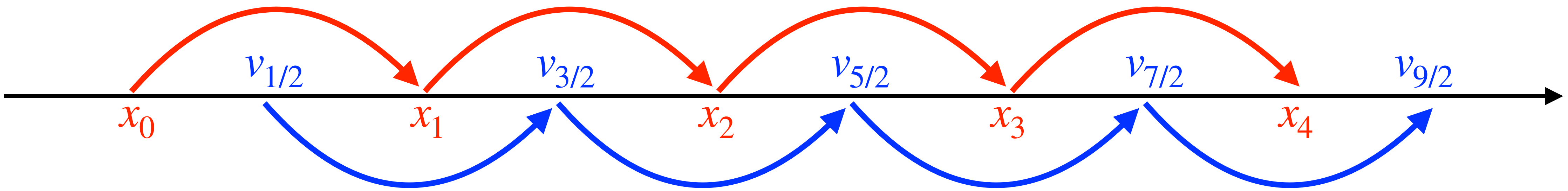




(Staggered) LeapFrog

Order dt^2





(Staggered) LeapFrog

$$\textit{Iterative scheme for } \tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a \text{ and scale factor } a(n_0)$$

(Staggered) LeapFrog

$$\textit{Iterative scheme for } \tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a \text{ and scale factor } a(n_0)$$

(@ semi-integer times)

(@ integer times)

(Staggered) LeapFrog

Iterative scheme for $\underline{\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2} \widetilde{\Delta}^{+}_{0} \tilde{\phi}_a}$ and $\underline{\text{scale factor } a(n_0)}$

(@ semi-integer times) (@ integer times)

$\boxed{\text{e.g. ACTION Approach (Full)}}$

(Staggered) LeapFrog

Iterative scheme for $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

EOM: $\widetilde{\Delta}^{-}_{0}[a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b] = a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$

e.g. **ACTION Approach (Full)**

(Staggered) LeapFrog

$$\textit{Iterative scheme for } \tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a \text{ and scale factor } a(n_0)$$

EOM: $$\widetilde{\Delta}^{-}_{0}[\underbrace{a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b}_{\equiv\, \tilde{\pi}^{(b)}_{+0/2}}] = a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

(canonical variables)

(Staggered) LeapFrog

$Iterative\ scheme\ for\ \tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

EOM: $$\widetilde{\Delta}^{-}_{0}[\underbrace{a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b}_{\equiv\, \tilde{\pi}^{(b)}_{+0/2}}] = \underbrace{a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}}_{\mathscr{K}_{\tilde{\phi}_a}[\{\tilde{\phi}^{(b)}\},\, a]}\, ; \qquad b = 1,2,\ldots,N_s$$

(canonical variables)

Kernel Conservative !

(Staggered) LeapFrog

Iterative scheme for $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2} \widetilde{\Delta}^{+}_{0} \tilde{\phi}_a$ and scale factor $a(n_0)$

EOM: $$\widetilde{\Delta}^{-}_{0}[\underbrace{a^{3-\alpha}_{+0/2} \widetilde{\Delta}^{+}_{0} \tilde{\phi}_b}_{\equiv \tilde{\pi}^{(b)}_{+0/2}}] = \underbrace{a^{1+\alpha} \sum_i \widetilde{\Delta}^{-}_{i} \widetilde{\Delta}^{+}_{i} \tilde{\phi}_b - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}_b}}_{\mathscr{K}_{\tilde{\phi}_a}[\{\tilde{\phi}^{(b)}\}, a]} ; \qquad b = 1,2,\ldots,N_s$$

(canonical variables)

Kernel Conservative !

$$\widetilde{\Delta}^{-}_{0}[\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha} \sum_i \widetilde{\Delta}^{-}_{i} \widetilde{\Delta}^{+}_{i} \tilde{\phi}_b - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}_b} ; \qquad b = 1,2,\ldots,N_s$$

(Staggered) LeapFrog

$\textit{Iterative scheme for } \tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2} \widetilde{\Delta}_0^+ \tilde{\phi}_a \text{ and scale factor } a(n_0)$

$$\widetilde{\Delta}_0^-[\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha} \sum_i \widetilde{\Delta}_i^- \widetilde{\Delta}_i^+ \tilde{\phi}_b - a^{3+\alpha} \widetilde{V}_{,\tilde{\phi}_b} ; \qquad b = 1,2,\ldots,N_s$$

(Staggered) LeapFrog

Iterative scheme for $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

$IC \quad : \quad \{\tilde{\phi}^{(a)}, a, \}$ at $\tilde{\eta}_0$, $\quad \{\tilde{\pi}^{(a)}_{-0/2}, b_{-0/2}\}$ at $\tilde{\eta}_0 - 0.5\delta\tilde{\eta}$

$$\widetilde{\Delta}^{-}_{0}[\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

(Staggered) LeapFrog

$$\textit{Iterative scheme for } \tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a \text{ and scale factor } a(n_0)$$

$$IC \quad : \quad \{\tilde{\phi}^{(a)}, a, \} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

$$\begin{cases} \tilde{\pi}^{(a)}_{+0/2} = \tilde{\pi}^{(a)}_{-0/2} + \left(a^{1+\alpha} \sum_k \widetilde{\Delta}^{-}_{k}\widetilde{\Delta}^{+}_{k}\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}} \right)\delta\tilde{\eta} \\ \\ \end{cases}$$

$$\widetilde{\Delta}^{-}_{0}[\tilde{\pi}^{(b)}_{+0/2}] = a^{1+\alpha}\sum_i \widetilde{\Delta}^{-}_{i}\widetilde{\Delta}^{+}_{i}\tilde{\phi}_b - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}_b}\,; \qquad b = 1,2,\ldots,N_s$$

(Staggered) LeapFrog

Iterative scheme for $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a$ and scale factor $a(n_0)$

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$$\begin{cases} \tilde{\pi}^{(a)}_{+0/2} & = \tilde{\pi}^{(a)}_{-0/2} + \left(a^{1+\alpha}\sum_k \widetilde{\Delta}^{-}_{k}\widetilde{\Delta}^{+}_{k}\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}}\right)\delta\tilde{\eta} \\ b_{+0/2} & = b_{-0/2} + \frac{\delta\tilde{\eta}}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \end{cases}$$

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b \left\langle(\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^{+}_{k}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

(Staggered) LeapFrog

$$\textit{Iterative scheme for } \tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a \text{ and scale factor } a(n_0)$$

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$$\begin{cases} \tilde{\pi}^{(a)}_{+0/2} = \tilde{\pi}^{(a)}_{-0/2} + \left(a^{1+\alpha}\sum_k \widetilde{\Delta}^{-}_{k}\widetilde{\Delta}^{+}_{k}\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}}\right)\delta\tilde{\eta} \\ b_{+0/2} = b_{-0/2} + \frac{\delta\tilde{\eta}}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \end{cases}$$

$$\begin{cases} a_{+0} = a + b_{+0/2}\,\delta\tilde{\eta} \quad \longrightarrow \quad a_{+0/2} \equiv (a_{+0} + a_0)/2\,, \\ \tilde{\phi}^{(a)}_{+0} = \tilde{\phi}^{(a)} + \delta\tilde{\eta}\,\tilde{\pi}^{(a)}_{+0/2}a^{-(3-\alpha)}_{+0/2}\,, \end{cases}$$

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^{+}_{k}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\ \right]$$

(Staggered) LeapFrog

$$\textit{Iterative scheme for } \tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^{+}_{0}\tilde{\phi}_a \text{ and scale factor } a(n_0)$$

$$IC \quad : \quad \{\tilde{\phi}^{(a)}, a,\} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

$$\begin{cases} \tilde{\pi}^{(a)}_{+0/2} &= \tilde{\pi}^{(a)}_{-0/2} + \left(a^{1+\alpha} \sum_k \widetilde{\Delta}^{-}_{k}\widetilde{\Delta}^{+}_{k}\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}} \right)\delta\tilde{\eta} \\ b_{+0/2} &= b_{-0/2} + \dfrac{\delta\tilde{\eta}}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \end{cases}$$

$$\begin{cases} a_{+0} &= a + b_{+0/2}\,\delta\tilde{\eta} \quad \longrightarrow \quad a_{+0/2} \equiv (a_{+0} + a_0)/2\,, \\ \tilde{\phi}^{(a)}_{+0} &= \tilde{\phi}^{(a)} + \delta\tilde{\eta}\,\tilde{\pi}^{(a)}_{+0/2}a^{-(3-\alpha)}_{+0/2}\,, \end{cases}$$

$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b \left\langle(\widetilde{\Delta}^{+}_{0}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^{+}_{k}\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

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Iterative scheme for $\tilde{\pi}^{(a)}_{+0/2} \equiv a^{3-\alpha}_{+0/2}\widetilde{\Delta}^+_0\tilde{\phi}_a$ and scale factor $a(n_0)$

$$IC \quad : \quad \{\tilde{\phi}^{(a)}, a, \} \text{ at } \tilde{\eta}_0, \quad \{\tilde{\pi}^{(a)}_{-0/2}, b_{-0/2}\} \text{ at } \tilde{\eta}_0 - 0.5\delta\tilde{\eta}$$

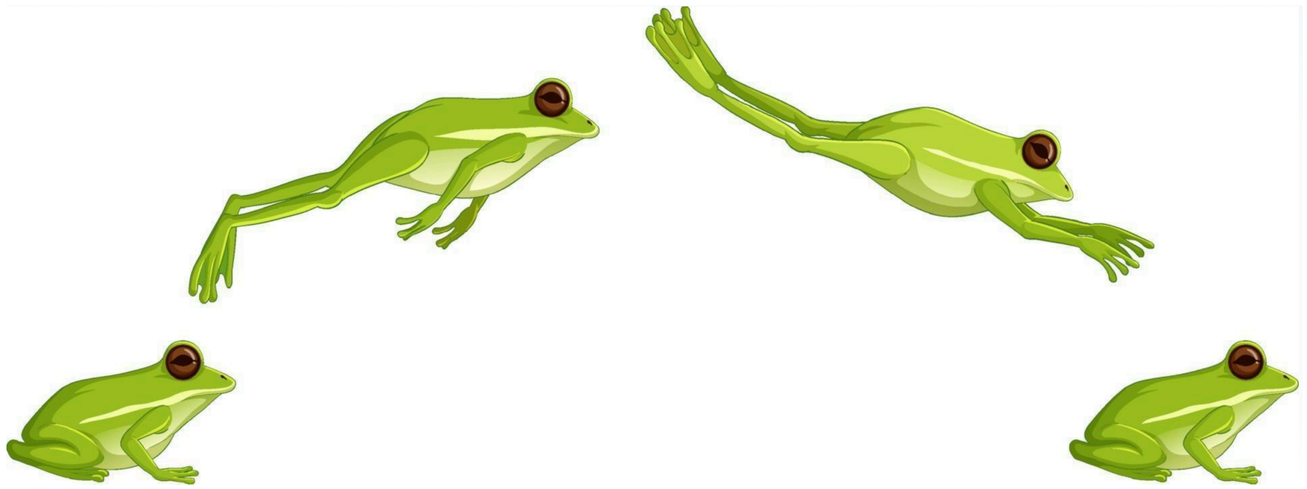
$$\left\{\begin{aligned}
\tilde{\pi}^{(a)}_{+0/2} &= \tilde{\pi}^{(a)}_{-0/2} + \left(a^{1+\alpha}\sum_k \widetilde{\Delta}^-_k\widetilde{\Delta}^+_k\tilde{\phi}^{(a)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(a)}}\right)\delta\tilde{\eta} \\
b_{+0/2} &= b_{-0/2} + \frac{\delta\tilde{\eta}}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right],
\end{aligned}\right.$$

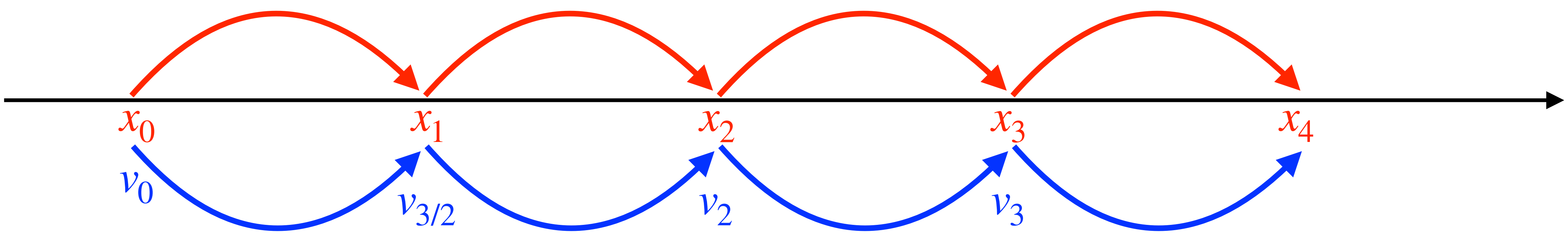
$$\left\{\begin{aligned}
a_{+0} &= a + b_{+0/2}\,\delta\tilde{\eta} \quad \longrightarrow \quad a_{+0/2} \equiv (a_{+0} + a_0)/2\,, \\
\tilde{\phi}^{(a)}_{+0} &= \tilde{\phi}^{(a)} + \delta\tilde{\eta}\,\tilde{\pi}^{(a)}_{+0/2}a^{-(3-\alpha)}_{+0/2}\,,
\end{aligned}\right.$$

$$HC \quad : \quad b^2_{+0/2} = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}_{+0/2}\left(\widetilde{E}_K + \overline{\widetilde{E}_G} + \overline{\widetilde{E}_V}\right),$$

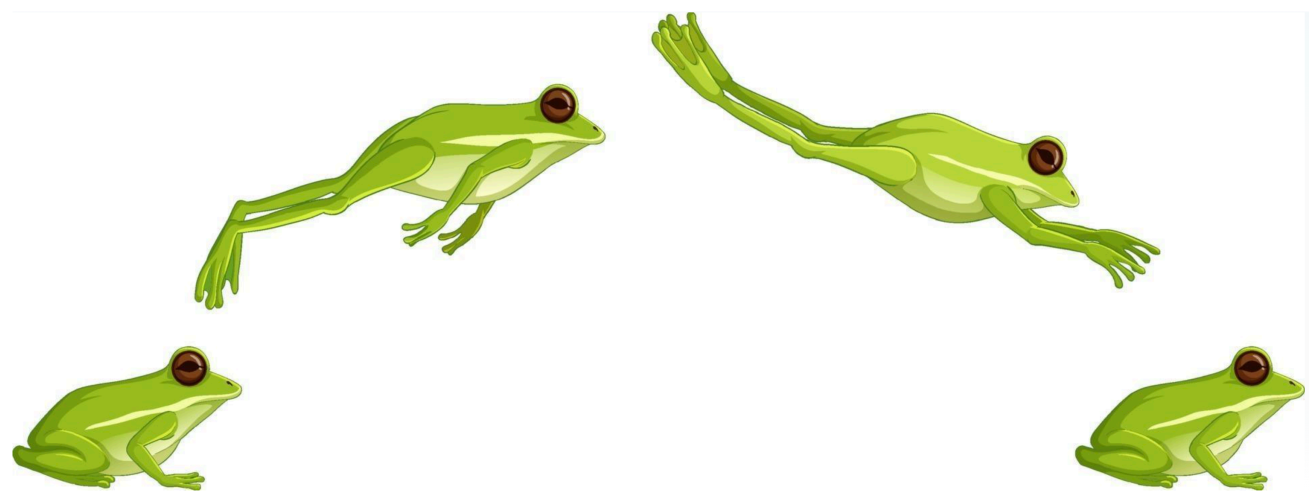
$$\left[\quad \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}^+_0\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^+_k\tilde{\phi}_b)^2\right\rangle, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle \quad\right]$$

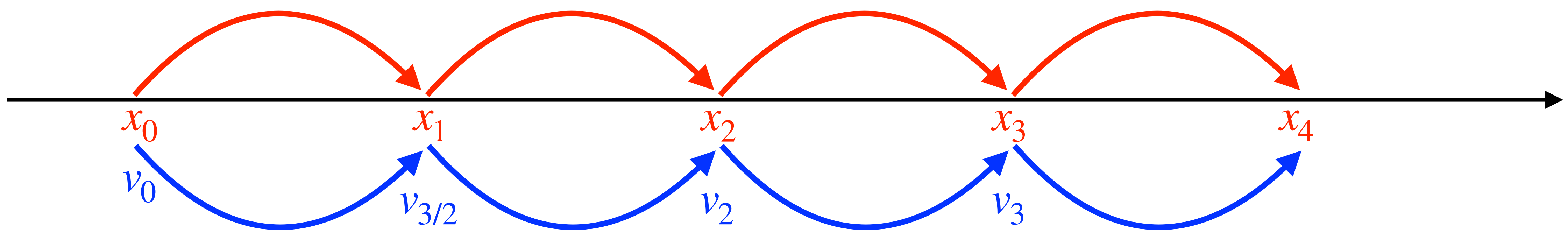
‘Synchronized' LeapFrog





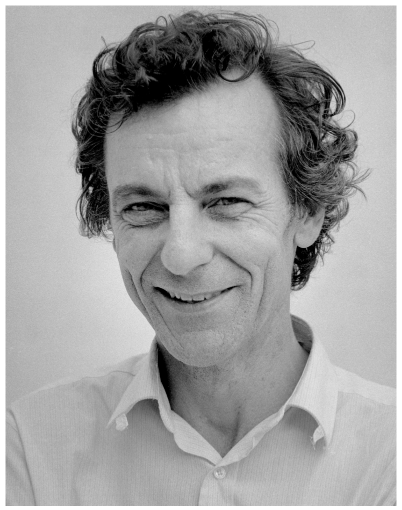
‘Synchronized' LeapFrog



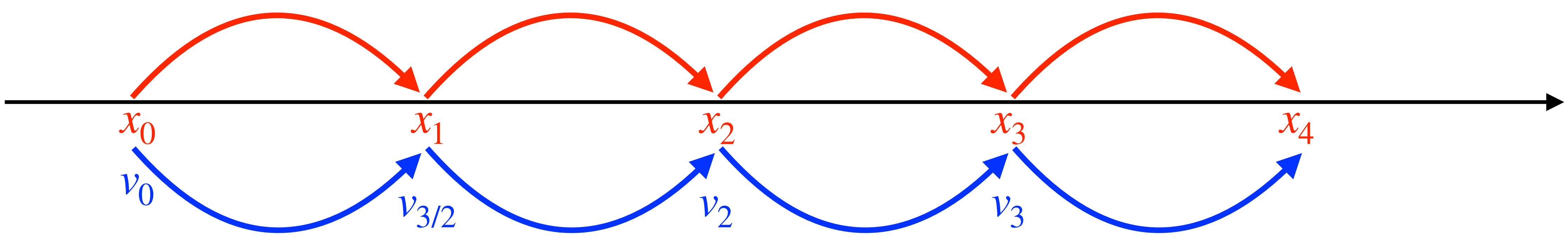


‘Synchronized' LeapFrog
Verlet Algorithms

Loup Verlet



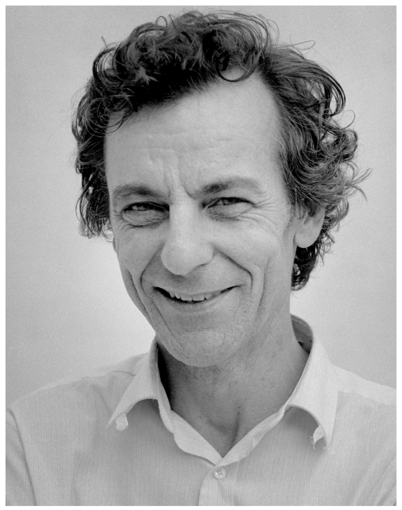
RIP 13th/06/2019



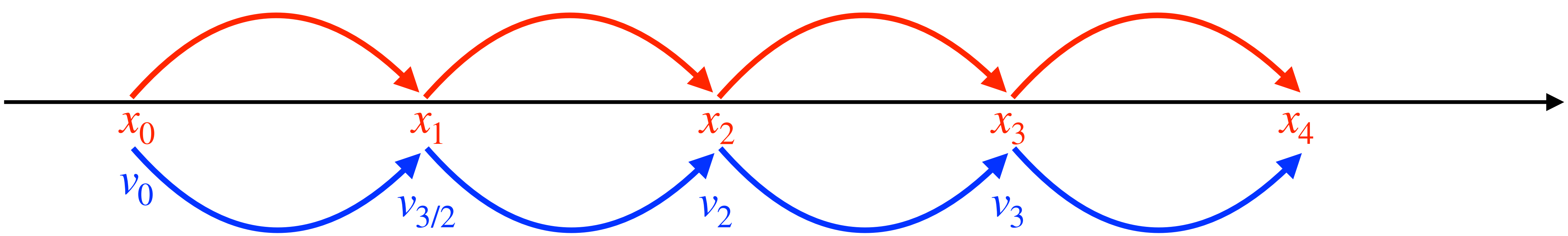
'Synchronized' LeapFrog
Verlet Algorithms

Order dt^2

Loup Verlet



RIP 13th/06/2019



Verlet Algorithms

Continuum:

Fld dynamics:

$$(a^{(3-\alpha)}\tilde{\phi}_i')' - a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i + a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} = 0\,, \qquad i = 1,2,\ldots,N_s$$

Verlet Algorithms

Continuum:

Fld dynamics:

$$\underbrace{(a^{(3-\alpha)}\tilde{\phi}_i')' - a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i + a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} = 0}\,, \qquad i = 1,2,\ldots,N_s$$

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \end{cases}$$

Verlet Algorithms

Continuum:

Fld dynamics:

$$\underbrace{(a^{(3-\alpha)}\tilde{\phi}_i')' - a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i + a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} = 0}\,, \qquad i = 1,2,\ldots,N_s$$

$$\left\{\begin{aligned} \tilde{\phi}_i' &= \overbrace{a^{-(3-\alpha)}\tilde{\pi}_i}^{\mathcal{D}_{\tilde{\phi}_i}(\tilde{\pi}_i,\, a)\ \textbf{(Drift)}}\,, \\ \tilde{\pi}_i' &= \underbrace{a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}}_{\textbf{(Kernel)}\ \mathcal{K}_{\phi_i}(\{\phi_j\},\, a)} \end{aligned}\right\}$$

Verlet Algorithms

Continuum:

Fld dynamics:

$$\underbrace{(a^{(3-\alpha)}\tilde{\phi}_i')' - a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i + a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} = 0}\,, \qquad i = 1,2,\ldots,N_s$$

$$\left\{\begin{array}{l} \tilde{\phi}_i' = \overbrace{a^{-(3-\alpha)}\tilde{\pi}_i,}^{\mathscr{D}_{\tilde{\phi}_i}(\tilde{\pi}_i,\, a)\ \textbf{(Drift)}} \\ \tilde{\pi}_i' = \underbrace{a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}}_{\textbf{(Kernel)}\ \mathscr{K}_{\phi_i}(\{\phi_j\},\, a)} \end{array}\right\} \Rightarrow \textbf{Conservative}$$

Verlet Algorithms

Continuum:

Fld dynamics $(\tilde{\pi}_i \equiv a^{(3-\alpha)}\tilde{\phi}_i')$ **:**

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = a^{1+\alpha}\,\tilde{\nabla}^{\,2}\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \end{cases} \qquad i = 1,2,\ldots,N_s$$

Verlet Algorithms

Continuum:

Fld dynamics $(\tilde{\pi}_i \equiv a^{(3-\alpha)}\tilde{\phi}_i')$ **:**

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = a^{1+\alpha}\,\tilde{\nabla}^{\,2}\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \end{cases} \qquad i = 1,2,\ldots,N_s$$

Exp. Background:

$$\begin{cases} a' = b\,, \\ b' = \dfrac{a^{1+2\alpha}}{3}\left(\dfrac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \dfrac{1}{2a^6}\displaystyle\sum_i \left\langle(\tilde{\pi}_i)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \dfrac{1}{2a^2}\displaystyle\sum_{i,k}\left\langle(\tilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle \tilde{V}(\{\tilde{\phi}_j\})\right\rangle\,, \end{cases}$$

Verlet Algorithms

position-Verlet **velocity-Verlet**

Continuum:

Fld dynamics $(\tilde{\pi}_i \equiv a^{(3-\alpha)}\tilde{\phi}_i')$ **:**

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = a^{1+\alpha}\,\tilde{\nabla}^{\,2}\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \end{cases} \qquad i = 1,2,\ldots,N_s$$

Exp. Background:

$$\begin{cases} a' = b\,, \\ b' = \dfrac{a^{1+2\alpha}}{3}\left(\dfrac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \dfrac{1}{2a^6}\displaystyle\sum_i\left\langle(\tilde{\pi}_i)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \dfrac{1}{2a^2}\displaystyle\sum_{i,k}\left\langle(\tilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_j\})\right\rangle\,, \end{cases}$$

Verlet Algorithms

position-Verlet | **velocity-Verlet**

Continuum:

Fld dynamics $(\tilde{\pi}_i \equiv a^{(3-\alpha)}\tilde{\phi}_i')$ **:**

$$\begin{cases} \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' = a^{1+\alpha}\,\tilde{\nabla}^{\,2}\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}} \end{cases} \quad i = 1,2,\ldots N$$

Exp.

$$\begin{cases} a' = \ldots, \\ b' = \dfrac{a^{1+2\alpha}}{3}\left(\dfrac{f_*}{m_p}\right)^2\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \dfrac{1}{2a^6}\displaystyle\sum_i\left\langle(\tilde{\pi}_i)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \dfrac{1}{2a^2}\displaystyle\sum_{i,k}\left\langle(\tilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_j\})\right\rangle\,, \end{cases}$$

Lattice EOM Approach

Velocity-Verlet Integration

I) Velocity-Verlet scheme for interacting scalar fields in an expanding background

$$IC \quad : \quad \{\tilde{\phi}^{(i)}, \tilde{\pi}^{(i)}, a, b\} \text{ at } \tilde{\eta}_0\,,$$

Velocity-Verlet Integration

I) Velocity-Verlet scheme for interacting scalar fields in an expanding background

$$IC \quad : \quad \{\tilde{\phi}^{(i)}, \tilde{\pi}^{(i)}, a, b\} \text{ at } \tilde{\eta}_0\,,$$

$$\tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \qquad\qquad \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,,$$

Velocity-Verlet Integration

I) Velocity-Verlet scheme for interacting scalar fields in an expanding background

$$IC \quad : \quad \{\tilde{\phi}^{(i)}, \tilde{\pi}^{(i)}, a, b\} \text{ at } \tilde{\eta}_0 \,,$$

$$\begin{cases} b_{+0/2} = b + \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]\frac{\delta\tilde{\eta}}{2} \,, \\ \tilde{\pi}^{(i)}_{+0/2} = \tilde{\pi}^{(i)} + \left(a^{1+\alpha}\sum_k \widetilde{\Delta}_k^- \widetilde{\Delta}_k^+ \tilde{\phi}^{(i)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(i)}}\right)\frac{\delta\tilde{\eta}}{2} \,, \end{cases}$$

$$\tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \qquad\qquad \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i \,,$$

Velocity-Verlet Integration

I) Velocity-Verlet scheme for interacting scalar fields in an expanding background

$$
\begin{aligned}
IC \quad &: \quad \{\tilde{\phi}^{(i)}, \tilde{\pi}^{(i)}, a, b\} \text{ at } \tilde{\eta}_0\,, \\
\begin{cases} b_{+0/2} = b + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]\dfrac{\delta\tilde{\eta}}{2}\,, \\
\tilde{\pi}^{(i)}_{+0/2} = \tilde{\pi}^{(i)} + \left(a^{1+\alpha}\displaystyle\sum_k \widetilde{\Delta}^-_k\widetilde{\Delta}^+_k\tilde{\phi}^{(i)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(i)}}\right)\dfrac{\delta\tilde{\eta}}{2}\,, \end{cases} \\
\begin{cases} a_{+0} = a + b_{+0/2}\delta\tilde{\eta}\,, \\
\left[a_{+0/2} = \dfrac{a_{+0}+a}{2}\,,\right] \\
\tilde{\phi}^{(i)}_{+0} = \tilde{\phi}^{(i)} + \delta\tilde{\eta}\,\tilde{\pi}^{(i)}_{+0/2}a^{-(3-\alpha)}_{+0/2}\,, \end{cases}
\end{aligned}
$$

$$\tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \qquad\qquad \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,,$$

Velocity-Verlet Integration

I) Velocity-Verlet scheme for interacting scalar fields in an expanding background

$$IC \quad : \quad \{\tilde{\phi}^{(i)}, \tilde{\pi}^{(i)}, a, b\} \text{ at } \tilde{\eta}_0\,,$$

$$\begin{cases} b_{+0/2} = b + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]\dfrac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}^{(i)}_{+0/2} = \tilde{\pi}^{(i)} + \left(a^{1+\alpha}\displaystyle\sum_k \widetilde{\Delta}^-_k\widetilde{\Delta}^+_k\tilde{\phi}^{(i)} - a^{3+\alpha}\widetilde{V}_{,\tilde{\phi}^{(i)}}\right)\dfrac{\delta\tilde{\eta}}{2}\,, \end{cases}$$

$$\begin{cases} a_{+0} = a + b_{+0/2}\delta\tilde{\eta}\,, \\ \left[a_{+0/2} = \dfrac{a_{+0}+a}{2}\,,\right] \\ \tilde{\phi}^{(i)}_{+0} = \tilde{\phi}^{(i)} + \delta\tilde{\eta}\,\tilde{\pi}^{(i)}_{+0/2}a^{-(3-\alpha)}_{+0/2}\,, \end{cases}$$

$$\begin{cases} \tilde{\pi}^{(i)}_{+0} = \tilde{\pi}^{(i)}_{+0/2} + \left(a^{1+\alpha}_{+0}\displaystyle\sum_k \widetilde{\Delta}^-_k\widetilde{\Delta}^+_k\tilde{\phi}^{(i)}_{+0} - a^{3+\alpha}_{+0}\widetilde{V}_{,\tilde{\phi}^{(i)}}\Big|_{+0}\right)\dfrac{\delta\tilde{\eta}}{2}\,, \\ b_{+0} = b_{+0/2} + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}_{+0}\left[(\alpha-2)\widetilde{E}_{K,+0} + \alpha\widetilde{E}_{G,+0} + (\alpha+1)\widetilde{E}_{V,+0}\right]\dfrac{\delta\tilde{\eta}}{2}\,, \end{cases}$$

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}_{+0/2}}\sum_b\left\langle(\widetilde{\Delta}^+_0\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde{\Delta}^+_k\tilde{\phi}_b)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\tilde{V}(\{\tilde{\phi}_b\})\right\rangle\ \right]$$

Velocity-Verlet Integration

I) Velocity-Verlet scheme for interacting scalar fields in an expanding background

$$IC \quad : \quad \{\tilde\phi^{(i)}, \tilde\pi^{(i)}, a, b\} \text{ at } \tilde\eta_0\,,$$

$$\begin{cases} b_{+0/2} = b + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right]\dfrac{\delta\tilde\eta}{2}\,, \\ \tilde\pi^{(i)}_{+0/2} = \tilde\pi^{(i)} + \left(a^{1+\alpha}\displaystyle\sum_k \widetilde\Delta_k^-\widetilde\Delta_k^+\tilde\phi^{(i)} - a^{3+\alpha}\widetilde{V}_{,\tilde\phi^{(i)}}\right)\dfrac{\delta\tilde\eta}{2}\,, \end{cases}$$

$$\begin{cases} a_{+0} = a + b_{+0/2}\delta\tilde\eta\,, \\ \left[a_{+0/2} = \dfrac{a_{+0}+a}{2}\,,\right] \\ \tilde\phi^{(i)}_{+0} = \tilde\phi^{(i)} + \delta\tilde\eta\,\tilde\pi^{(i)}_{+0/2}a_{+0/2}^{-(3-\alpha)}\,, \end{cases}$$

$$\begin{cases} \tilde\pi^{(i)}_{+0} = \tilde\pi^{(i)}_{+0/2} + \left(a_{+0}^{1+\alpha}\displaystyle\sum_k \widetilde\Delta_k^-\widetilde\Delta_k^+\tilde\phi^{(i)}_{+0} - a_{+0}^{3+\alpha}\widetilde{V}_{,\tilde\phi^{(i)}}\Big|_{+0}\right)\dfrac{\delta\tilde\eta}{2}\,, \\ b_{+0} = b_{+0/2} + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a_{+0}^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_{K,+0} + \alpha\widetilde{E}_{G,+0} + (\alpha+1)\widetilde{E}_{V,+0}\right]\dfrac{\delta\tilde\eta}{2}\,, \end{cases}$$

$$HC \quad : \quad b^2 = \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{2(\alpha+1)}\left(\widetilde{E}_K + \widetilde{E}_G + \widetilde{E}_V\right),$$

$$\left[\ \widetilde{E}_K \equiv \frac{1}{2a_{+0/2}^{2\alpha}}\sum_b\left\langle(\widetilde\Delta_0^+\tilde\phi_b)^2\right\rangle\,,\quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{b,k}\left\langle(\widetilde\Delta_k^+\tilde\phi_b)^2\right\rangle\,,\quad \widetilde{E}_V \equiv \left\langle\tilde V(\{\tilde\phi_b\})\right\rangle\ \right]$$

Velocity-Verlet Integration

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$$IC \quad : \quad \{\tilde\phi^{(i)}, \tilde\pi^{(i)}, a, b\} \text{ at } \tilde\eta_0\,,$$

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$$\begin{cases} a_{+0} = a + b_{+0/2}\delta\tilde\eta\,, \\ \left[a_{+0/2} = \dfrac{a_{+0}+a}{2}\,,\right] \\ \tilde\phi^{(i)}_{+0} = \tilde\phi^{(i)} + \delta\tilde\eta\,\tilde\pi^{(i)}_{+0/2}a_{+0/2}^{-(3-\alpha)}\,, \end{cases}$$

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Position-Verlet Integration

II) Position-Verlet scheme for interacting scalar fields in an expanding background

$$IC \quad : \quad \{\tilde{\phi}^{(i)}, \tilde{\pi}^{(i)}, a, b\} \text{ at } \tilde{\eta}_0\,,$$

$$\tilde{\pi}_i' = a^{1+\alpha}\tilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i} \qquad \tilde{\phi}_i' = a^{-(3-\alpha)}\tilde{\pi}_i\,,$$

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$$\begin{cases} \tilde{\pi}^{(i)}_{+0} &= \tilde{\pi}^{(i)} + \left(a^{1+\alpha}_{+0/2} \displaystyle\sum_k \widetilde{\Delta}^{-}_k \widetilde{\Delta}^{+}_k \tilde{\phi}^{(i)}_{+0/2} - a^{3+\alpha}_{+0/2} \widetilde{V}_{,\tilde{\phi}^{(i)}}\Big|_{+0/2} \right) \delta\tilde{\eta}\,, \\ b_{+0} &= b + \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}_{+0/2}\Big[(\alpha-2)\overline{\widetilde{E}_K} + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\Big]\delta\tilde{\eta}\,, \end{cases}$$

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$$\begin{cases} a_{+0} &= a_{+0/2} + b_{+0}\,\dfrac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\phi}^{(i)}_{+0} &= \tilde{\phi}^{(i)}_{+0/2} + \dfrac{\delta\tilde{\eta}}{2}\,\tilde{\pi}^{(i)}_{+0}a^{-(3-\alpha)}_{+0}\,. \end{cases}$$

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$$\begin{cases} a_{+0} &= a_{+0/2} + b_{+0}\,\dfrac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\phi}^{(i)}_{+0} &= \tilde{\phi}^{(i)}_{+0/2} + \dfrac{\delta\tilde{\eta}}{2}\,\tilde{\pi}^{(i)}_{+0} a^{-(3-\alpha)}_{+0}\,. \end{cases}$$

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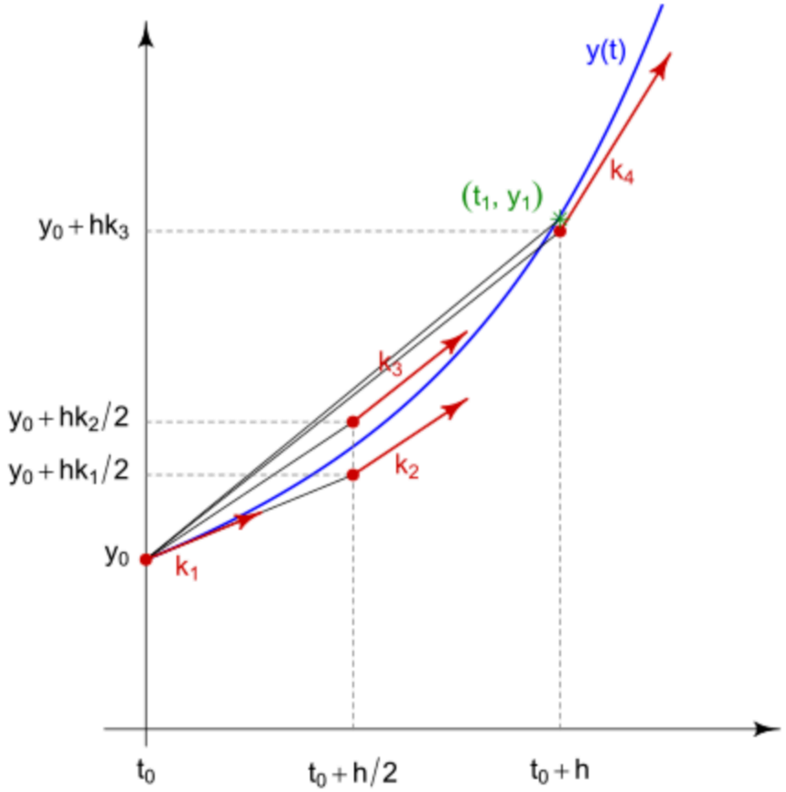
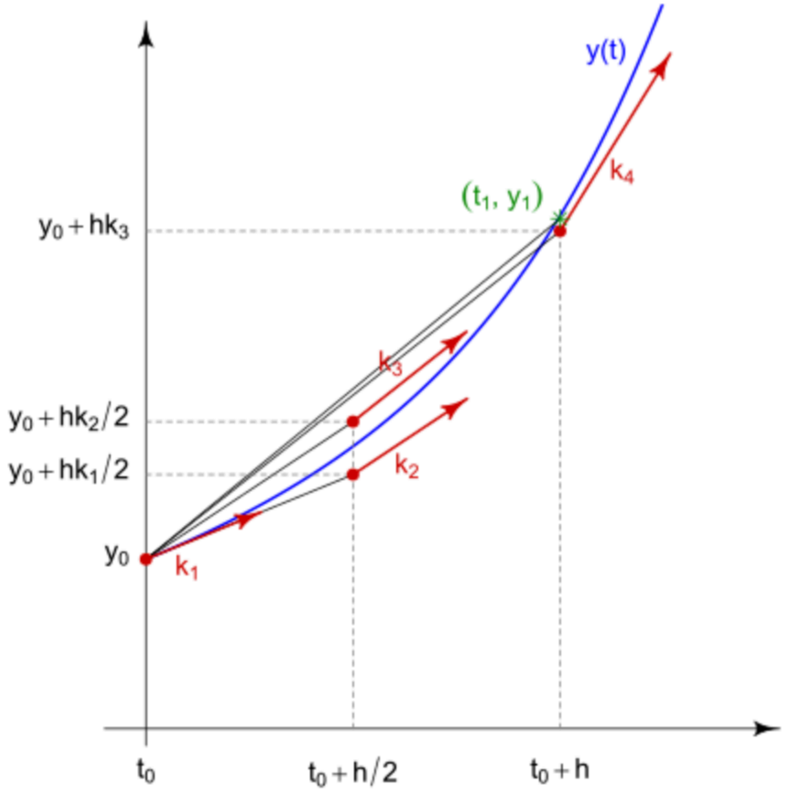
Runge-Kutta



Runge-Kutta

Order $dt^2, dt^3, dt^4, \ldots$



Runge-Kutta

(Can solve non-symplectic systems)

Runge-Kutta

(Can solve non-symplectic systems)

non-conservative

Continuum Problem:

$$\begin{cases} a' &= b\,, \\ \tilde{\phi}_i' &= \tilde{\pi}_i\,, \\ \tilde{\pi}_i' &= \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i]\,, \quad \text{(fld Kernels)} \\ b' &= \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V]\,, \quad \text{(expansion Kernels)} \end{cases}$$

$$i = 1,2,....,N_s$$

Runge-Kutta
(Can solve non-symplectic systems)

Continuum Problem:

non-symplectic !

$$\begin{cases} a' &= b\,, \\ \tilde{\phi}_i' &= \tilde{\pi}_i\,, \\ \tilde{\pi}_i' &= \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i]\,, \quad \text{(fld Kernels)} \\ b' &= \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V]\,, \quad \text{(expansion Kernels)} \end{cases}$$

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$i = 1,2,....,N_s$

where
$$\begin{cases} \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i] \equiv a^{-2(1-\alpha)}\widetilde{\nabla}^2\tilde{\phi}_i - (3-\alpha)\frac{b}{a}\tilde{\pi}_i - a^{2\alpha}\tilde{V}_{,\tilde{\phi}_i}\,, \\ \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V] \equiv \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}}\sum_i \langle \tilde{\pi}_i^2 \rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{i,k}\left\langle (\widetilde{\nabla}_k\tilde{\phi}_i)^2 \right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle \widetilde{V}(\{\tilde{\phi}_j\}) \right\rangle \end{cases}$$

Runge-Kutta

(Can solve non-symplectic systems)

Many flavours: $\mathcal{O}(d\eta^p)$ + explicit OR implicit*

Continuum Problem:

non-symplectic !

$$\begin{cases} a' &= b\,, \\ \tilde{\phi}_i' &= \tilde{\pi}_i\,, \\ \tilde{\pi}_i' &= \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i]\,, \quad \text{(fld Kernels)} \\ b' &= \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V]\,, \quad \text{(expansion Kernels)} \end{cases} \qquad i = 1,2,\ldots.,N_s$$

$$\text{where} \quad \begin{cases} \mathcal{K}_i[a, b, \{\tilde{\phi}_j\}, \tilde{\pi}_i] \equiv a^{-2(1-\alpha)}\widetilde{\nabla}^2\tilde{\phi}_i - (3-\alpha)\frac{b}{a}\tilde{\pi}_i - a^{2\alpha}\tilde{V}_{,\tilde{\phi}_i}\,, \\ \mathcal{K}_a[a, \widetilde{E}_K, \widetilde{E}_G, \widetilde{E}_V] \equiv \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \frac{1}{2a^{2\alpha}}\sum_i \langle\tilde{\pi}_i^2\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{i,k}\left\langle(\widetilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\widetilde{V}(\{\tilde{\phi}_j\})\right\rangle \end{cases}$$

*Gauss-Legendre

Runge-Kutta
(Can solve non-symplectic systems)

Explicit RK2 [$\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left.\begin{array}{rcl}
\tilde{\pi}_i^{(p)} & \equiv & \tilde{\pi}_i(\mathbf{n}, n_0) + \delta\tilde{\eta} c_{p,p-1} k_i^{(p-1)}\,, \\
b^{(p)} & \equiv & b(n_0) + \delta\tilde{\eta} c_{p,p-1} k_a^{(p-1)}\,, \\
\tilde{\phi}_i^{(p)} & \equiv & \tilde{\phi}_i(\mathbf{n}, n_0) + \delta\tilde{\eta} c_{p,p-1} \tilde{\pi}_i^{(p-1)}\,, \\
a^{(p)} & \equiv & a(n_0) + \delta\tilde{\eta} c_{p,p-1} b^{(p-1)}\,, \\
k_i^{(p)} & \equiv & \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}, \tilde{\pi}_i^{(p)}]\,, \\
k_a^{(p)} & \equiv & \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}]\,,
\end{array}\right\}_{p=1,2} \Longrightarrow \left\{\begin{array}{rcl}
\widetilde{\Delta}_0^+ \tilde{\phi}_i(\mathbf{n}, n_0) & = & \frac{1}{2}(\tilde{\pi}_i^{(1)} + \tilde{\pi}_i^{(2)}) \\
\Delta_0^+ a(n_0) & = & \frac{1}{2}(b^{(1)} + b^{(2)}) \\
\widetilde{\Delta}_0^+ \tilde{\pi}_i(\mathbf{n}, n_0) & = & \frac{1}{2}(k_i^{(1)} + k_i^{(2)}) \\
\widetilde{\Delta}_0^+ b(n_0) & = & \frac{1}{2}(k_a^{(1)} + k_a^{(2)})
\end{array}\right.$$

$$\left(\ c_{10} \equiv 0\,, \quad c_{21} \equiv 1\ \right)$$

Runge-Kutta
(Can solve non-symplectic systems)

Explicit RK4 [$\mathcal{O}(d\eta^4)$]

Lattice Problem:

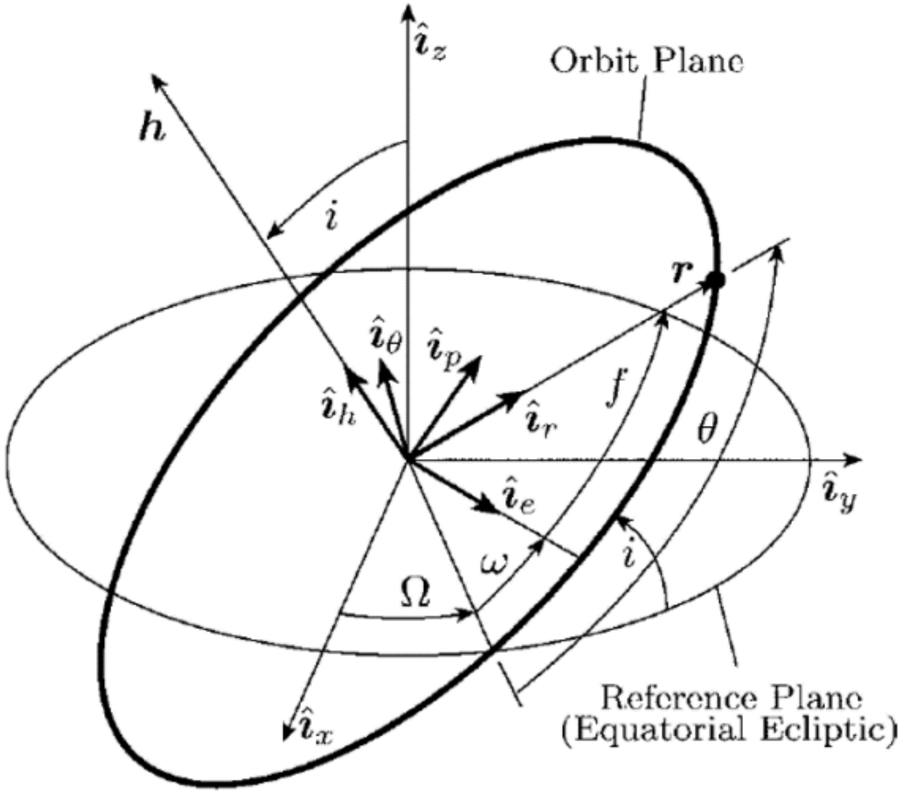
$$
\left.\begin{aligned}
\tilde{\pi}_i^{(p)} &\equiv \tilde{\pi}_i(\mathbf{n}, n_0) + \delta\tilde{\eta} c_{p,p-1} k_i^{(p-1)}\,,\\
b^{(p)} &\equiv b(n_0) + \delta\tilde{\eta} c_{p,p-1} k_a^{(p-1)}\,,\\
\tilde{\phi}_i^{(p)} &\equiv \tilde{\phi}_i(\mathbf{n}, n_0) + \delta\tilde{\eta} c_{p,p-1} \tilde{\pi}_i^{(p-1)}\,,\\
a^{(p)} &\equiv a(n_0) + \delta\tilde{\eta} c_{p,p-1} b^{(p-1)}\,,\\
k_i^{(p)} &\equiv \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}, \tilde{\pi}_i^{(p)}]\,,\\
k_a^{(p)} &\equiv \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}]\,,
\end{aligned}\right\}_{p=1,2,3,4}
\Longrightarrow
\left\{\begin{aligned}
\widetilde{\Delta}_0^+ \tilde{\phi}_i(\mathbf{n}, n_0) &= \tfrac{1}{6}(\tilde{\pi}_i^{(1)} + 2\tilde{\pi}_i^{(2)} + 2\tilde{\pi}_i^{(3)} + \tilde{\pi}_i^{(4)})\,,\\
\Delta_0^+ a(n_0) &= \tfrac{1}{6}(b^{(1)} + 2b^{(2)} + 2b^{(3)} + b^{(4)})\\
\widetilde{\Delta}_0^+ \tilde{\pi}_i(\mathbf{n}, n_0) &= \tfrac{1}{6}(k_i^{(1)} + 2k_i^{(2)} + 2k_i^{(3)} + k_i^{(4)})\\
\widetilde{\Delta}_0^+ b(n_0) &= \tfrac{1}{6}(k_a^{(1)} + 2k_a^{(2)} + 2k_a^{(3)} + k_a^{(4)})
\end{aligned}\right.
$$

$$
\left(c_{10} \equiv 0\,, \quad c_{21} \equiv \frac{1}{2}\,, \quad c_{32} \equiv \frac{1}{2}\,, \quad c_{43} \equiv 1 \right)
$$

Higher Order Symplectic Integrators (Yoshida Method)

Haruo Yoshida

National Astronomical Observatory of Japan
NAOJ · Division of Theoretical Astronomy



Higher Order Symplectic Integrators (Yoshida Method)

Order $dt^2, dt^4, dt^6, dt^8, dt^{10}$

Haruo Yoshida

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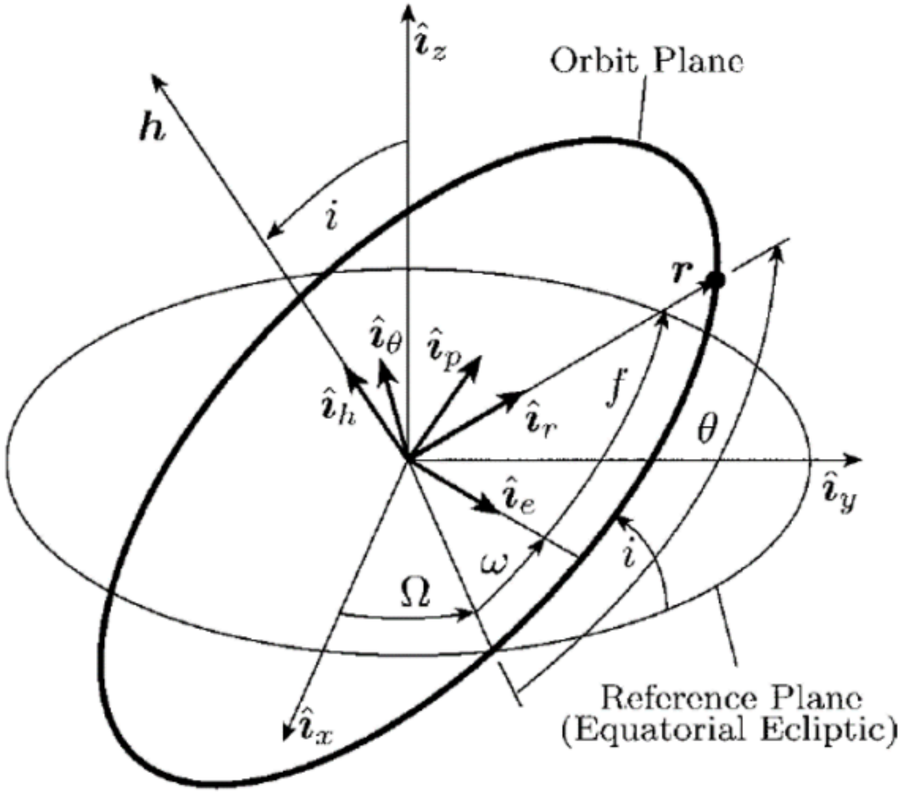

Higher Order Symplectic Integrators

Continuum Problem:

symplectic !

$$\left\{\begin{array}{lcl} a' & = & b\,, \\ \tilde{\phi}_i' & = & a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' & = & \mathcal{K}_i[a,\{\tilde{\phi}_j\}]\,, \\ b' & = & \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V]\,, \end{array}\right.$$

Higher Order Symplectic Integrators

Continuum Problem:

symplectic !

$$\begin{cases} a' &= b\,, \\ \tilde{\phi}_i' &= a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' &= \mathcal{K}_i[a,\{\tilde{\phi}_j\}]\,, \\ b' &= \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V]\,, \end{cases}$$

where

$$\begin{cases} \mathcal{K}_i[a,\{\tilde{\phi}_j\}] \equiv a^{1+\alpha}\widetilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}\,, \\ \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V] \equiv \frac{1}{3}\left(\frac{f_*}{m_p}\right)^2 a^{1+2\alpha}\left[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\right], \\ \widetilde{E}_K \equiv \frac{1}{2a^6}\sum_i\langle\tilde{\pi}_i^2\rangle\,, \quad \widetilde{E}_G \equiv \frac{1}{2a^2}\sum_{i,k}\left\langle(\widetilde{\nabla}_k\tilde{\phi}_i)^2\right\rangle\,, \quad \widetilde{E}_V \equiv \left\langle\widetilde{V}(\{\tilde{\phi}_j\})\right\rangle\,. \end{cases}$$

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Continuum Problem:

symplectic !

$$\begin{cases} a' &= b\,, \\ \tilde{\phi}_i' &= a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' &= \mathcal{K}_i[a,\{\tilde{\phi}_j\}]\,, \\ b' &= \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V]\,, \end{cases}$$

where

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Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Continuum Problem:

symplectic !

$$\left\{\begin{array}{rcl} a' & = & b\,, \\ \tilde{\phi}_i' & = & a^{-(3-\alpha)}\tilde{\pi}_i\,, \\ \tilde{\pi}_i' & = & \mathcal{K}_i[a,\{\tilde{\phi}_j\}]\,, \\ b' & = & \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V]\,, \end{array}\right.$$

where

$$\left\{\begin{array}{l} \mathcal{K}_i[a,\{\tilde{\phi}_j\}] \equiv a^{1+\alpha}\widetilde{\nabla}^2\tilde{\phi}_i - a^{3+\alpha}\tilde{V}_{,\tilde{\phi}_i}\,, \\ \mathcal{K}_a[a,\widetilde{E}_K,\widetilde{E}_G,\widetilde{E}_V] \equiv \dfrac{1}{3}\left(\dfrac{f_*}{m_p}\right)^2 a^{1+2\alpha}\Big[(\alpha-2)\widetilde{E}_K + \alpha\widetilde{E}_G + (\alpha+1)\widetilde{E}_V\Big]\,, \\ \widetilde{E}_K \equiv \dfrac{1}{2a^6}\displaystyle\sum_i\langle\tilde{\pi}_i^2\rangle\,, \quad \widetilde{E}_G \equiv \dfrac{1}{2a^2}\displaystyle\sum_{i,k}\Big\langle(\widetilde{\nabla}_k\tilde{\phi}_i)^2\Big\rangle\,, \quad \widetilde{E}_V \equiv \Big\langle\widetilde{V}(\{\tilde{\phi}_j\})\Big\rangle\,. \end{array}\right.$$

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} & = & b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} & = & \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} & = & a^{(p-1)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} & = & \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} & = & a_{1/2}^{(p)} + b_{1/2}^{(p)} \omega_p \frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} & = & \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} & = & b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2} \mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\dots,s} \Longrightarrow$$

(s-copies)

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left\{\boxed{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}}\right. \Longrightarrow \left\{\begin{array}{lcl} b_{1/2}^{(p)} & = & b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} & = & \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} & = & a^{(p-1)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} & = & \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} & = & a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} & = & \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} & = & b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\dots,s} \Longrightarrow$$

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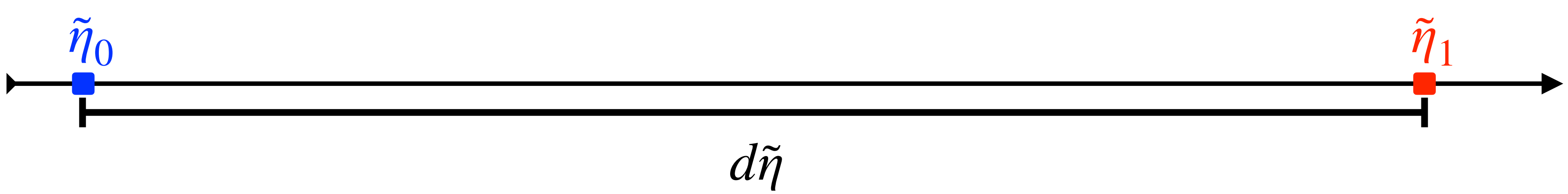


Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left\{\boxed{\begin{array}{l}\tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0)\\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0)\\ a^{(0)} \equiv a(n_0)\\ b^{(0)} \equiv b(n_0)\end{array}}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}]\\ \tilde{\pi}_{i,1/2}^{(p)} &=& \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}]\\ a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2}\\ \tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)}(a_{1/2}^{(p)})^{-(3-\alpha)}\,,\\ a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2}\,,\\ \tilde{\pi}_i^{(p)} &=& \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}]\\ b^{(p)} &=& b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}]\end{array}\right\}_{p=1,\dots,s} \Longrightarrow$$

(s-copies)

$\tilde{\eta}_0$ $\tilde{\eta}_1$

$$d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}$$

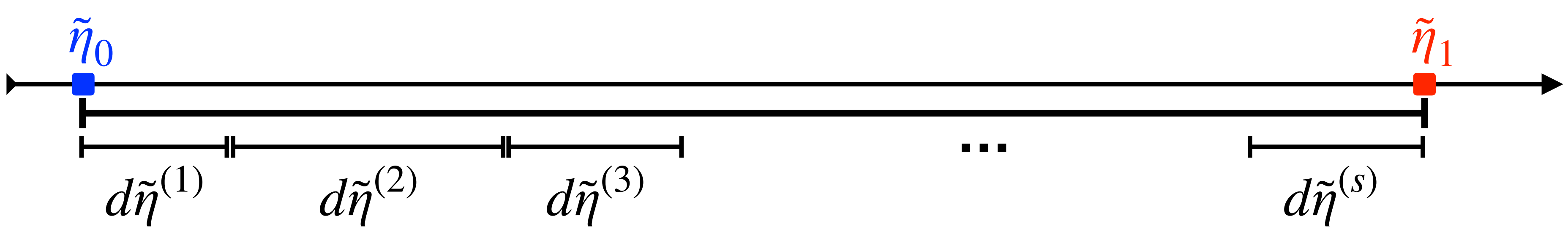
Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left\{\boxed{\begin{array}{l} \tilde\pi_i^{(0)} \equiv \tilde\pi_i(\mathbf{n},n_0) \\ \tilde\phi_i^{(0)} \equiv \tilde\phi_i(\mathbf{n},n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p\frac{\delta\tilde\eta}{2}\mathcal{K}_a[a^{(p-1)},\widetilde{E}_K^{(p-1)},\widetilde{E}_G^{(p-1)},\widetilde{E}_V^{(p-1)}] \\ \tilde\pi_{i,1/2}^{(p)} &=& \tilde\pi_i^{(p-1)} + \omega_p\frac{\delta\tilde\eta}{2}\mathcal{K}_i[a^{(p-1)},\{\tilde\phi_j^{(p-1)}\}] \\ a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)}\omega_p\frac{\delta\tilde\eta}{2} \\ \tilde\phi_i^{(p)} &=& \tilde\phi_i^{(p-1)} + \omega_p\delta\tilde\eta\,\tilde\pi_{i,1/2}^{(p)}(a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p\frac{\delta\tilde\eta}{2}\,, \\ \tilde\pi_i^{(p)} &=& \tilde\pi_{i,1/2}^{(p)} + \omega_p\frac{\delta\tilde\eta}{2}\mathcal{K}_i[a^{(p)},\{\tilde\phi_j^{(p)}\}] \\ b^{(p)} &=& b_{1/2}^{(p)} + \omega_p\frac{\delta\tilde\eta}{2}\mathcal{K}_a[a^{(p)},\widetilde{E}_K^{(p)},\widetilde{E}_G^{(p)},\widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\dots,s} \Longrightarrow$$

(s-copies)



$$d\tilde\eta \equiv d\tilde\eta^{(1)} + d\tilde\eta^{(2)} + \ldots + d\tilde\eta^{(s)} \quad ; \quad \left\{\begin{array}{l} d\tilde\eta^{(p)} \equiv \omega_p d\tilde\eta \\ \sum_{p=1}^{s}\omega_p = 1 \end{array}\right.$$

($\omega_p < 0$ **allowed)**

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left\{\boxed{\begin{array}{l} \tilde\pi_i^{(0)} \equiv \tilde\pi_i(\mathbf{n}, n_0) \\ \tilde\phi_i^{(0)} \equiv \tilde\phi_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde\eta}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde\pi_{i,1/2}^{(p)} &=& \tilde\pi_i^{(p-1)} + \omega_p \frac{\delta\tilde\eta}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde\phi_j^{(p-1)}\}] \\ a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde\eta}{2} \\ \tilde\phi_i^{(p)} &=& \tilde\phi_i^{(p-1)} + \omega_p\delta\tilde\eta\, \tilde\pi_{i,1/2}^{(p)}(a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde\eta}{2}\,, \\ \tilde\pi_i^{(p)} &=& \tilde\pi_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde\eta}{2}\mathcal{K}_i[a^{(p)}, \{\tilde\phi_j^{(p)}\}] \\ b^{(p)} &=& b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde\eta}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\ldots,s} \Longrightarrow$$

(s-copies)

$$d\tilde\eta \equiv d\tilde\eta^{(1)} + d\tilde\eta^{(2)} + \ldots + d\tilde\eta^{(s)}$$

$\tilde\eta_0$ $\tilde\eta_1$

$$\left\{\begin{array}{l} d\tilde\eta^{(p)} \equiv \omega_p d\tilde\eta \\ \displaystyle\sum_{p=1}^{s} \omega_p = 1 \end{array}\right.$$

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\begin{cases} \tilde\pi_i^{(0)} \equiv \tilde\pi_i(\mathbf{n}, n_0) \\ \tilde\phi_i^{(0)} \equiv \tilde\phi_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{cases} \Longrightarrow \left\{ \begin{array}{rcl} b^{(p)}_{1/2} & = & b^{(p-1)} + \omega_p \frac{\delta\tilde\eta}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde\pi^{(p)}_{i,1/2} & = & \tilde\pi_i^{(p-1)} + \omega_p \frac{\delta\tilde\eta}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde\phi_j^{(p-1)}\}] \\ a^{(p)}_{1/2} & = & a^{(p-1)} + b^{(p)}_{1/2}\omega_p\frac{\delta\tilde\eta}{2} \\ \tilde\phi_i^{(p)} & = & \tilde\phi_i^{(p-1)} + \omega_p \delta\tilde\eta\, \tilde\pi^{(p)}_{i,1/2}(a^{(p)}_{1/2})^{-(3-\alpha)}\,, \\ a^{(p)} & = & a^{(p)}_{1/2} + b^{(p)}_{1/2}\omega_p\frac{\delta\tilde\eta}{2}\,, \\ \tilde\pi_i^{(p)} & = & \tilde\pi^{(p)}_{i,1/2} + \omega_p\frac{\delta\tilde\eta}{2}\mathcal{K}_i[a^{(p)}, \{\tilde\phi_j^{(p)}\}] \\ b^{(p)} & = & b^{(p)}_{1/2} + \omega_p\frac{\delta\tilde\eta}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array} \right\}_{p=1,\ldots,s} \Longrightarrow$$

(s-copies)

$$\tilde\eta_0 \qquad d\tilde\eta \equiv d\tilde\eta^{(1)} + d\tilde\eta^{(2)} + \ldots + d\tilde\eta^{(s)} \qquad \tilde\eta_1$$

$$\Longrightarrow \begin{cases} \tilde\pi_i(\mathbf{n}, n_0+1) \equiv \tilde\pi_i^{(s)} \\ \tilde\phi_i(\mathbf{n}, n_0+1) \equiv \tilde\phi_i^{(s)} \\ a(n_0+1) \equiv a^{(s)} \\ b(n_0+1) \equiv b^{(s)}\,. \end{cases}$$

$$\begin{cases} d\tilde\eta^{(p)} \equiv \omega_p d\tilde\eta \\ \sum_{p=1}^{s} \omega_p = 1 \end{cases}$$

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left\{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}\right. \Longrightarrow \left\{\begin{array}{rcl} b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}] \\ \tilde{\pi}_{i,1/2}^{(p)} &=& \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}] \\ a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2} \\ \tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p \delta\tilde{\eta}\, \tilde{\pi}_{i,1/2}^{(p)} (a_{1/2}^{(p)})^{-(3-\alpha)}\,, \\ a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p \frac{\delta\tilde{\eta}}{2}\,, \\ \tilde{\pi}_i^{(p)} &=& \tilde{\pi}_{i,1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}] \\ b^{(p)} &=& b_{1/2}^{(p)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}] \end{array}\right\}_{p=1,\ldots,s} \Longrightarrow$$

(s-copies)

$\tilde{\eta}_0$ $\qquad d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}$ $\qquad \tilde{\eta}_1$

Appropriate $\{\omega_p\}_{p=1,\ldots,s}$ **: Errors intermediate steps cancel to** $\mathcal{O}(\delta\eta^r)$

$$s_n \equiv 2s_{n-1} + 1 = 1,3,7,15,\ldots \quad \Rightarrow \quad \mathcal{O}(\delta\eta^{2n})$$

$$(s_1 \equiv 1\;;\; n = 1,\,2,\,\ldots) \qquad (r = 2,4,6,8,\ldots)$$

$$\left\{\begin{array}{l} d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \\ \sum_{p=1}^{s} \omega_p = 1 \end{array}\right.$$

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$
\left\{\begin{array}{l}
\tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0)\\
\tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0)\\
a^{(0)} \equiv a(n_0)\\
b^{(0)} \equiv b(n_0)
\end{array}\right.
\Longrightarrow
\left\{\begin{array}{rcl}
b_{1/2}^{(p)} &=& b^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p-1)}, \widetilde{E}_K^{(p-1)}, \widetilde{E}_G^{(p-1)}, \widetilde{E}_V^{(p-1)}]\\
\tilde{\pi}_{i,1/2}^{(p)} &=& \tilde{\pi}_i^{(p-1)} + \omega_p \frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p-1)}, \{\tilde{\phi}_j^{(p-1)}\}]\\
a_{1/2}^{(p)} &=& a^{(p-1)} + b_{1/2}^{(p)}\omega_p\frac{\delta\tilde{\eta}}{2}\\
\tilde{\phi}_i^{(p)} &=& \tilde{\phi}_i^{(p-1)} + \omega_p\delta\tilde{\eta}\,\tilde{\pi}_{i,1/2}^{(p)}(a_{1/2}^{(p)})^{-(3-\alpha)}\,,\\
a^{(p)} &=& a_{1/2}^{(p)} + b_{1/2}^{(p)}\omega_p\frac{\delta\tilde{\eta}}{2}\,,\\
\tilde{\pi}_i^{(p)} &=& \tilde{\pi}_{i,1/2}^{(p)} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_i[a^{(p)}, \{\tilde{\phi}_j^{(p)}\}]\\
b^{(p)} &=& b_{1/2}^{(p)} + \omega_p\frac{\delta\tilde{\eta}}{2}\mathcal{K}_a[a^{(p)}, \widetilde{E}_K^{(p)}, \widetilde{E}_G^{(p)}, \widetilde{E}_V^{(p)}]
\end{array}\right\}_{p=1,\ldots,s}
\Longrightarrow
$$

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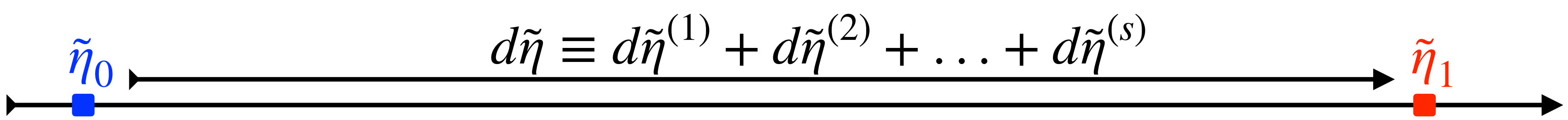

Appropriate $\{\omega_p\}_{p=1,\ldots,s}$ **: Errors intermediate steps cancel to** $\mathcal{O}(\delta\eta^r)$

$$
\left\{\begin{array}{l}
d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta}\\
\sum_{p=1}^{s}\omega_p = 1
\end{array}\right.
$$

e.g. $w_1 = w_3 = 1.351207191959657771818$
$w_2 = -1.702414403875838200264$ $\qquad \mathcal{O}(d\eta^4)$

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left\{\boxed{\begin{array}{l} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{array}}\right. \quad \underset{\text{(s-copies VV2)}}{\Longrightarrow} \quad \left\{\boxed{\begin{array}{l} \tilde{\pi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\pi}_i^{(s)} \\ \tilde{\phi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\phi}_i^{(s)} \\ a(n_0+1) \equiv a^{(s)} \\ b(n_0+1) \equiv b^{(s)}\,. \end{array}}\right.$$

$\tilde{\eta}_0$ $\qquad d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}$ $\qquad \tilde{\eta}_1$

$$\left[\ d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \quad ; \quad \sum_{p=1}^{s} \omega_p = 1\ \right]$$

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\left\{\boxed{\begin{aligned} \tilde{\pi}_i^{(0)} &\equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} &\equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} &\equiv a(n_0) \\ b^{(0)} &\equiv b(n_0) \end{aligned}}\right. \quad \underset{\text{(s-copies VV2)}}{\Longrightarrow} \quad \left\{\boxed{\begin{aligned} \tilde{\pi}_i(\mathbf{n}, n_0+1) &\equiv \tilde{\pi}_i^{(s)} \\ \tilde{\phi}_i(\mathbf{n}, n_0+1) &\equiv \tilde{\phi}_i^{(s)} \\ a(n_0+1) &\equiv a^{(s)} \\ b(n_0+1) &\equiv b^{(s)}\,. \end{aligned}}\right.$$

$\tilde{\eta}_0$ $\qquad d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}$ $\qquad \tilde{\eta}_1$

$$\left[\; d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \quad ; \quad \sum_{p=1}^{s} \omega_p = 1 \;\right]$$

Name	Order	$w_i = \frac{\delta t_i}{\delta t}$	s
$VV4$	$O(\delta t^4)$	$w_1 = w_3 = 1.3512071919596577718 18$ $w_2 = -1.7024143839193155436 4$	3
$VV6$	$O(\delta t^6)$	$w_1 = w_7 = 0.78451361047755726382$ $w_2 = w_6 = 0.23557321335935813368$ $w_3 = w_5 = -1.17767998417887100069$ $w_4 = 1.3151863206839112189$	7
$VV8$	$O(\delta t^8)$	$w_1 = w_{15} = 0.74167036435061295345$ $w_2 = w_{14} = -0.40910082580003159400$ $w_3 = w_{13} = 0.19075471029623837995$ $w_4 = w_{12} = -0.57386247111608226666$ $w_5 = w_{11} = 0.29906418130365592384$ $w_6 = w_{10} = 0.33462491824529818378$ $w_7 = w_9 = 0.31529309239676659663$ $w_8 = -0.79688793935291635402$	15
$VV10$	$O(\delta t^{10})$	$w_1 = w_{31} = -0.48159895600253002870$ $w_2 = w_{30} = 0.0036303931544595926879$ $w_3 = w_{29} = 0.50180317558723140279$ $w_4 = w_{28} = 0.28298402624506254868$ $w_5 = w_{27} = 0.80702967895372223806$ $w_6 = w_{26} = -0.026090580538592205447$ $w_7 = w_{25} = -0.87286590146318071547$ $w_8 = w_{24} = -0.52373568062510581643$ $w_9 = w_{23} = 0.44521844299952789252$ $w_{10} = w_{22} = 0.18612289547097907887$ $w_{11} = w_{21} = 0.23137327866438360633$ $w_{12} = w_{20} = -0.52191036590418628905$ $w_{13} = w_{19} = 0.74866113714499296793$ $w_{14} = w_{18} = 0.066736511890604057532$ $w_{15} = w_{17} = -0.80360324375670830316$ $w_{16} = 0.91249037635867994571$	31

Higher Order Symplectic Integrators

[based on position- or velocity-Verlet $\mathcal{O}(d\eta^2)$]

Lattice Problem:

$$\begin{cases} \tilde{\pi}_i^{(0)} \equiv \tilde{\pi}_i(\mathbf{n}, n_0) \\ \tilde{\phi}_i^{(0)} \equiv \tilde{\phi}_i(\mathbf{n}, n_0) \\ a^{(0)} \equiv a(n_0) \\ b^{(0)} \equiv b(n_0) \end{cases} \quad \xRightarrow{\text{(s-copies VV2)}} \quad \begin{cases} \tilde{\pi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\pi}_i^{(s)} \\ \tilde{\phi}_i(\mathbf{n}, n_0+1) \equiv \tilde{\phi}_i^{(s)} \\ a(n_0+1) \equiv a^{(s)} \\ b(n_0+1) \equiv b^{(s)} \,. \end{cases}$$

$\tilde{\eta}_0$ $\qquad d\tilde{\eta} \equiv d\tilde{\eta}^{(1)} + d\tilde{\eta}^{(2)} + \ldots + d\tilde{\eta}^{(s)}$ $\qquad \tilde{\eta}_1$

$$\left[\; d\tilde{\eta}^{(p)} \equiv \omega_p d\tilde{\eta} \quad ; \quad \sum_{p=1}^{s} \omega_p = 1 \;\right]$$

Name	Order	$w_i = \frac{\delta t_i}{\delta t}$	s
$VV4$	$O(\delta t^4)$	$w_1 = w_3 = 1.3512071919596577771818$ $w_2 = -1.7024144038758382002 64$	3
$VV6$	$O(\delta t^6)$	$w_1 = w_7 = 0.78451361047755726382$ $w_2 = w_6 = 0.23557321335935813368$ $w_3 = w_5 = -1.17767998417887100 69$ $w_4 = 1.31518632068391121 89$	7
$VV8$	$O(\delta t^8)$	$w_1 = w_{15} = 0.74167036435061295345$ $w_2 = w_{14} = -0.40910082580003159400$ $w_3 = w_{13} = 0.19075471029623837995$ $w_4 = w_{12} = -0.57386247111608226666$ $w_5 = w_{11} = 0.29906418130365592384$ $w_6 = w_{10} = 0.33462491824529818378$ $w_7 = w_9 = 0.31529309239676659663$ $w_8 = -0.79688793935291635402$	15
$VV10$	$O(\delta t^{10})$	$w_1 = w_{31} = -0.48159895600253002870$ $w_2 = w_{30} = 0.00363039315445959268 79$ $w_3 = w_{29} = 0.50180317558723140279$ $w_4 = w_{28} = 0.28298402624506254868$ $w_5 = w_{27} = 0.80702967895372223806$ $w_6 = w_{26} = -0.02609058053859220544 7$ $w_7 = w_{25} = -0.87286590146318071547$ $w_8 = w_{24} = -0.52373568062510581643$ $w_9 = w_{23} = 0.44521844299952789252$ $w_{10} = w_{22} = 0.18612289547097907887$ $w_{11} = w_{21} = 0.23137327866438360633$ $w_{12} = w_{20} = -0.52191036590418628905$ $w_{13} = w_{19} = 0.74866113714499296793$ $w_{14} = w_{18} = 0.06673651189060405753 2$ $w_{15} = w_{17} = -0.80360324375670830316$ $w_{16} = 0.91249037635867994571$	31

Summary

LeapFrog (LF)
Verlet methods (VV2,PV2)
Runge-Kutta methods (RK2, RK4)
Higher Order integrators (VV4, VV6, VV8, VV10)

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Thanks for your attention !

Back Slides

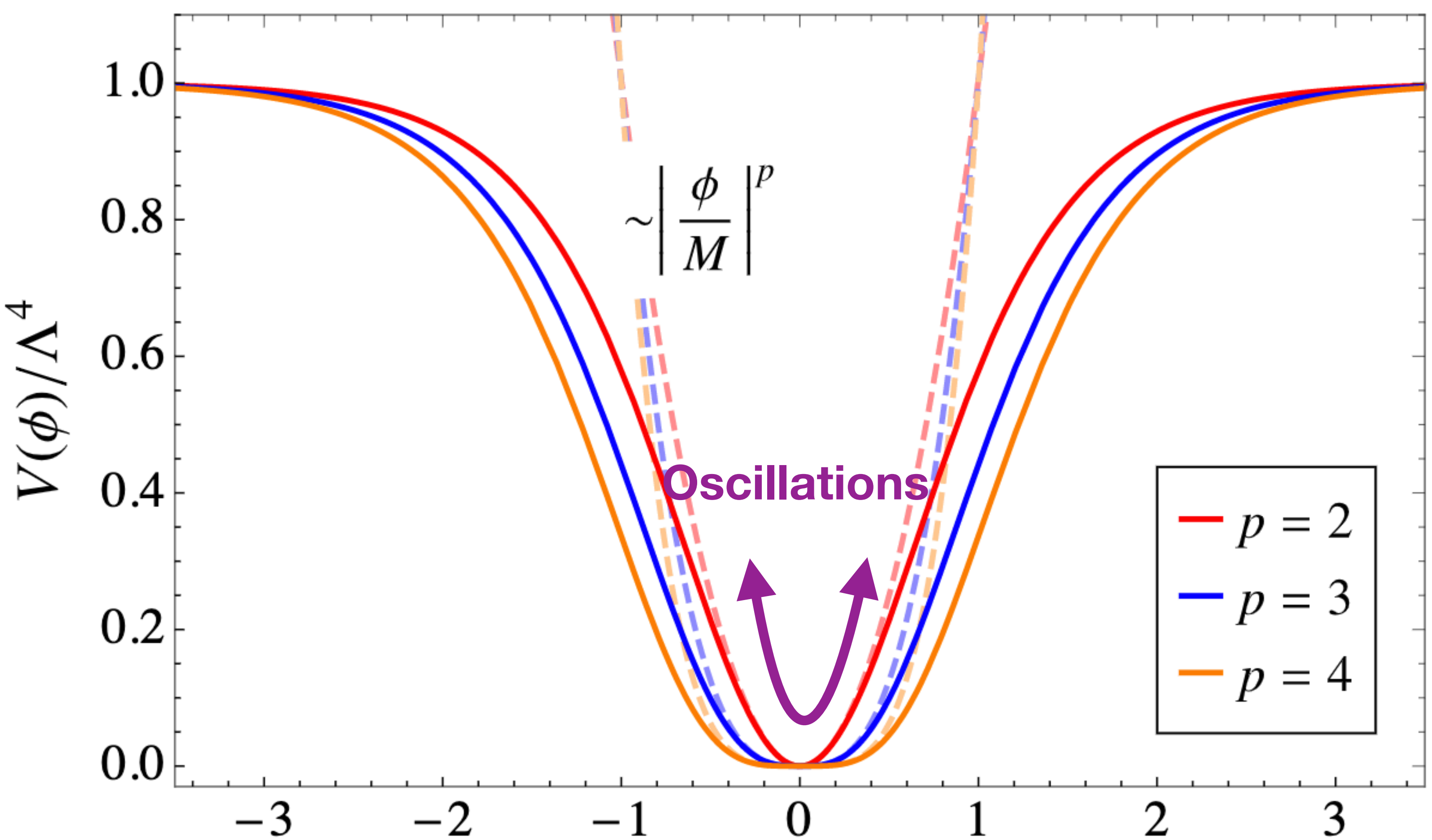
Models

$$\tanh^p(\phi/M)$$

Inflationary Models $\tanh^p(\phi/M)$

$$V(\phi,\chi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2 \qquad (p \geq 2)$$

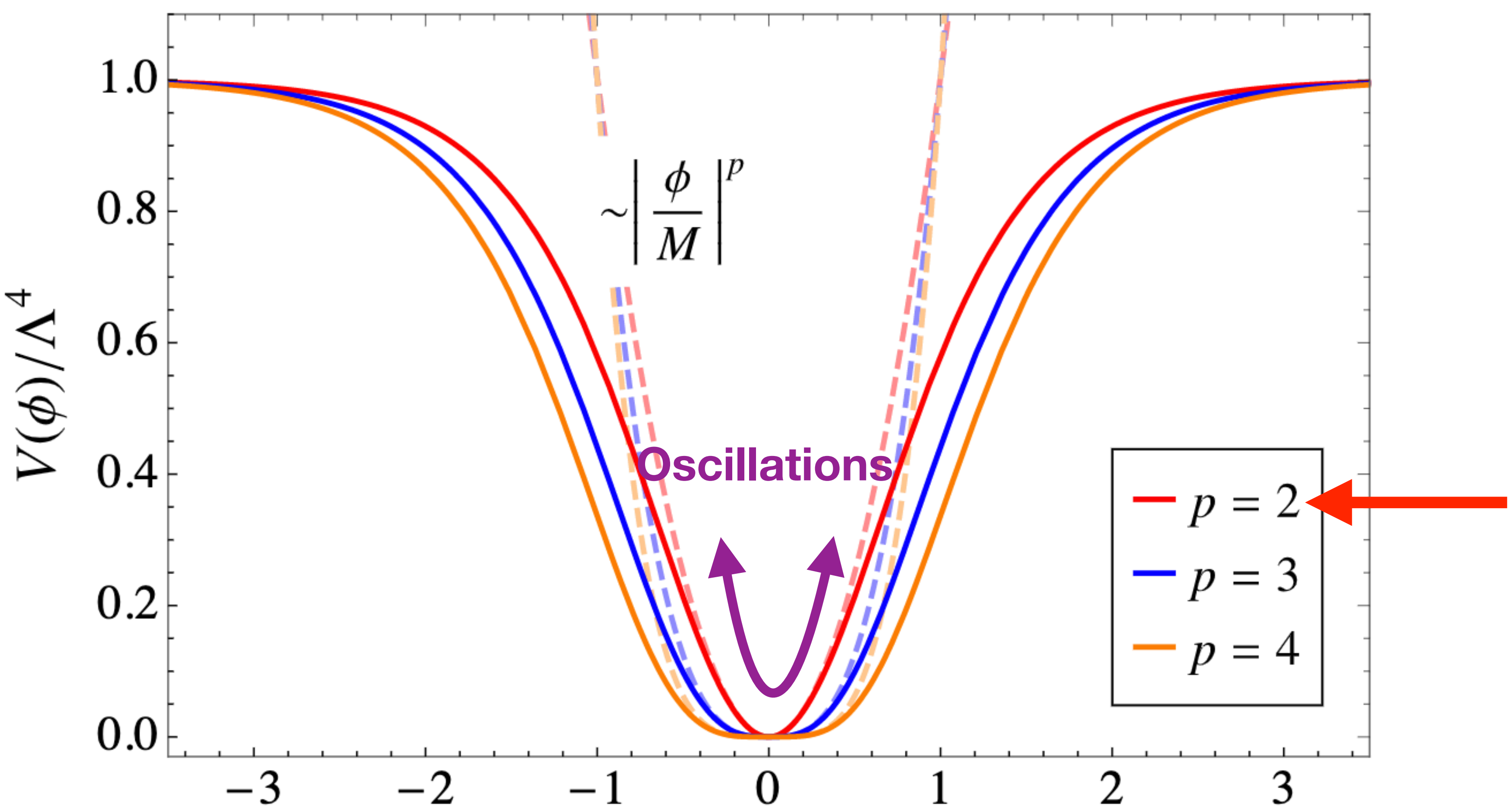
(*e.g.* $\alpha -$ attractors, Kallosh, Linde 2013)



Inflationary Models $\tanh^p(\phi/M)$

$$\boxed{V(\phi,\chi) = \frac{1}{2}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^2 + \frac{1}{2}g^2\chi^2\phi^2}$$

(*e.g.* α – attractors, Kallosh, Linde 2013)



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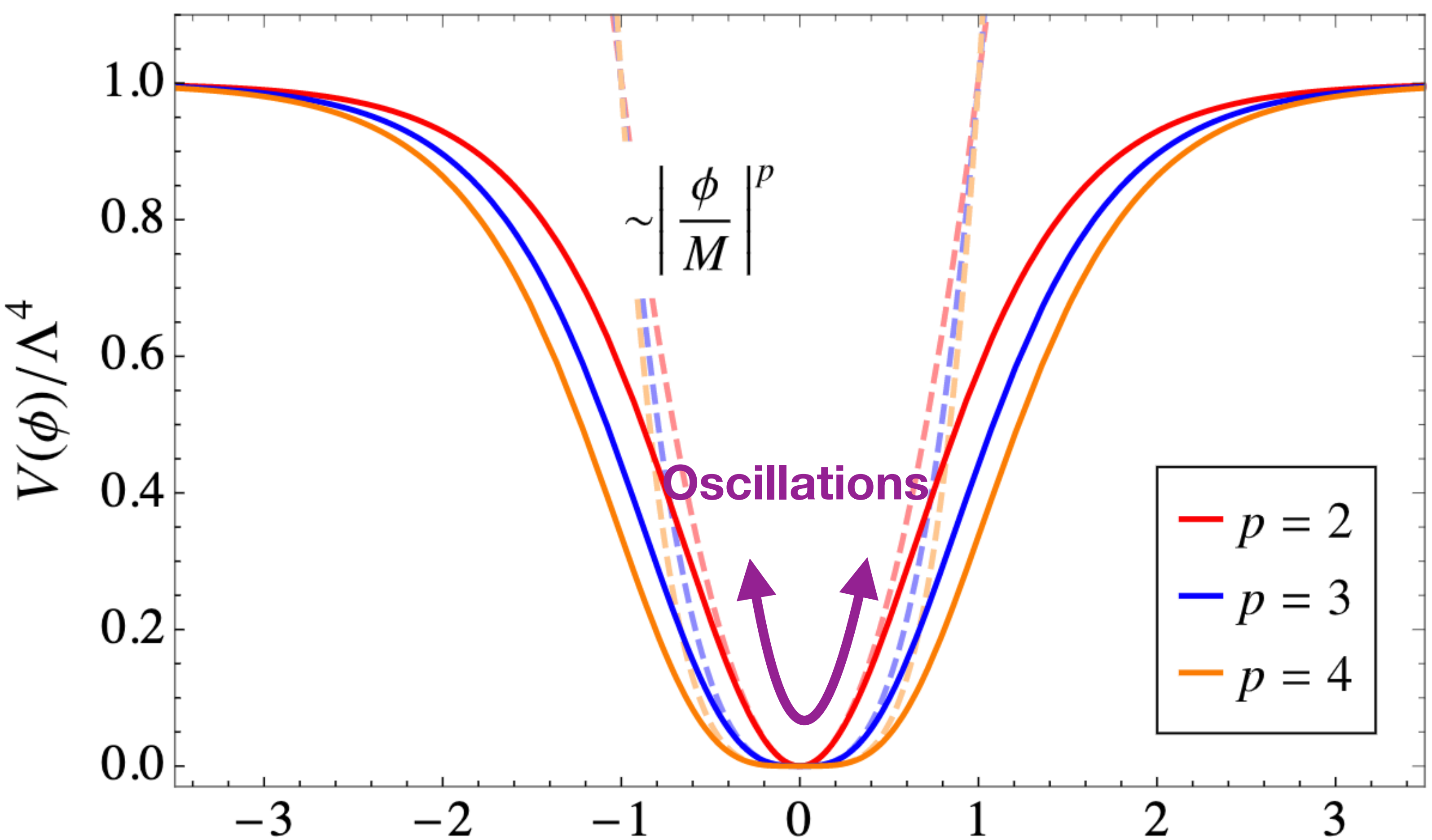

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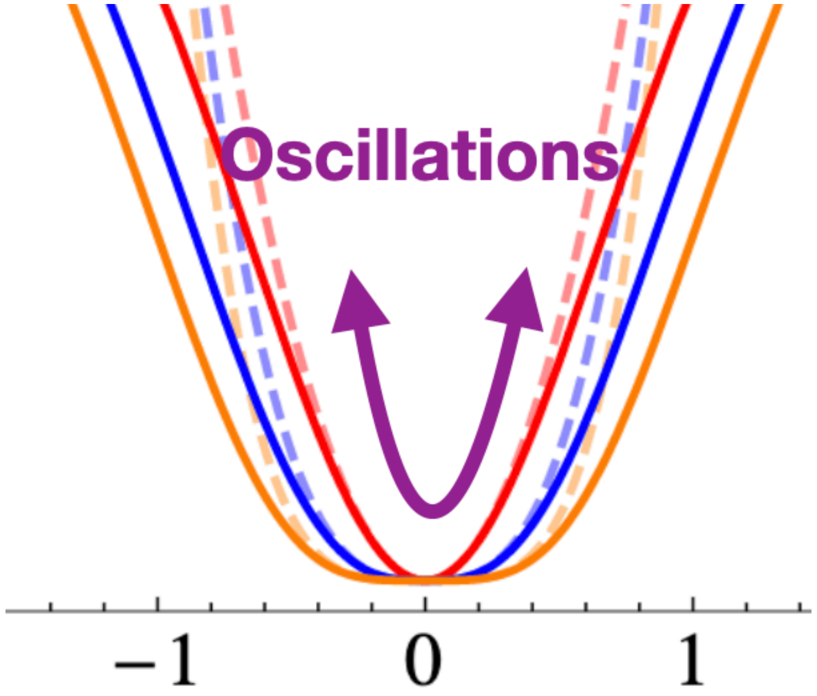
$$V_\phi(\phi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p \simeq \frac{1}{p}\lambda\mu^{4-p}|\phi|^p\,, \qquad \boxed{\lambda\mu^{4-p} \equiv \Lambda^4 M^{-p}}$$



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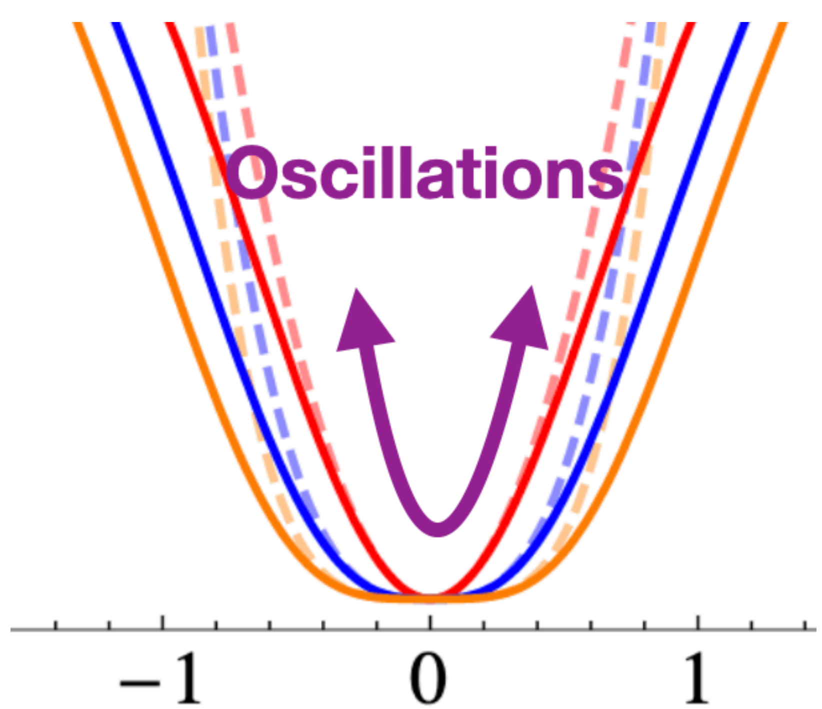

$$V_\phi \simeq \frac{1}{2}m_\phi^2\phi^2\ ; \quad m_\phi^2 \equiv \left(\frac{\Lambda}{M}\right)^2\Lambda^2$$

$$V_\phi \simeq \frac{1}{4}\lambda\phi^4\ ; \quad \lambda \equiv \left(\frac{\Lambda}{M}\right)^4$$

$$V_\phi \simeq \frac{1}{6}\frac{1}{\sigma^2}\phi^6\ ; \quad \sigma^2 \equiv \left(\frac{M}{\Lambda}\right)^4 M^2$$

Inflationary Models $\tanh^p(\phi/M)$

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Turner '83: $\Omega(\phi) \sim \omega_*\left(\frac{t}{t_*}\right)^{4/p-2} \sim \left(\frac{a}{a_*}\right)^{\frac{-3(p-2)}{(p+2)}}\,, \qquad \omega_* \equiv \sqrt{\lambda}\mu^{\frac{4-p}{2}}\phi_*^{\frac{p-2}{2}}$

Initial amplitude
(@ oscillations onset)

Inflationary Models $\tanh^p(\phi/M)$

$$\boxed{V(\phi,\chi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2} \qquad (p \geq 2)$$

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Choice Param.: $\boxed{\alpha = 3\left(\frac{p-2}{p+2}\right)\,,\qquad f_* \equiv \phi_*\,,\qquad \omega_* \equiv \Lambda^2 M^{-\frac{p}{2}}\phi_*^{\frac{p-2}{2}}}$

$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$
$$d\tilde{x}^i \equiv \omega_* dx^i$$
$$\tilde{\phi} = \frac{\phi}{f_*}$$

Inflationary Models $\tanh^p(\phi/M)$

$$\boxed{V(\phi,\chi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2} \quad (p \geq 2)$$

(*e.g.* $\alpha-$attractors, Kallosh, Linde 2013)

Exercise time !

Run for p = 2, p = 4, p = 6,

changing $k_{\rm IR}, k_{\rm UV}, N,$

choosing LF, VV2, VV4, VV6, VV8, VV10

checking energy conservation, field mean values, field variance, field spectra, …

$$\boxed{V(\phi,\chi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p + \frac{1}{2}g^2\chi^2\phi^2} \qquad (p \geq 2)$$

$$V_\phi(\phi) = \frac{1}{p}\Lambda^4\left[\tanh\left(\frac{\phi}{M}\right)\right]^p \simeq \frac{1}{p}\lambda\mu^{4-p}|\phi|^p \,, \qquad \boxed{\lambda\mu^{4-p} \equiv \Lambda^4 M^{-p}}$$

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$$d\tilde{\eta} \equiv a^{-\alpha}\omega_* dt$$

$$d\tilde{x}^i \equiv \omega_* dx^i$$

$$\tilde{\phi} = \frac{\phi}{f_*} \;;\; \tilde{\chi} = \frac{\chi}{f_*}$$

Potential: $\tilde{V}(\tilde{\phi},\tilde{\chi}) \equiv \frac{1}{f_*^2\omega_*^2}V(\tilde{\phi},\tilde{\chi}) = \frac{1}{p}\tilde{M}^p\tanh^p\left(\frac{|\tilde{\phi}|}{\tilde{M}}\right) + \frac{g^2}{2}\frac{\tilde{M}^p}{\tilde{\Lambda}^4}\tilde{\chi}^2\tilde{\phi}^2 \quad ; \quad \begin{cases} \tilde{M} \equiv \dfrac{M}{\phi_*} \\ \tilde{\Lambda} \equiv \dfrac{\Lambda}{\phi_*} \end{cases}$

Deriv. Potential: $\frac{\partial\tilde{V}}{\partial\tilde{\phi}} = \tilde{M}^{p-1}\frac{\tanh^{p-1}(|\tilde{\phi}|/\tilde{M})}{\cosh^2(|\tilde{\phi}|/M)}\mathrm{sgn}(\tilde{\phi}) + \left(\frac{g^2\tilde{M}^p}{\tilde{\Lambda}^4}\right)\tilde{\chi}^2\tilde{\phi} \quad ; \quad \frac{\partial\tilde{V}}{\partial\tilde{\chi}} = \left(\frac{g^2\tilde{M}^p}{\tilde{\Lambda}^4}\right)\tilde{\phi}^2\tilde{\chi}$

Deriv.^2 Potential: $\begin{cases} \dfrac{\partial^2\tilde{V}}{\partial\tilde{\phi}^2} = \dfrac{\tilde{M}^{p-2}}{\cosh^2(|\tilde{\phi}|/M)}\left[(p-1)\tanh^{p-2}(|\tilde{\phi}|/\tilde{M}) - (p+1)\tanh^p(|\tilde{\phi}|/\tilde{M})\right]\mathrm{sgn}(\tilde{\phi}) + \left(\dfrac{g^2\tilde{M}^p}{\tilde{\Lambda}^4}\right)\tilde{\chi}^2 \\ \dfrac{\partial^2\tilde{V}}{\partial\tilde{\chi}^2} = \left(\dfrac{g^2\tilde{M}^p}{\tilde{\Lambda}^4}\right)\tilde{\phi}^2 \end{cases}$