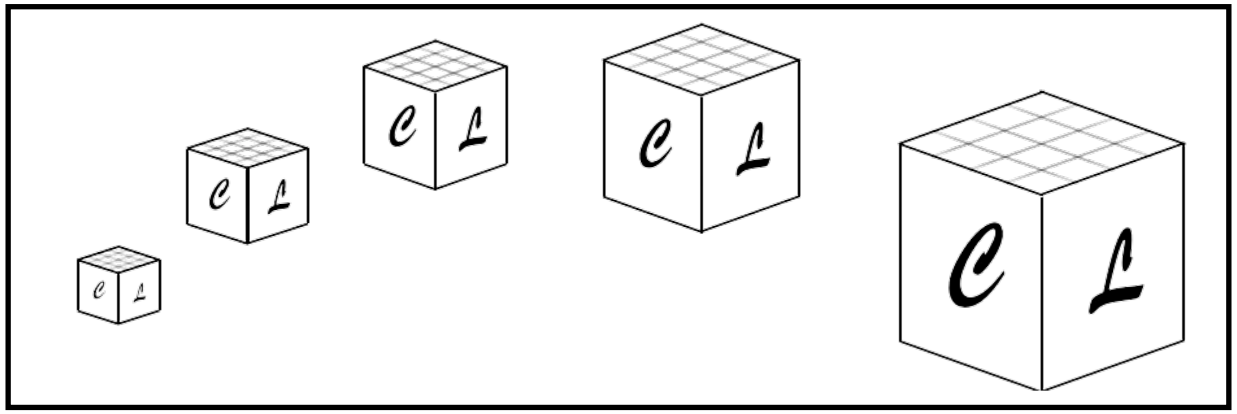
ICCUB
Institut de Ciències del Cosmos
UNIVERSITAT DE BARCELONA
EXCELENCIA MARÍA DE MAEZTU
04/2025-03/2031
MINISTERIO DE CIENCIA, INNOVACIÓN Y UNIVERSIDADES
AGENCIA ESTATAL DE INVESTIGACIÓN



CosmoLattice school:

Lecture 1: Welcome to the Lattice!

Francisco Torrentí
ICCUB & U. Barcelona

CosmoLattice

WHAT IS

CosmoLattice?

Lecture 1

WHAT IS

Cosmo?Lattice

Lecture 2

CosmoLattice

Lectures 3, 4, 5...

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Lattice: set of regular discrete coordinates

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1D Lattice

1. The Lattice

Lattice: set of regular discrete coordinates

1D Lattice

$$N = 8$$

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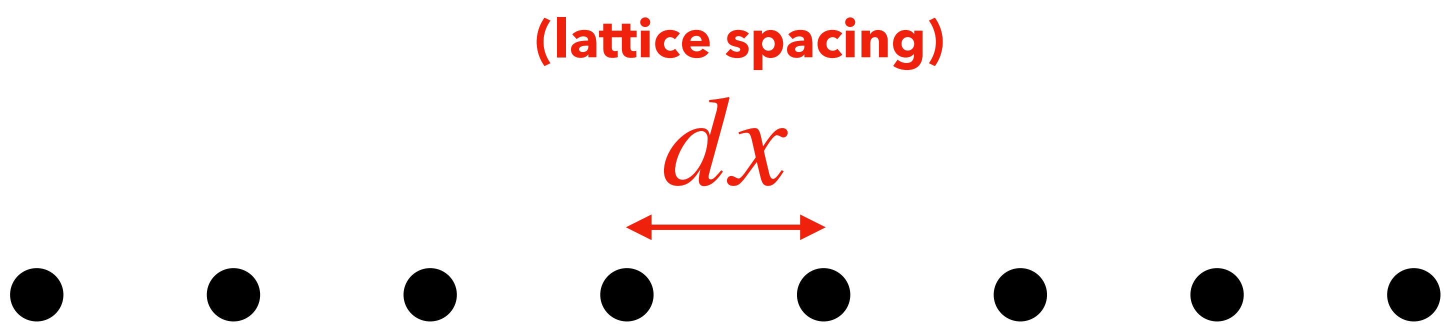
Lattice: set of regular discrete coordinates

1D Lattice

$N = 8$

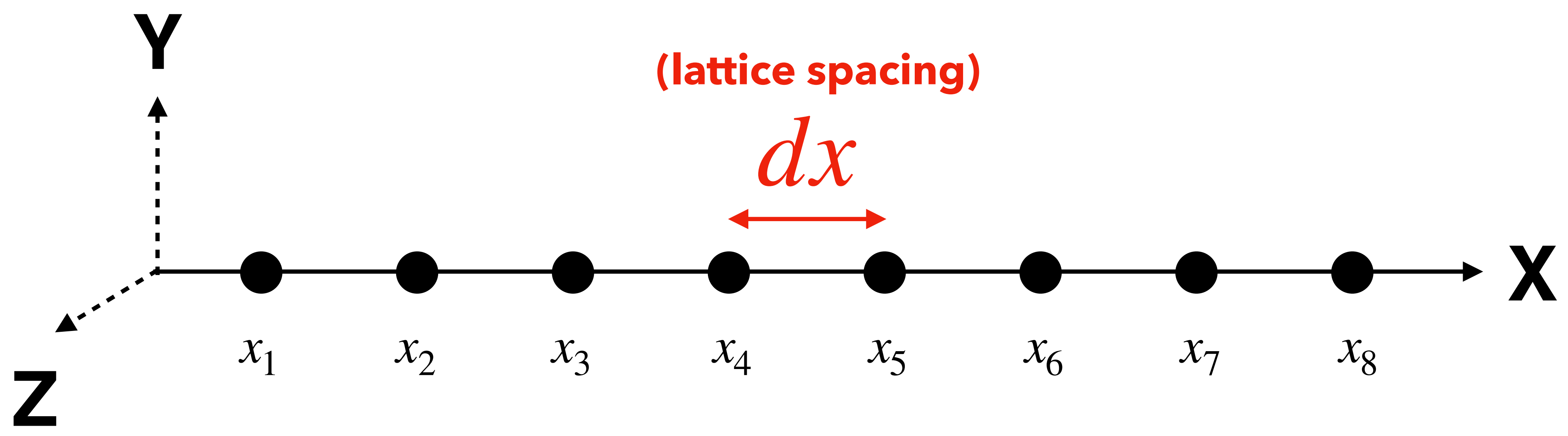


1. The Lattice

Lattice: set of regular discrete coordinates

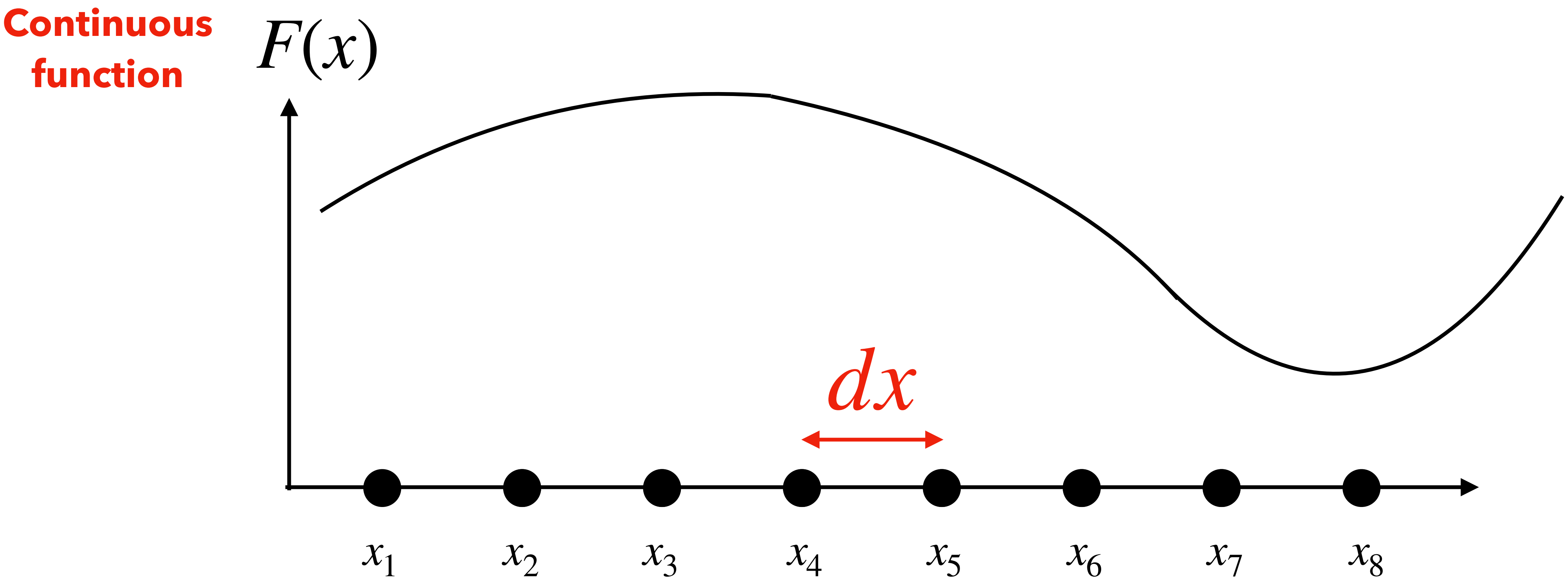
1D Lattice

$N = 8$



Coordinates: $\{x_n\};$ $\quad x_n \equiv x_* + n dx;$ $\quad n = 1, 2, \ldots, N$

1. The Lattice



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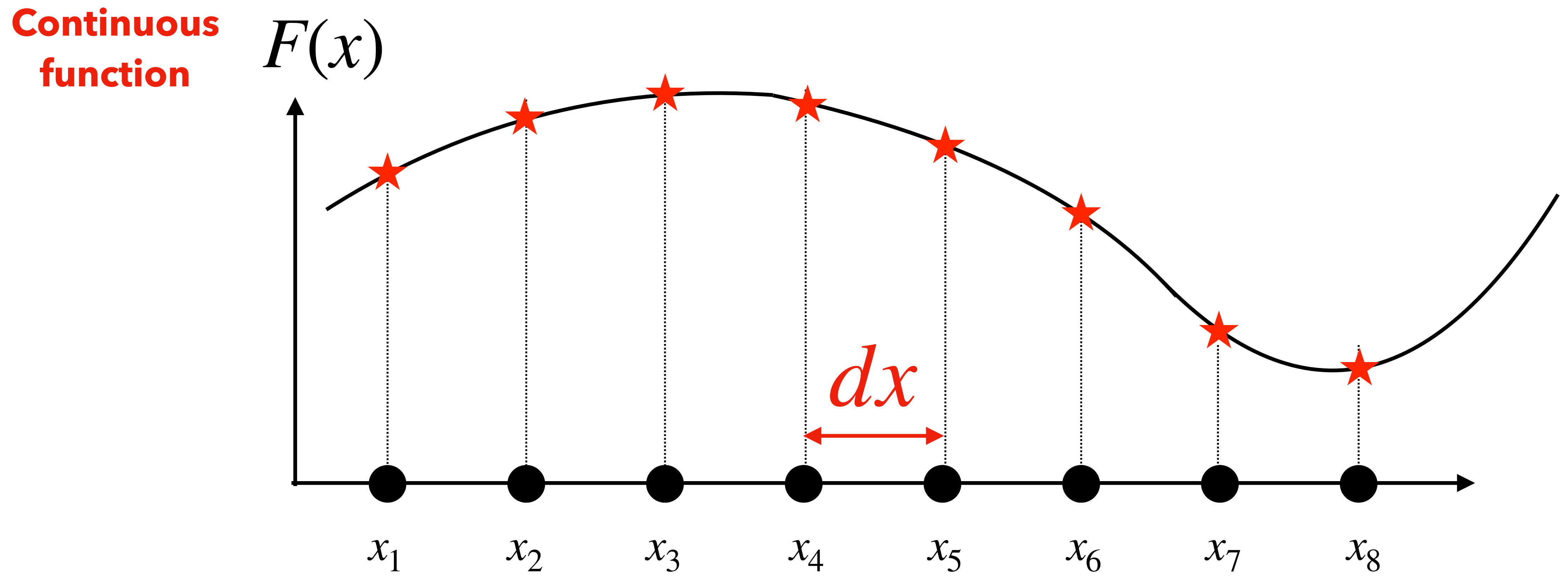

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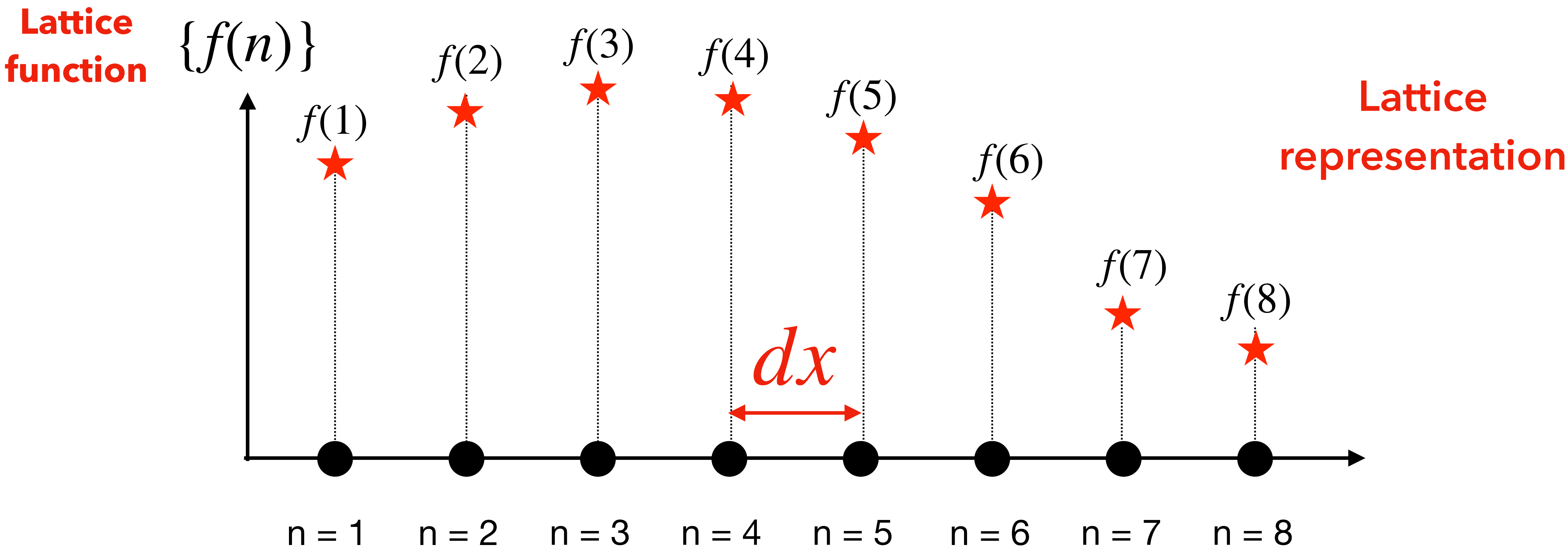
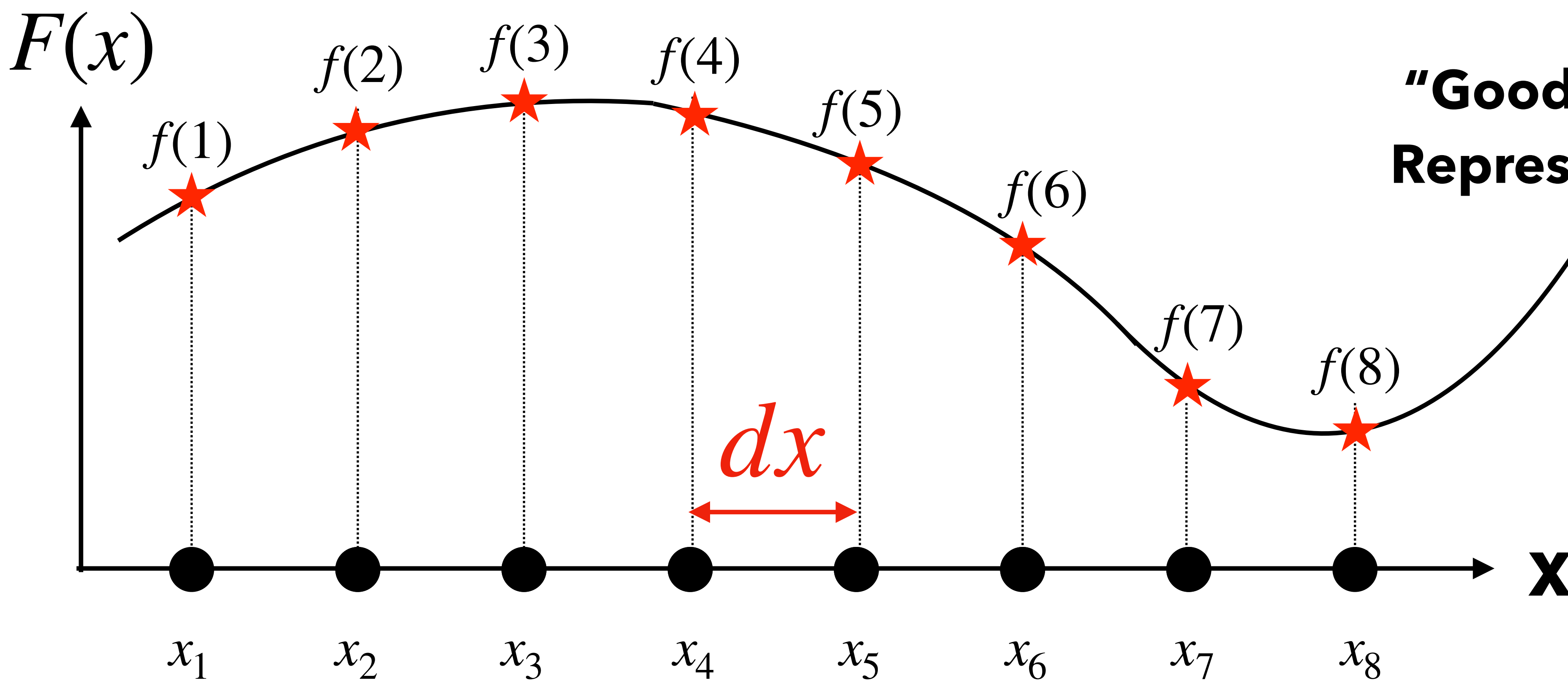


Coordinates: $\{x_n\}; \qquad x_n \equiv x_* + ndx; \qquad n = 1,\, 2,\, \ldots,\, N$

Lattice function: $\{f(n)\}; \qquad f(n) \equiv F(x_n \equiv x_* + ndx); \qquad n = 1,\, 2,\, \ldots,\, N$

1. The Lattice

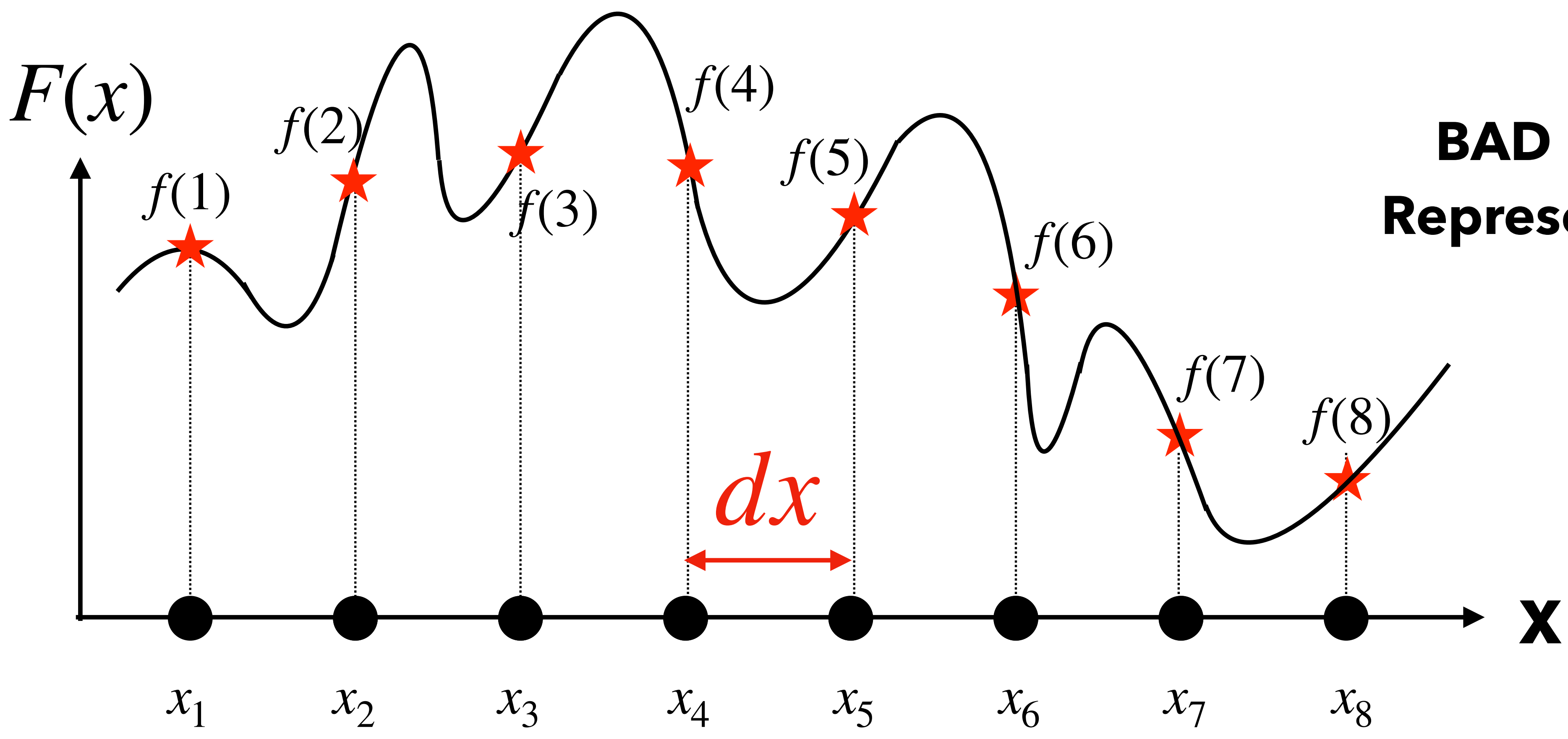


"Good" lattice Representation

Coordinates: $\{x_n\}$; $\quad x_n \equiv x_* + ndx$; $\quad n = 1, 2, \ldots, N$

Lattice function: $\{f(n)\}$; $\quad f(n) \equiv F(x_n \equiv x_* + ndx)$; $\quad n = 1, 2, \ldots, N$

1. The Lattice



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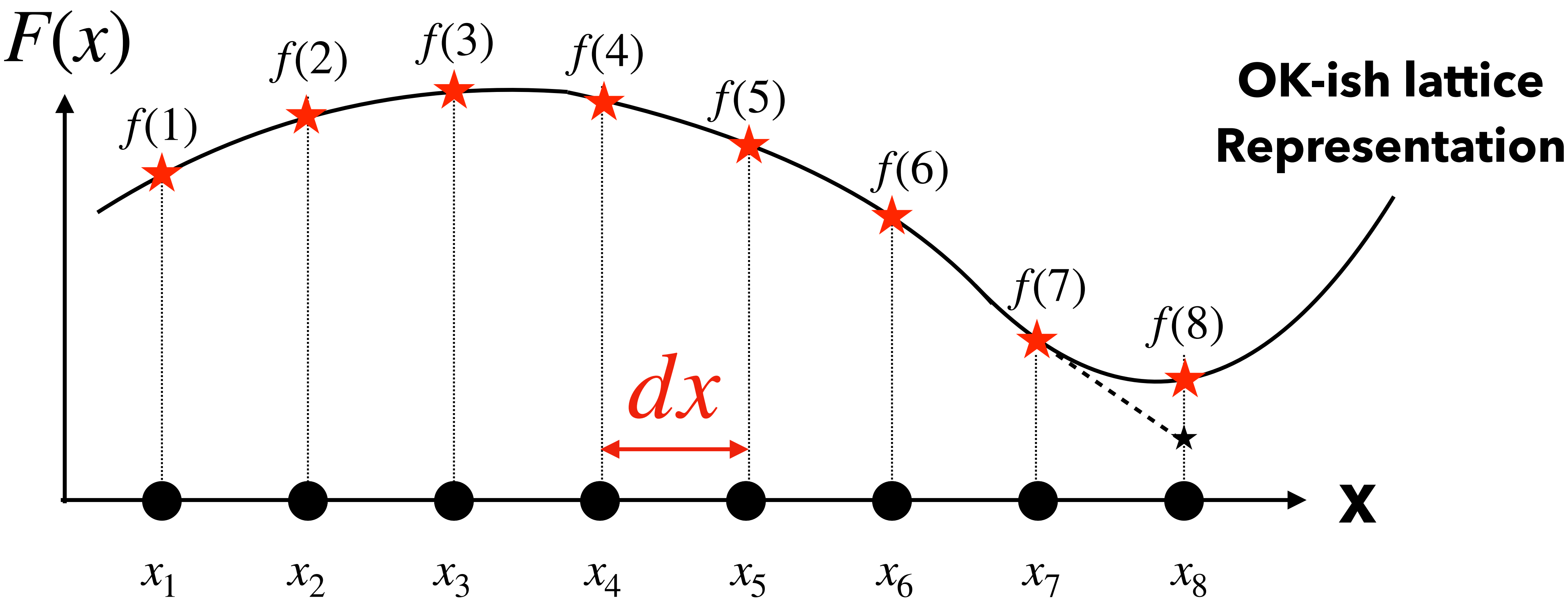

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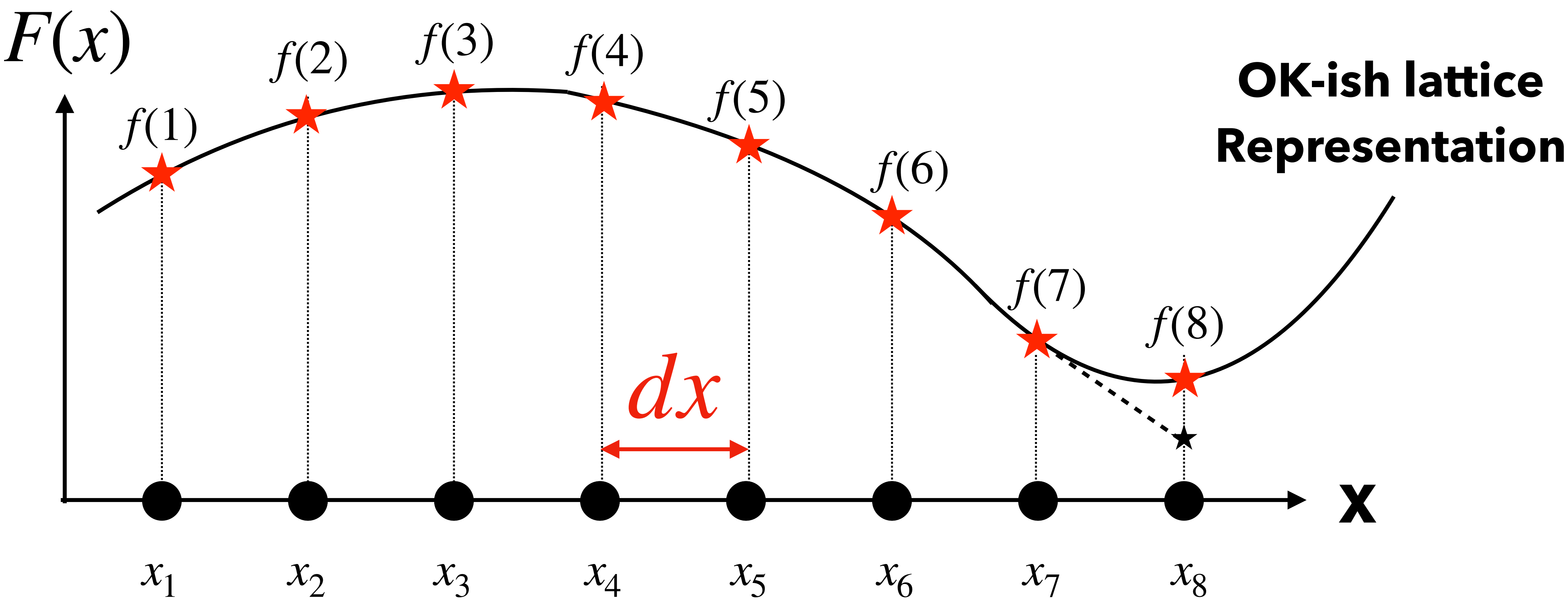


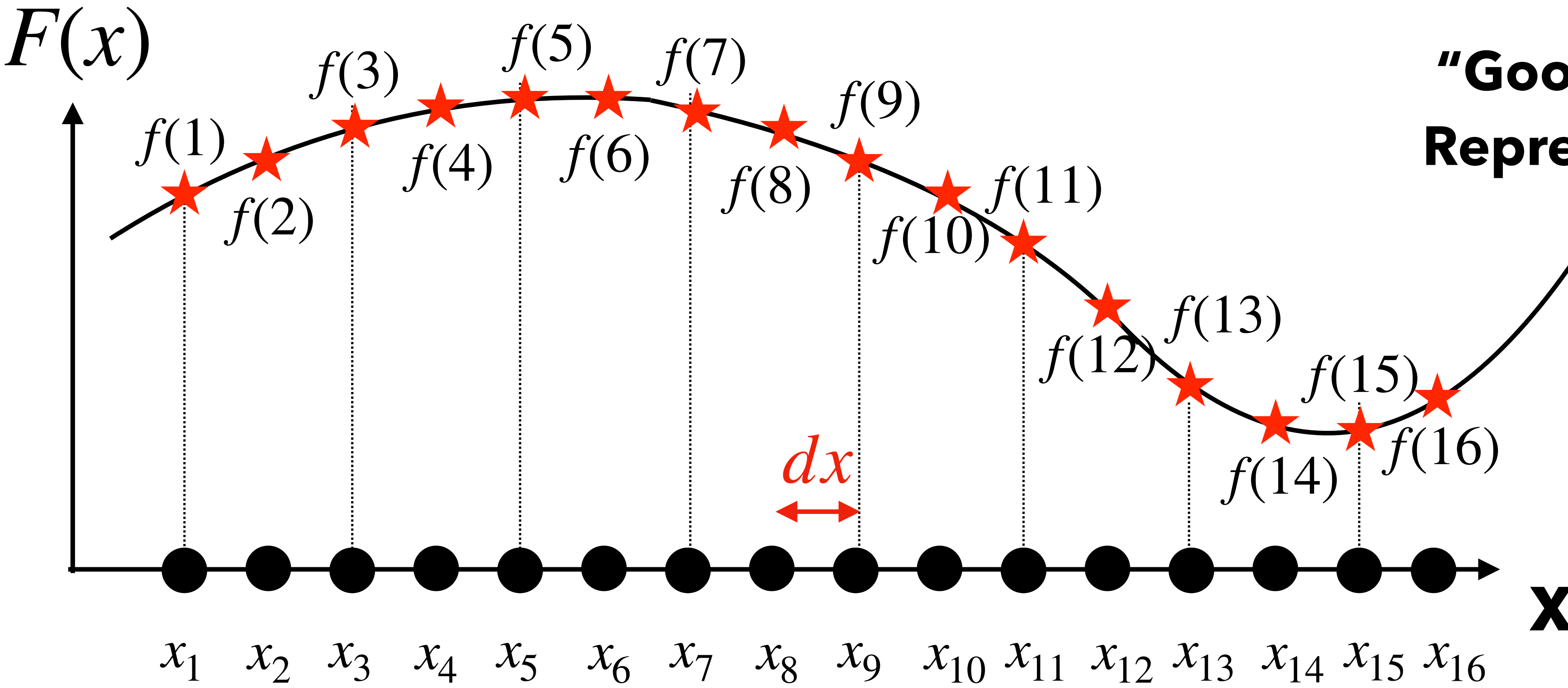
dx must be chosen such that:

$$|f(n+1) - f(n)| \sim |F'(x_{n+1/2})| dx$$

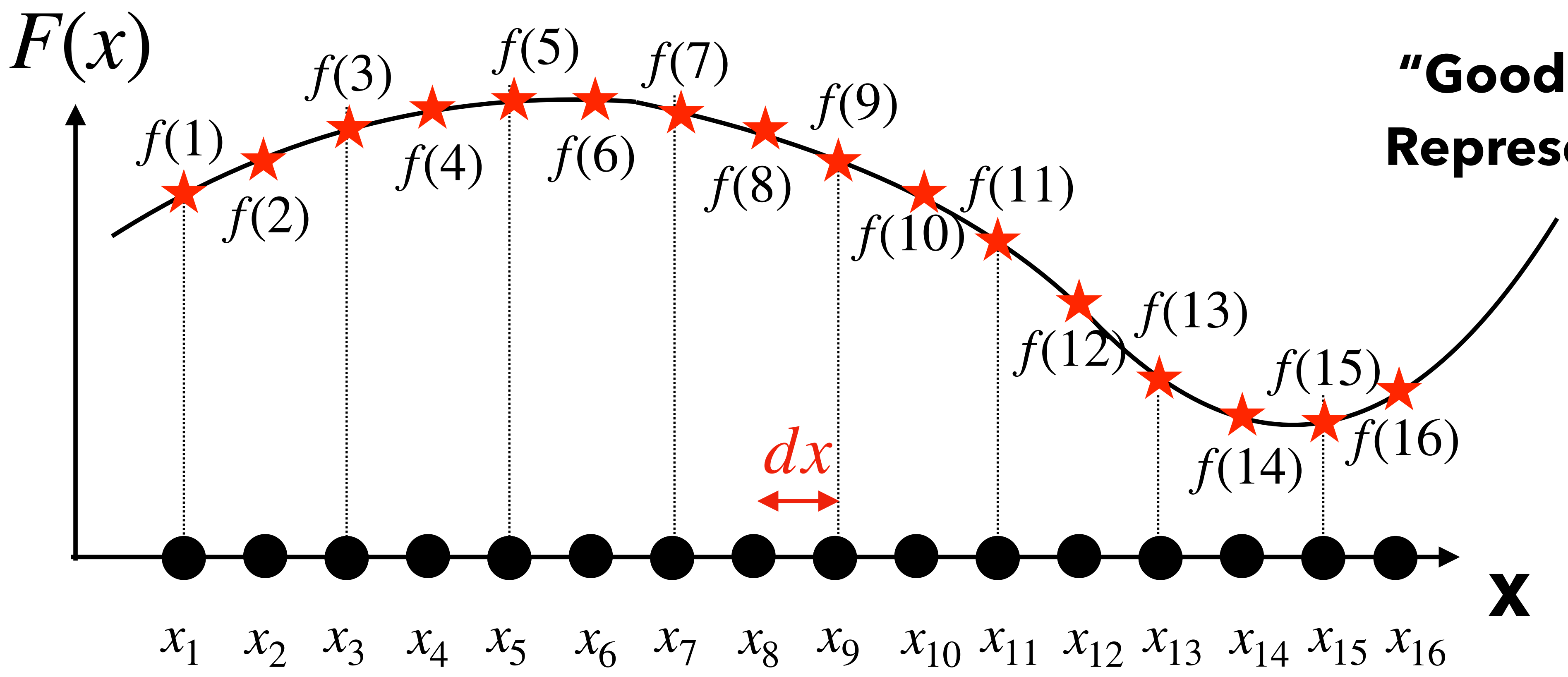
to represent well your derivatives!

(more on this later on)

1. The Lattice



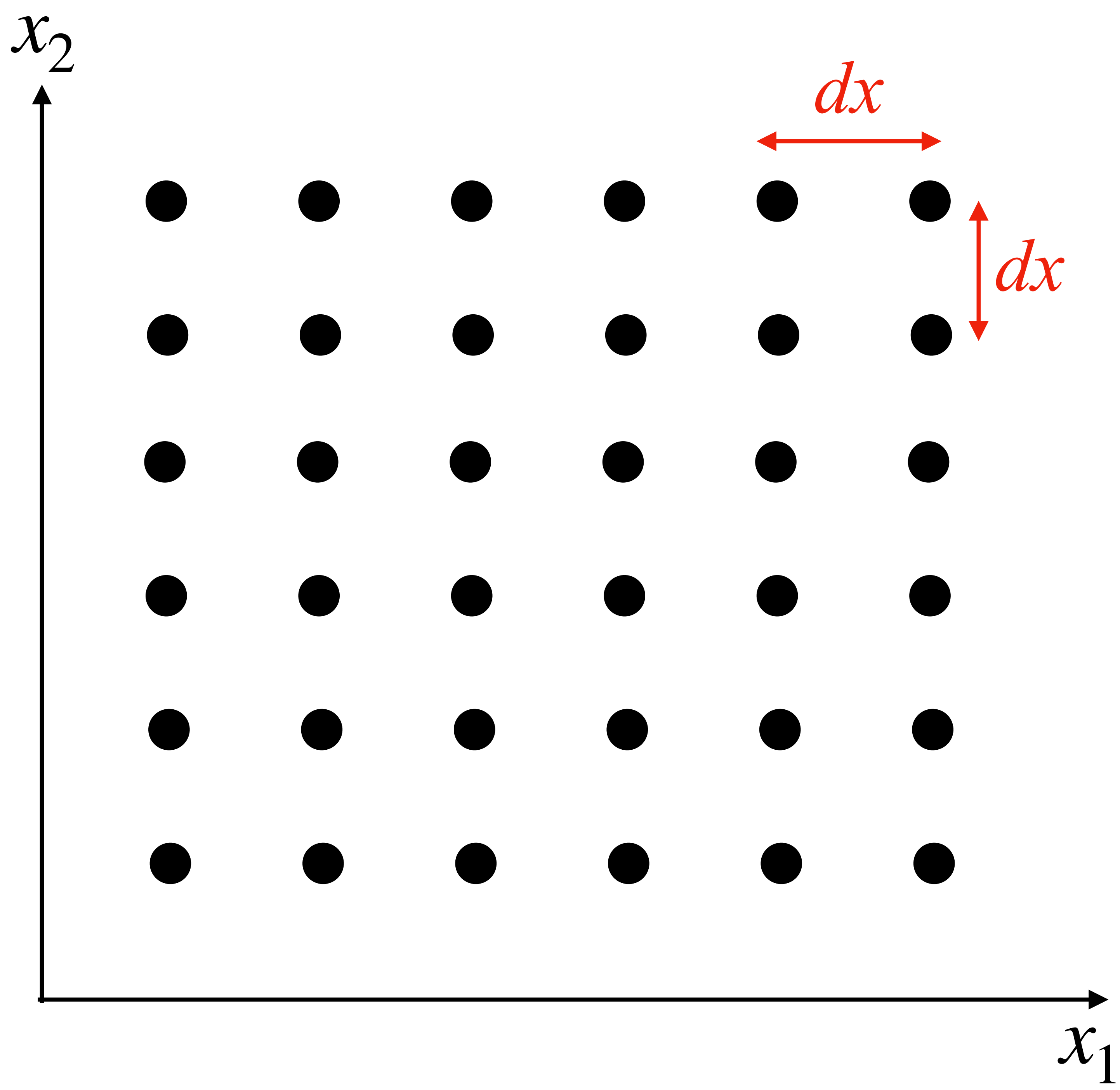
"Good" lattice Representation



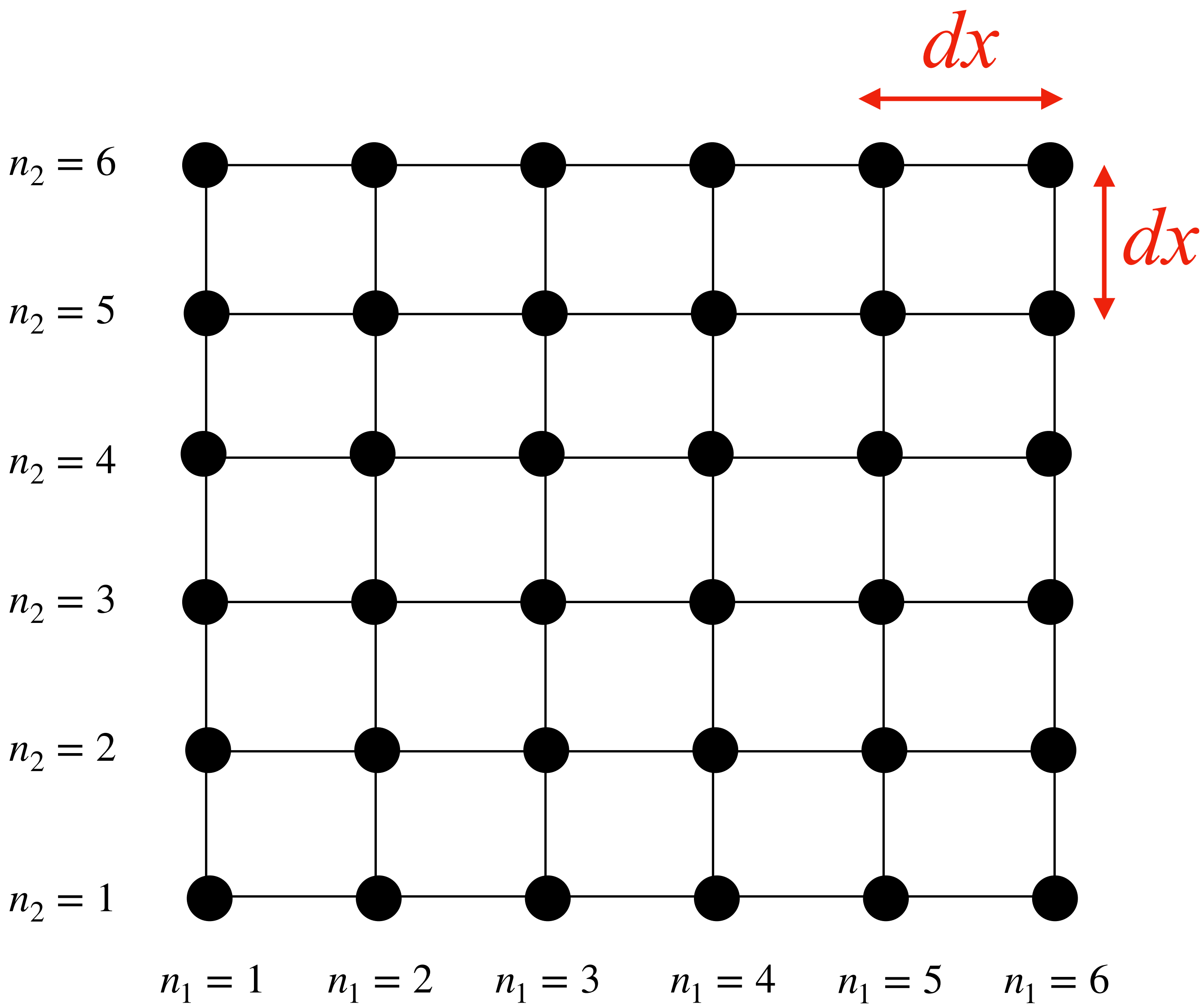
"Good" lattice Representation

Better lattice representation

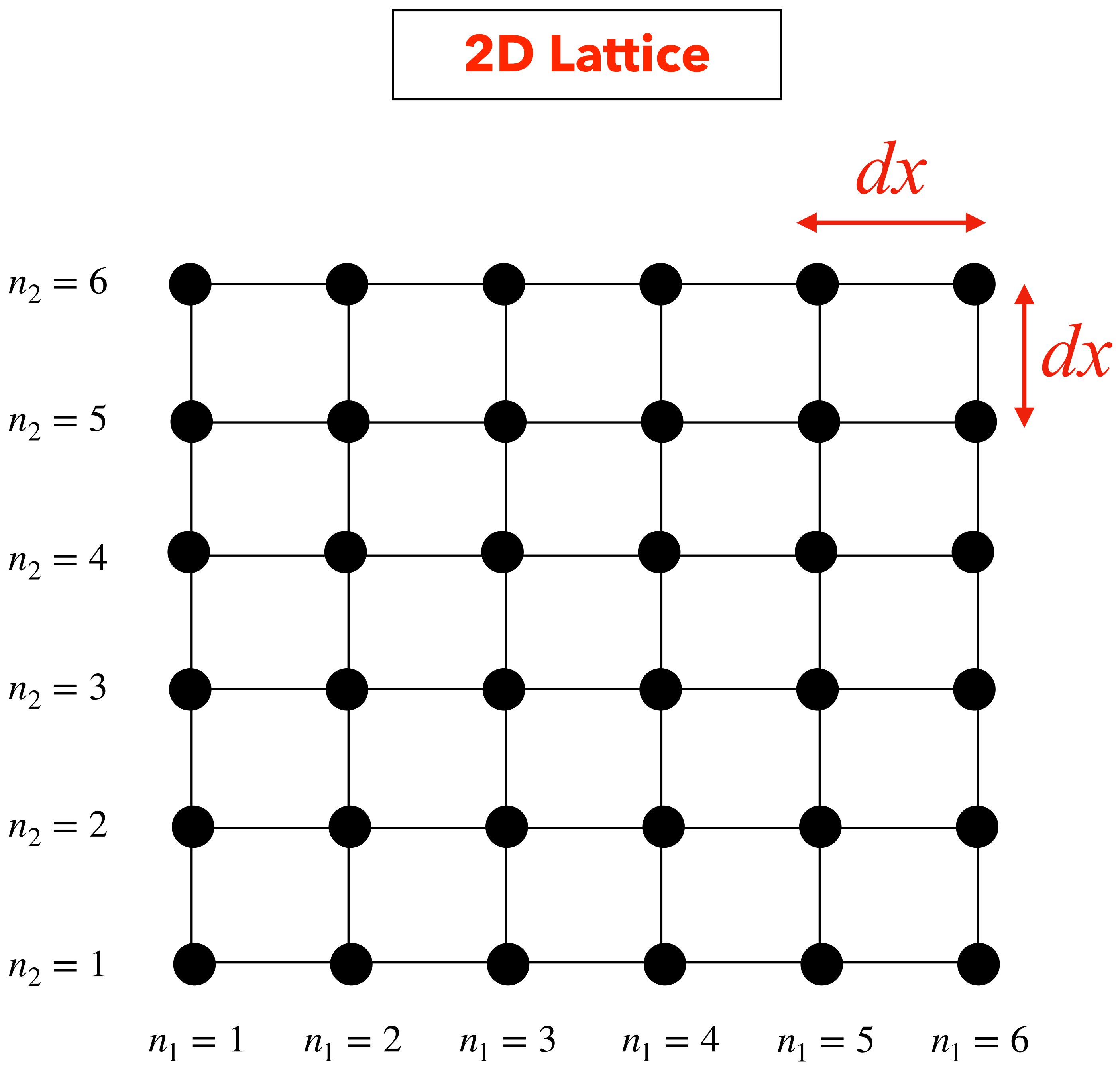
Price to pay: must store twice as many numbers


x_2
dx
dx
x_1

2D Lattice



1. The Lattice



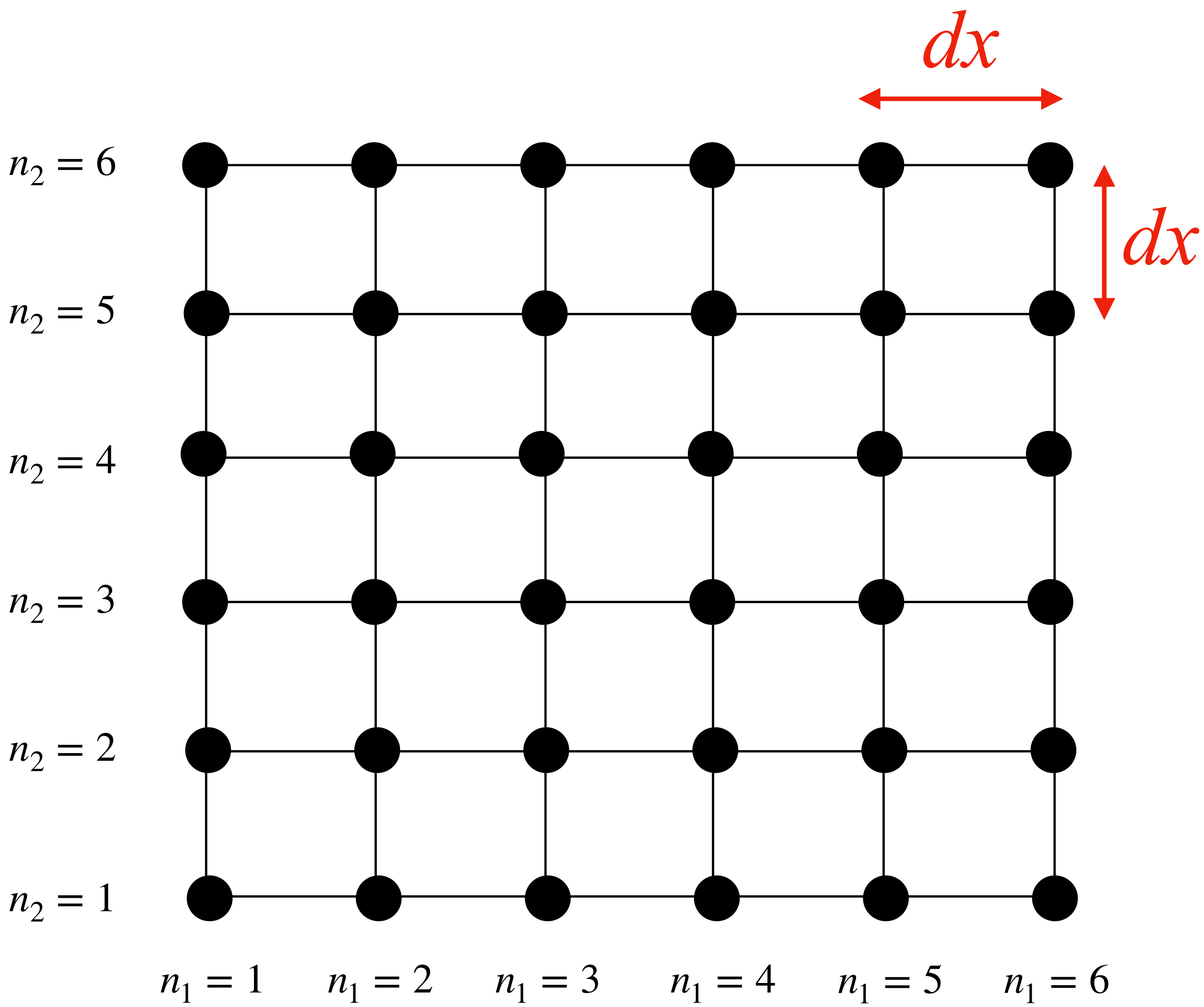
Coordinates:

$$\{\mathbf{x}_{n_1 n_2}\} \qquad (N^2 \text{ entries})$$

$$n_i = 1,\ 2,\ \ldots.,\ N \qquad (i = 1{,}2)$$

1. The Lattice

2D Lattice



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$$F(\mathbf{x}) \longrightarrow f(n_1, n_2) \equiv F(\mathbf{x}_{n_1 n_2})$$

$$\boxed{\{f(\mathbf{n})\}; \qquad \mathbf{n} \equiv (n_1, n_2)}$$

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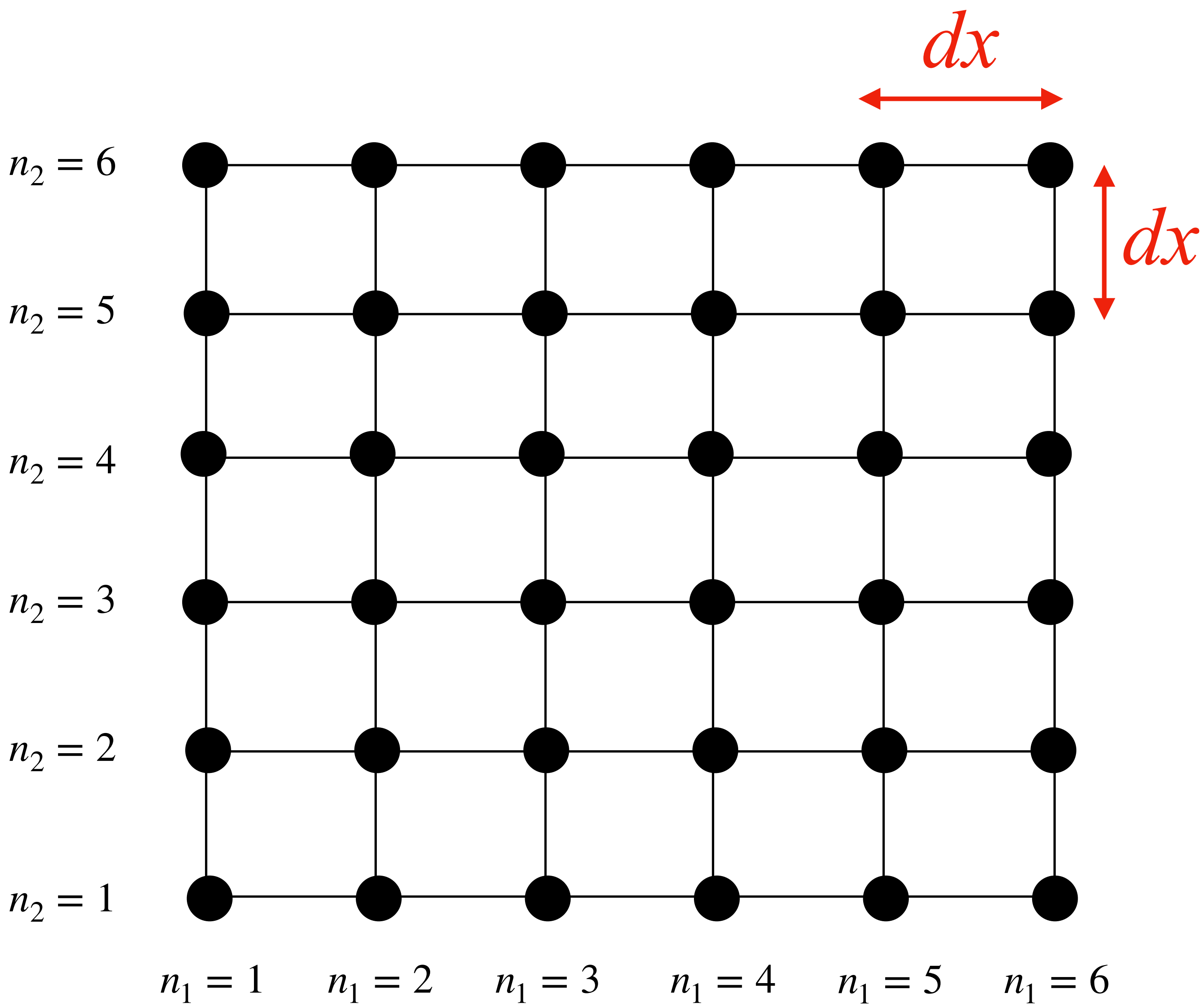

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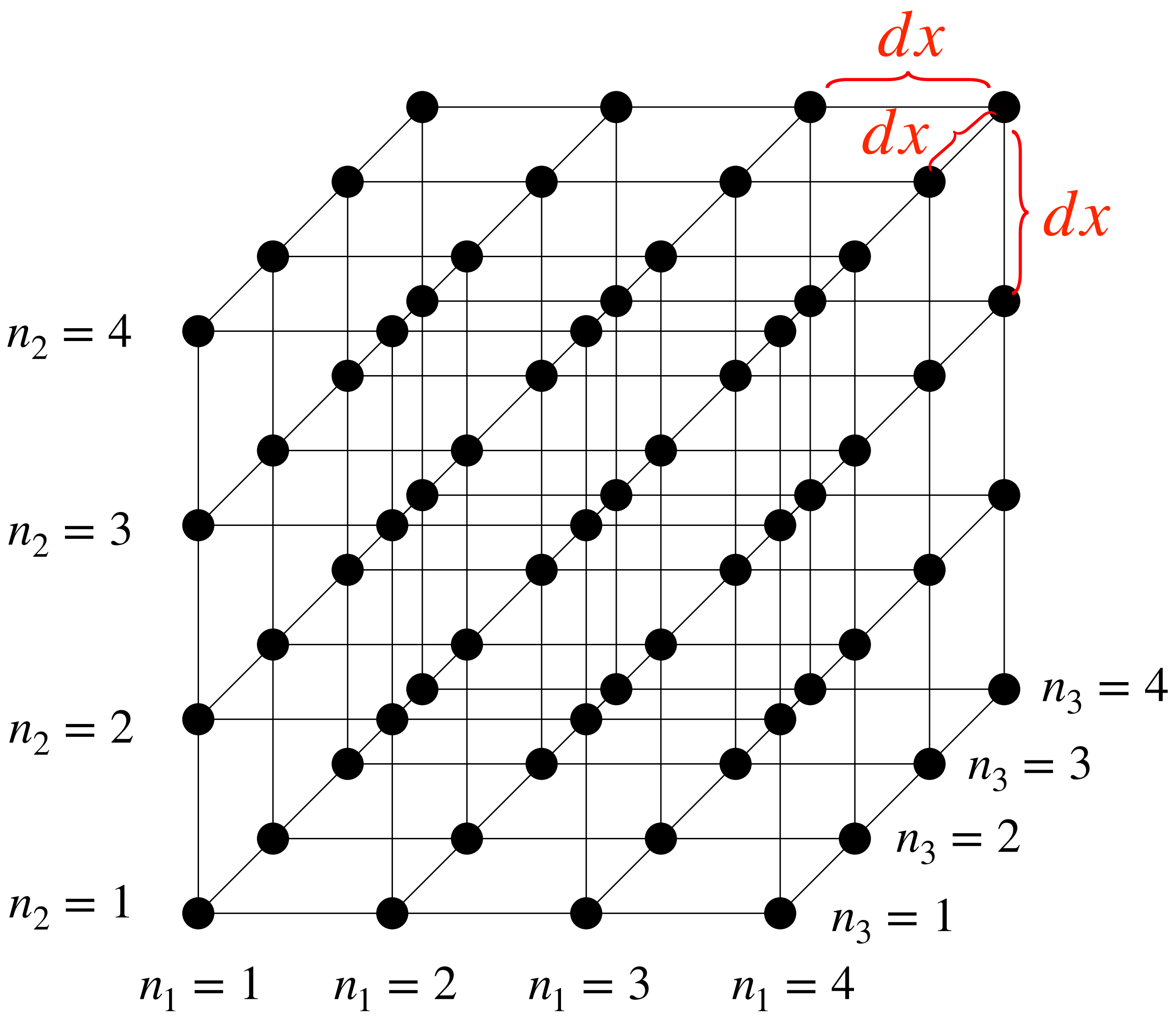
"Good" lattice spacing IF:

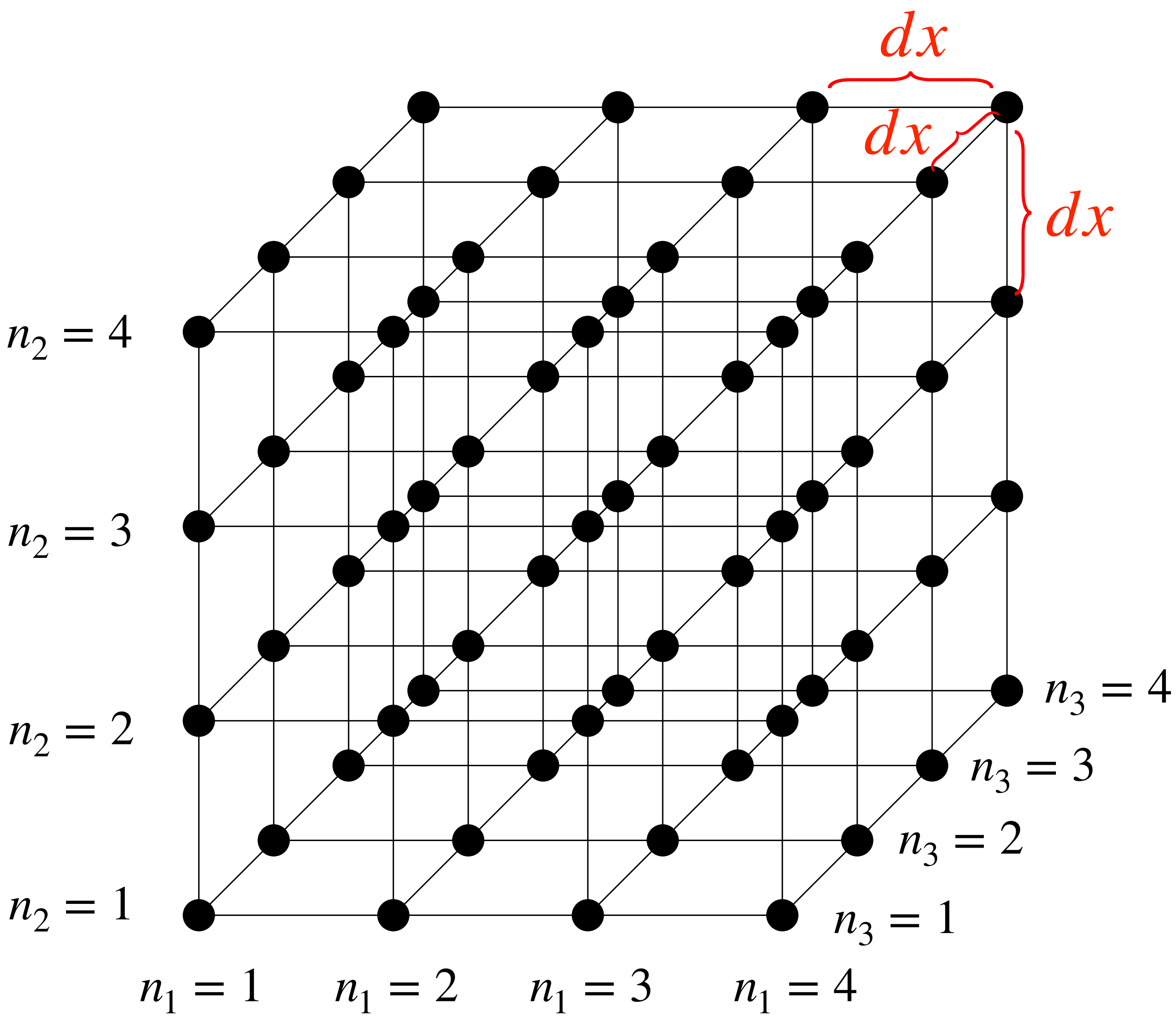
$$\frac{f(\mathbf{n}+\hat{i}) - f(\mathbf{n})}{dx} \approx \left.\frac{\partial F}{\partial x_i}\right|_{(\mathbf{n}+\hat{i}/2)dx}$$

$\hat{i} \equiv$ unit vector in i-direction (i = 1,2)

3D Lattice



3D Lattice



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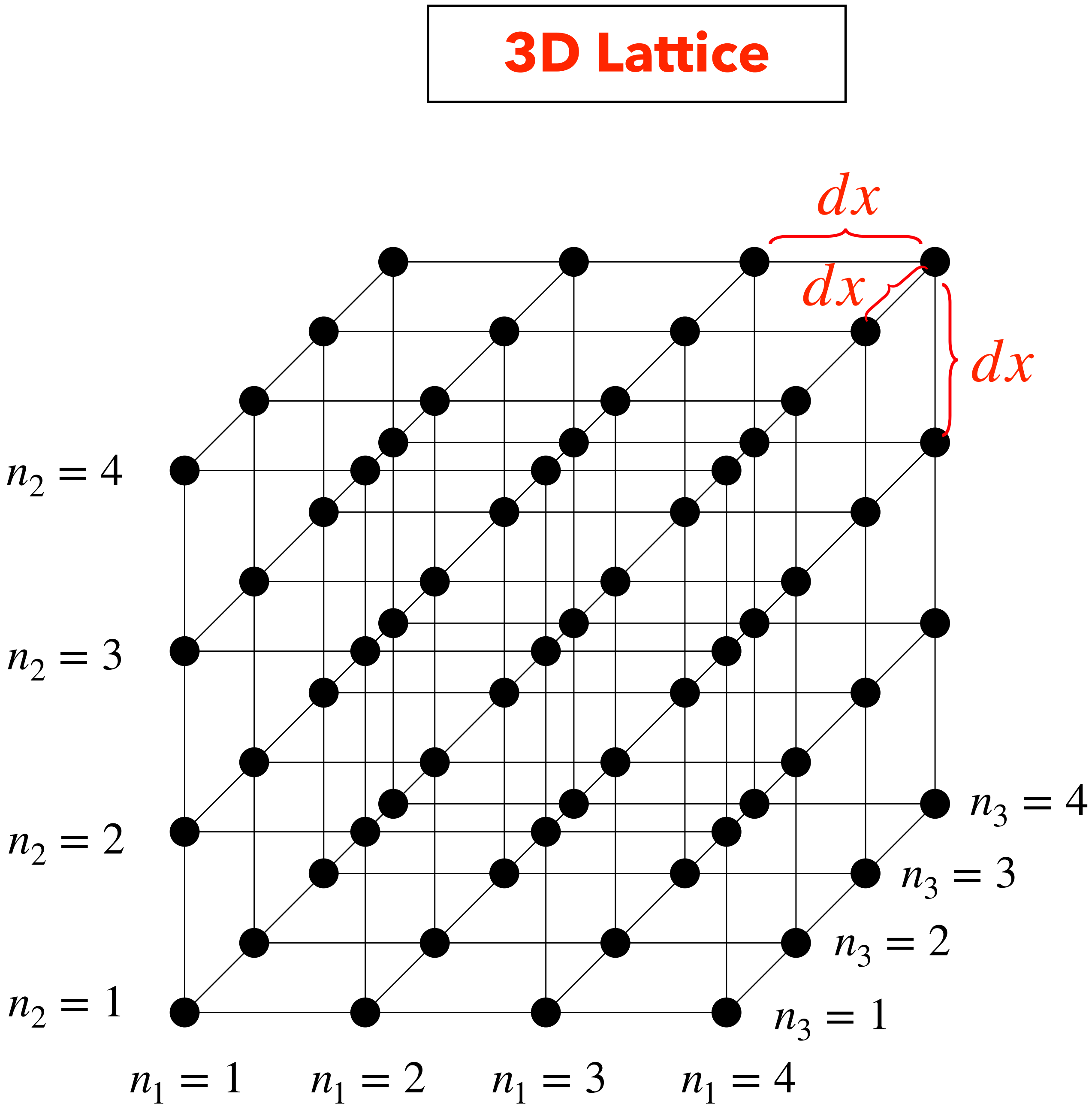

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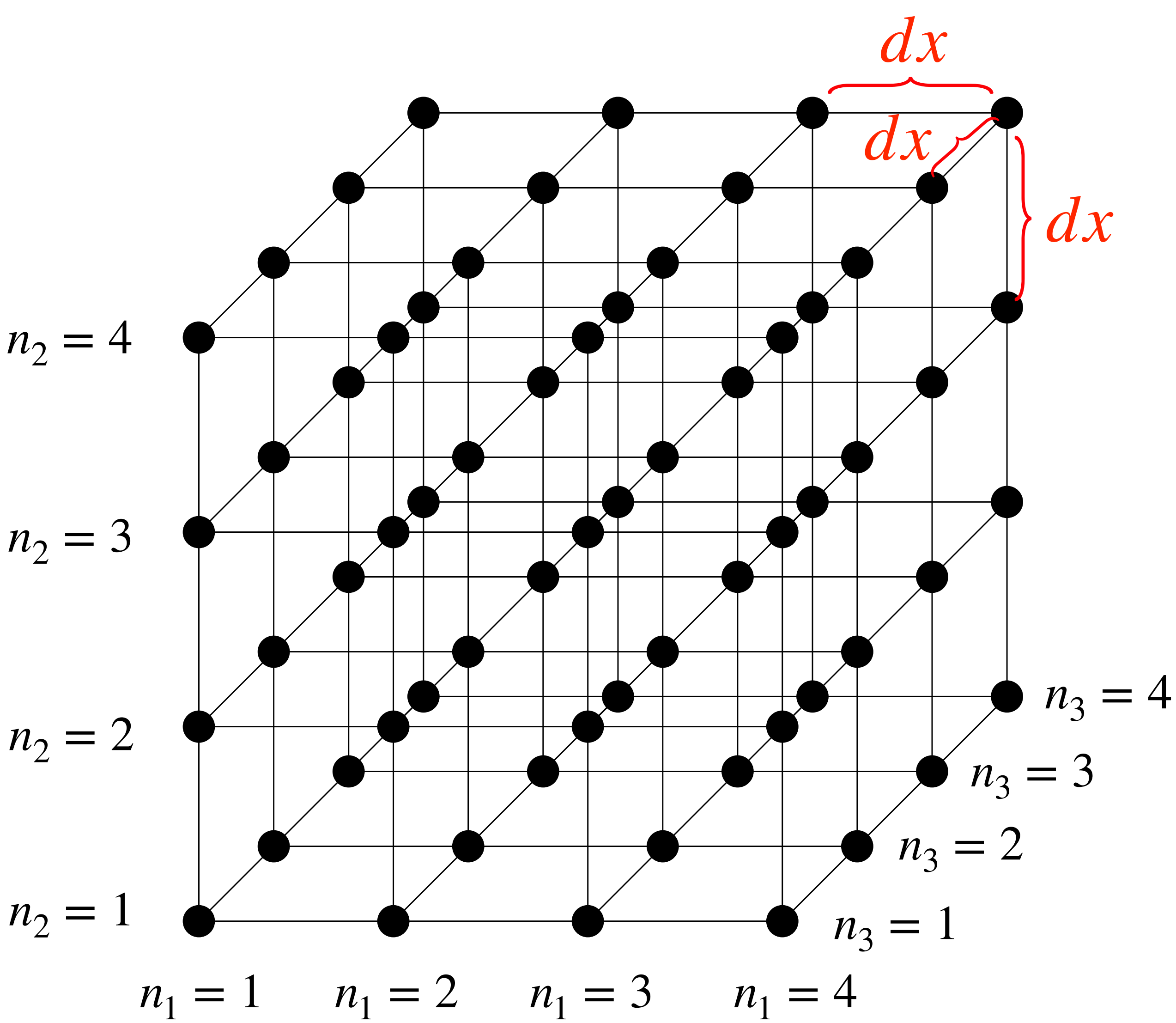

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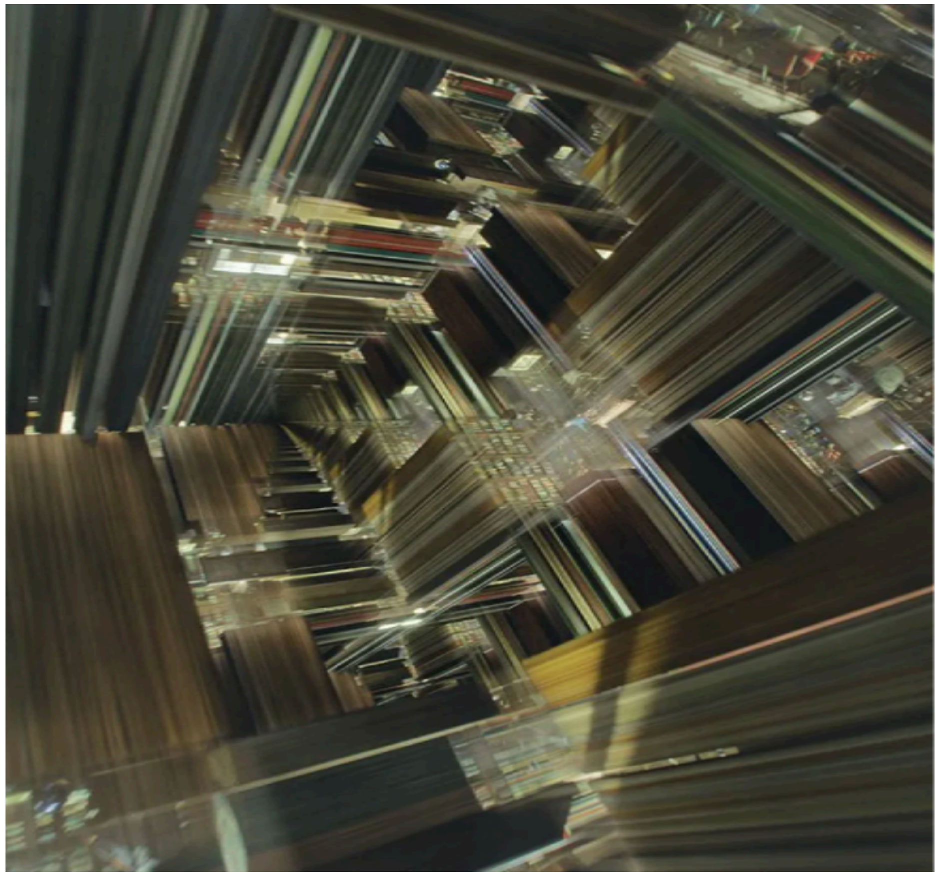
dD Lattice

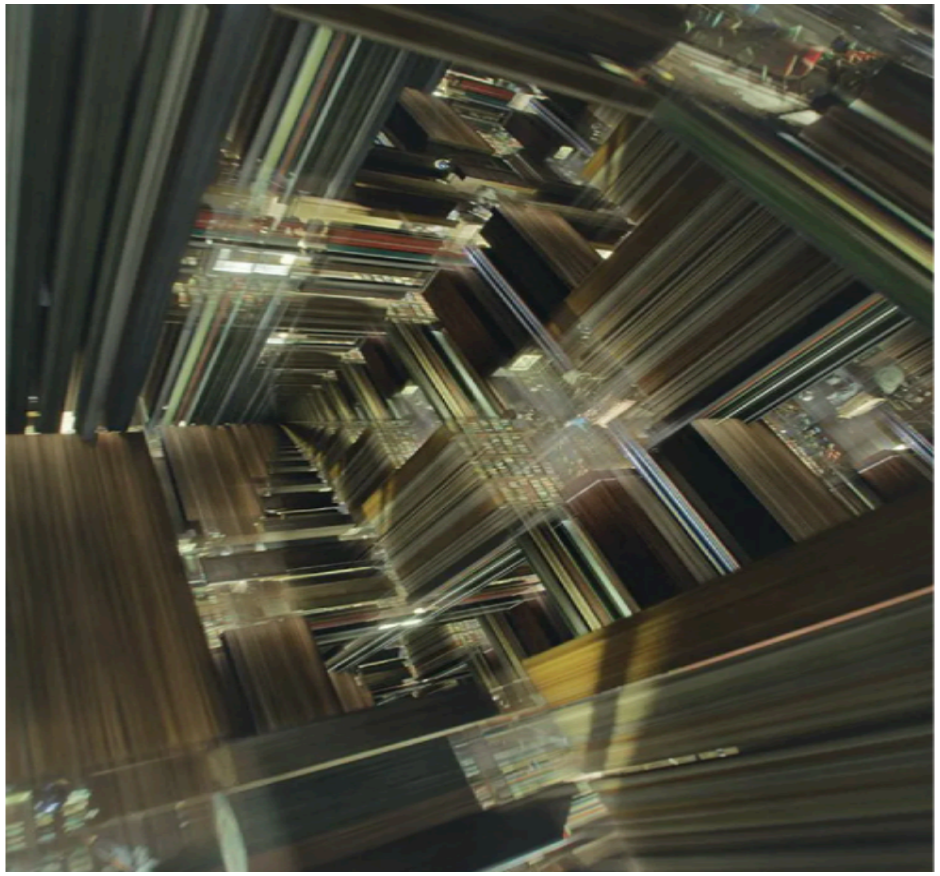


Coordinates:

$$\{\mathbf{x}_{n_1 n_2 \ldots n_d}\} \quad (N^d \text{ entries})$$

$$n_i = 1,\, 2,\, \ldots,\, N \qquad i = 1,2,\ldots,d$$

dD Lattice



Coordinates:

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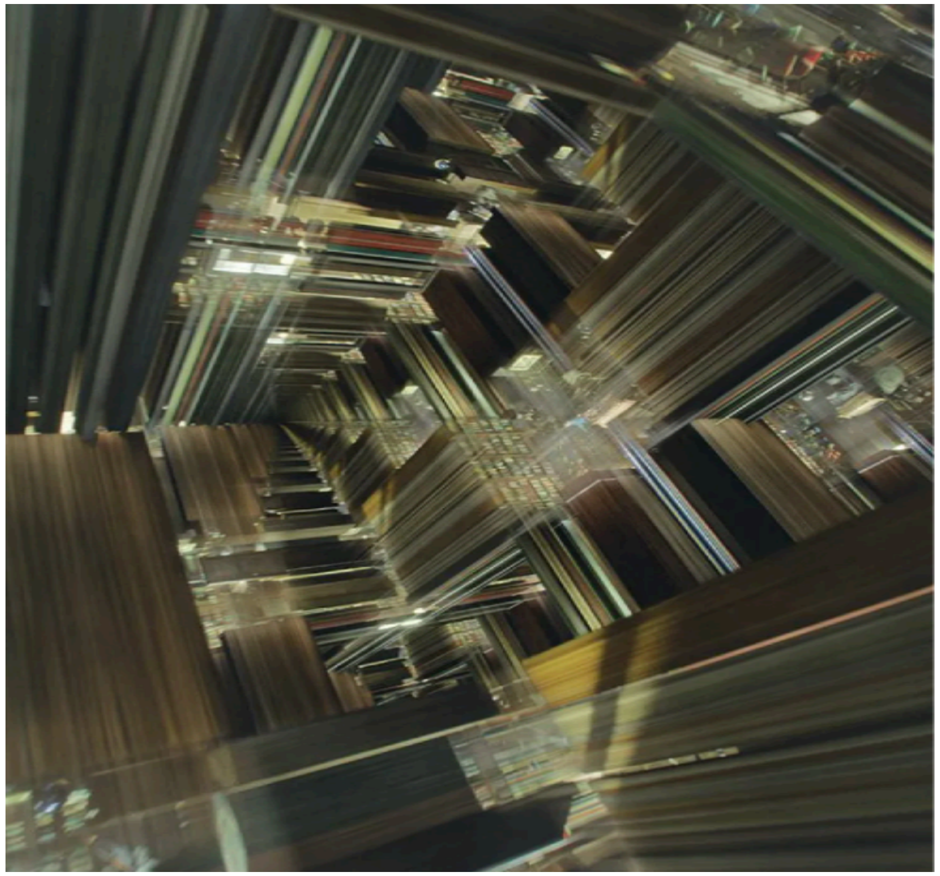

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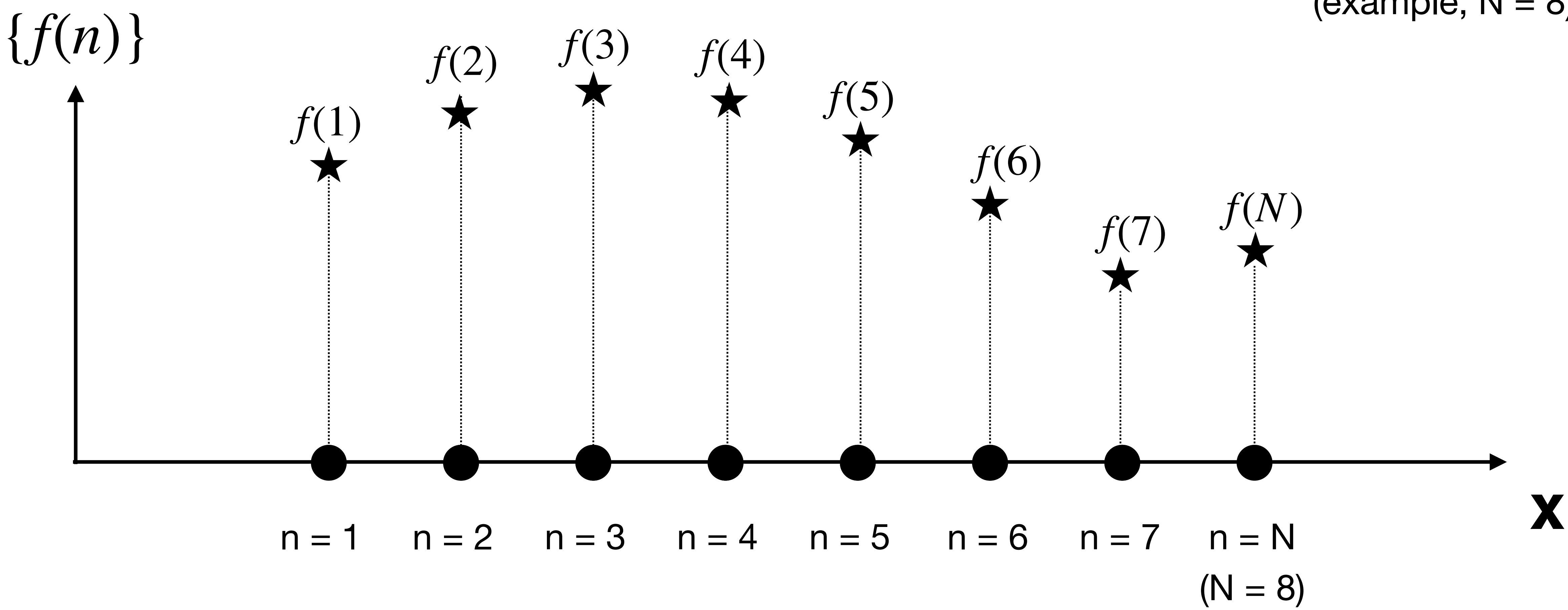
2. Boundary conditions

What about the boundaries?

(example, N = 8)



2. Boundary conditions

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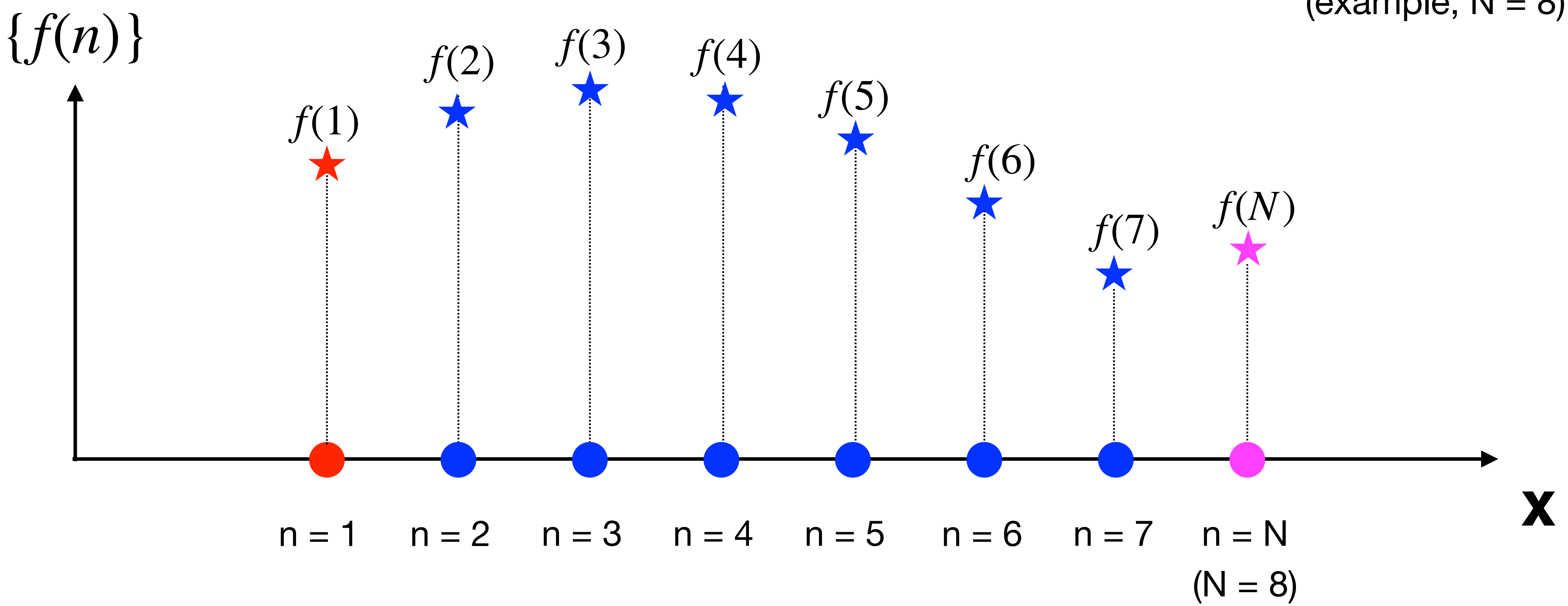
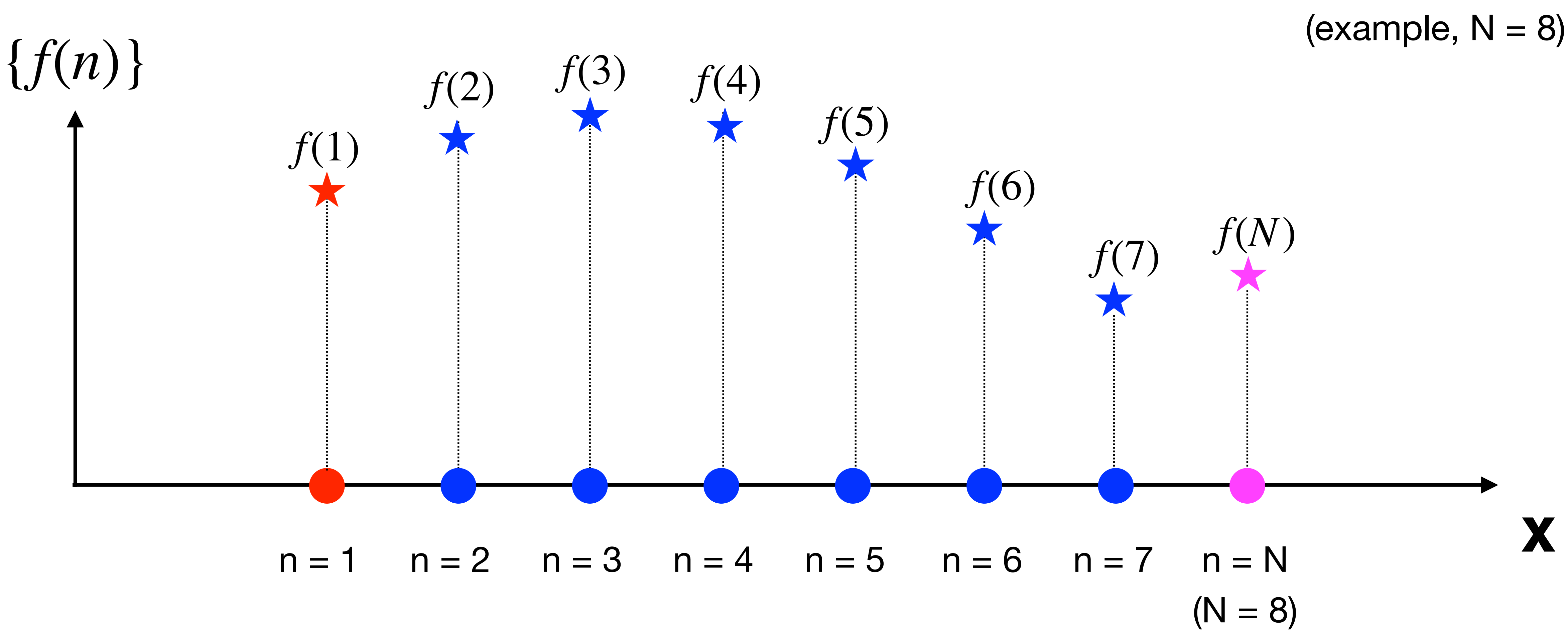


2. Boundary conditions

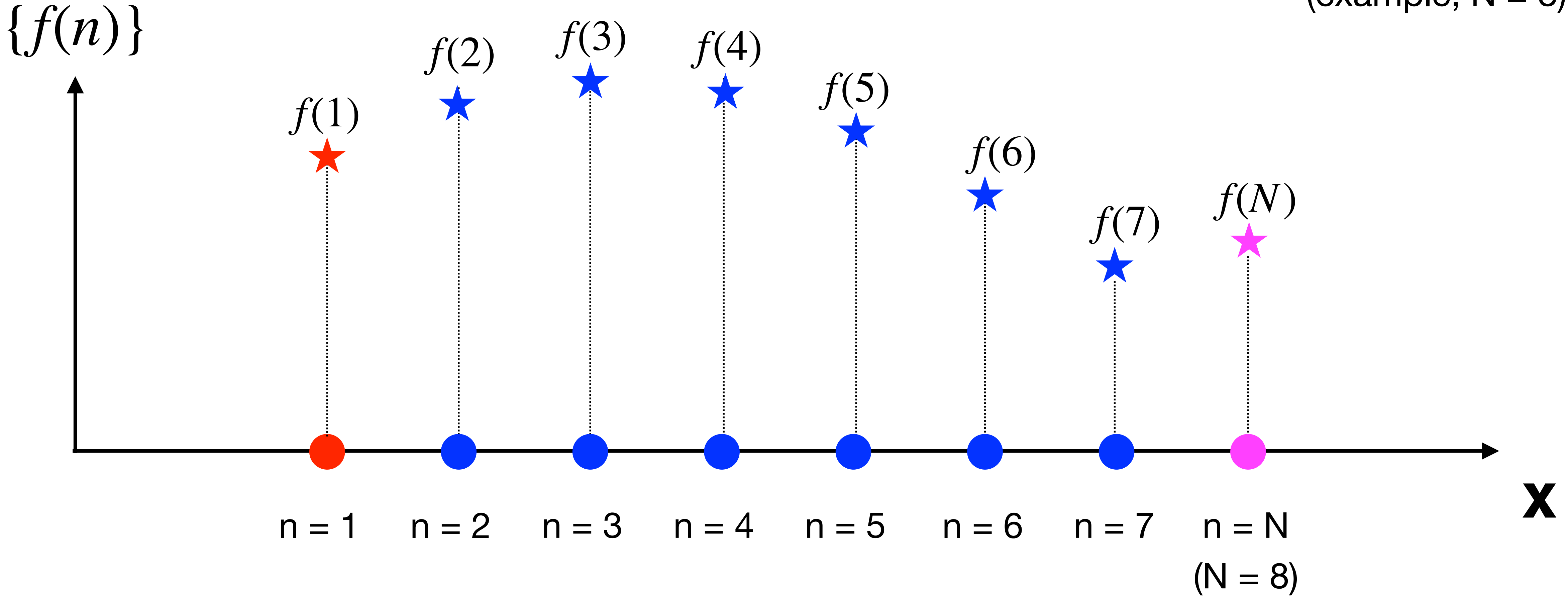
What about the boundaries?



Fixed Boundary Conditions $\left\{ \; f(1) \equiv F(x_1) \; ; \; f(N) \equiv F(x_N) \right.$

What about the boundaries?

(example, N = 8)



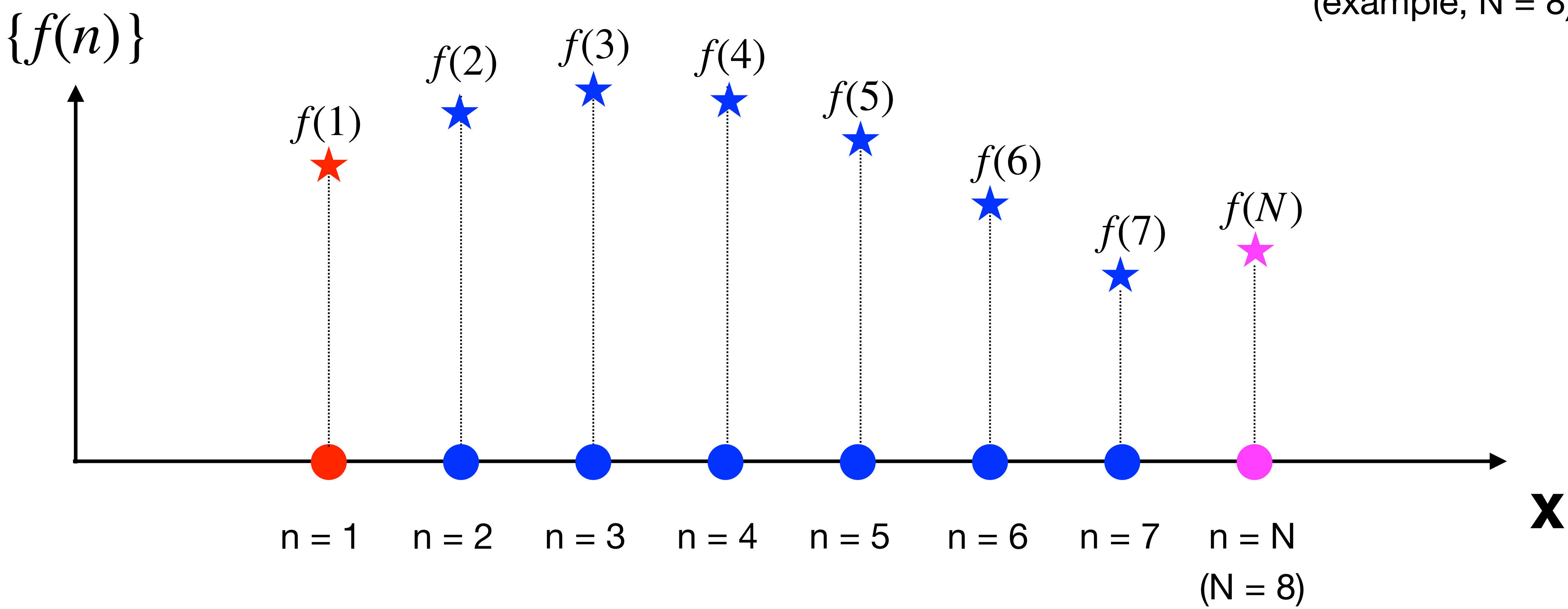
Fixed Boundary Conditions

$$\begin{cases} f(1) \equiv F(x_1) \;;\; f(N) \equiv F(x_N) \\ f(n,t) \equiv F(x_n,t), \quad n = 2,3,..,N-1 \end{cases}$$

2. Boundary conditions

What about the boundaries?

(example, N = 8)



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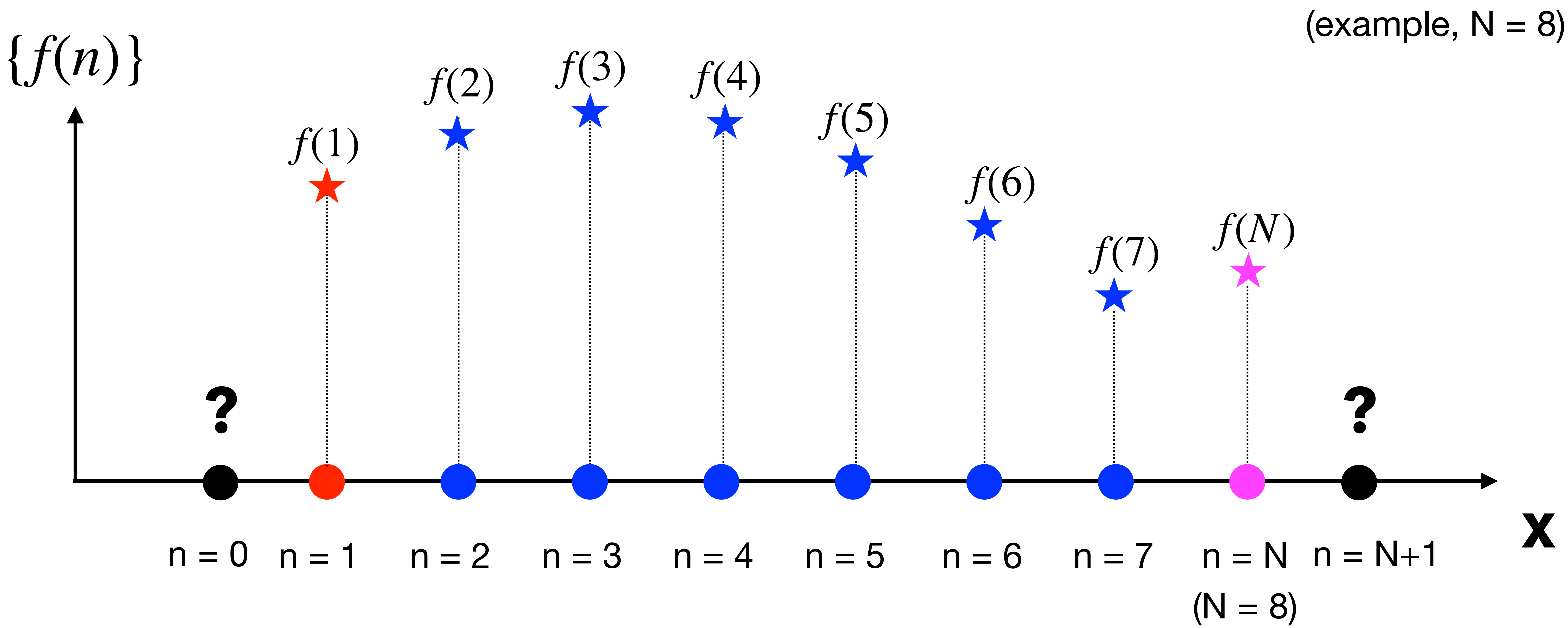
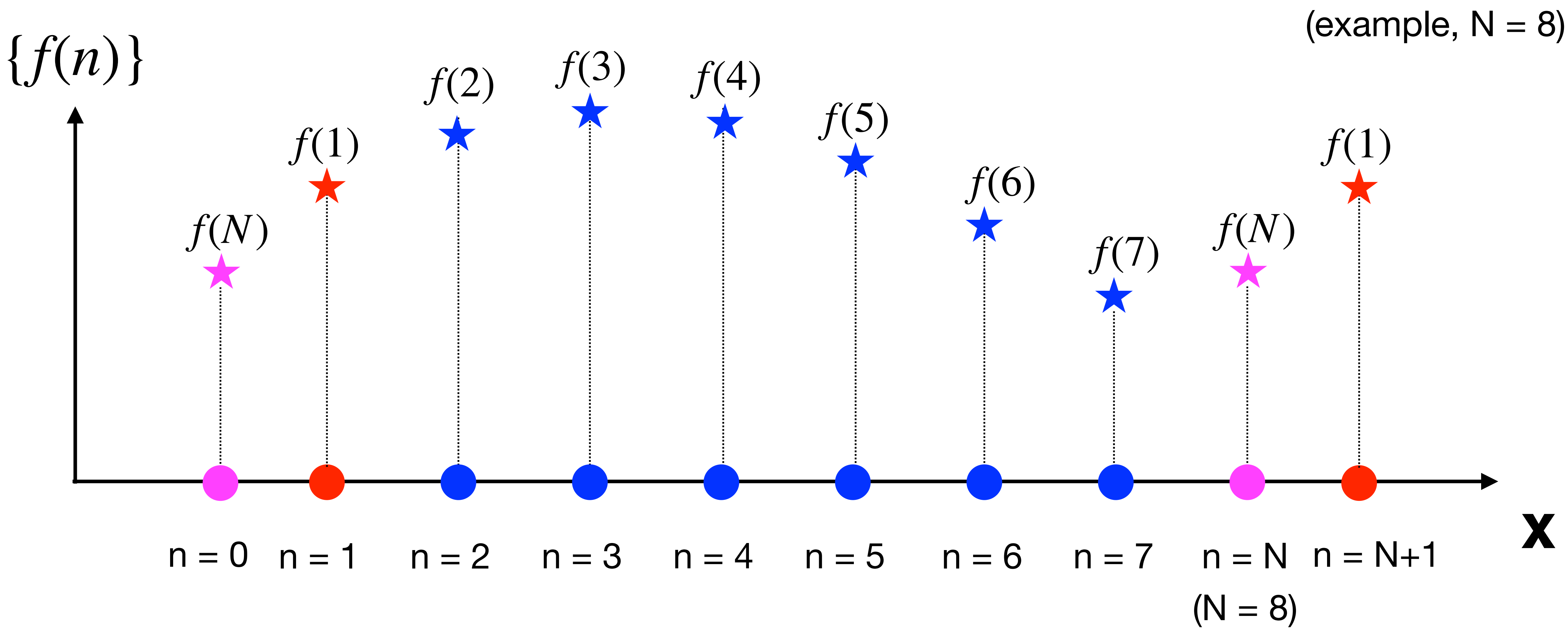


2. Boundary conditions

What about the boundaries?

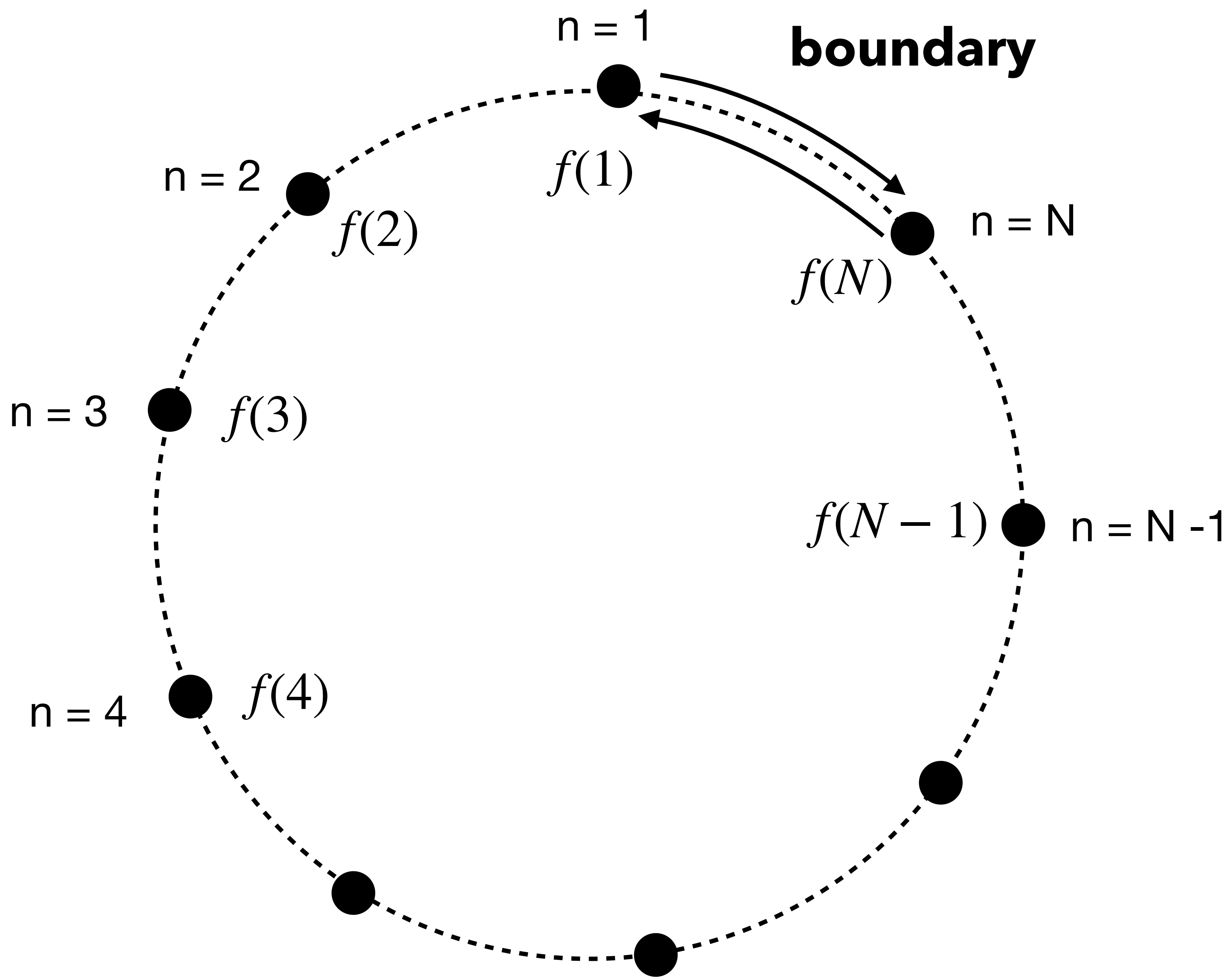


Periodic Boundary Conditions

$$\begin{cases} f(N+1,t) \equiv f(1,t) \ ; \ f(0,t) \equiv f(N,t) \\ f(n,t) \equiv F(x_n,t) \ , n = 1,2,3,..,N \end{cases}$$

2. Boundary conditions

Periodic Boundary Conditions: 1D

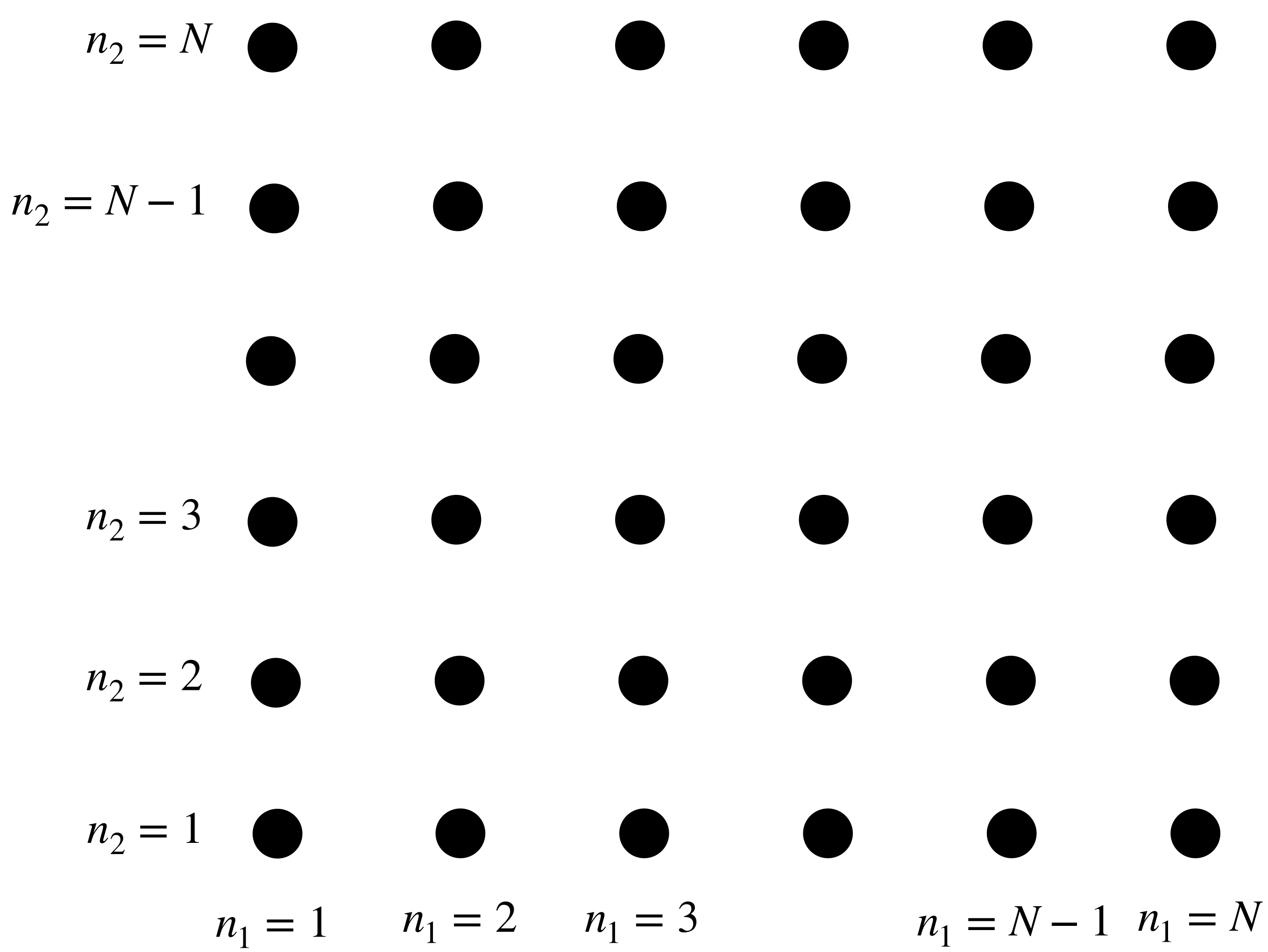


$$\left.\begin{aligned} f(N+1) &\equiv f(1) \\ f(0) &\equiv f(N) \end{aligned}\right\}$$

Torus (T1)

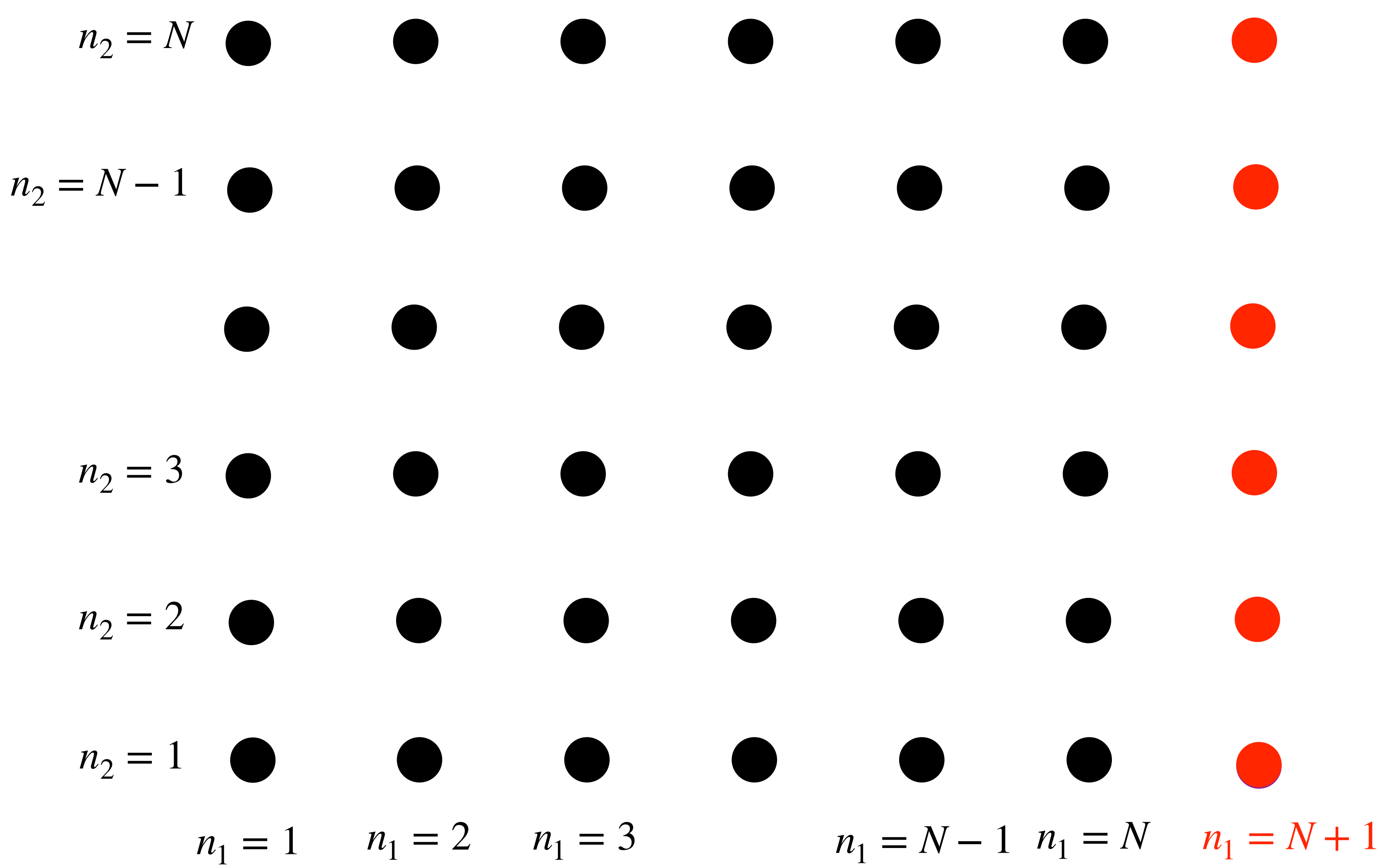
2. Boundary conditions

Periodic Boundary Conditions: 2D



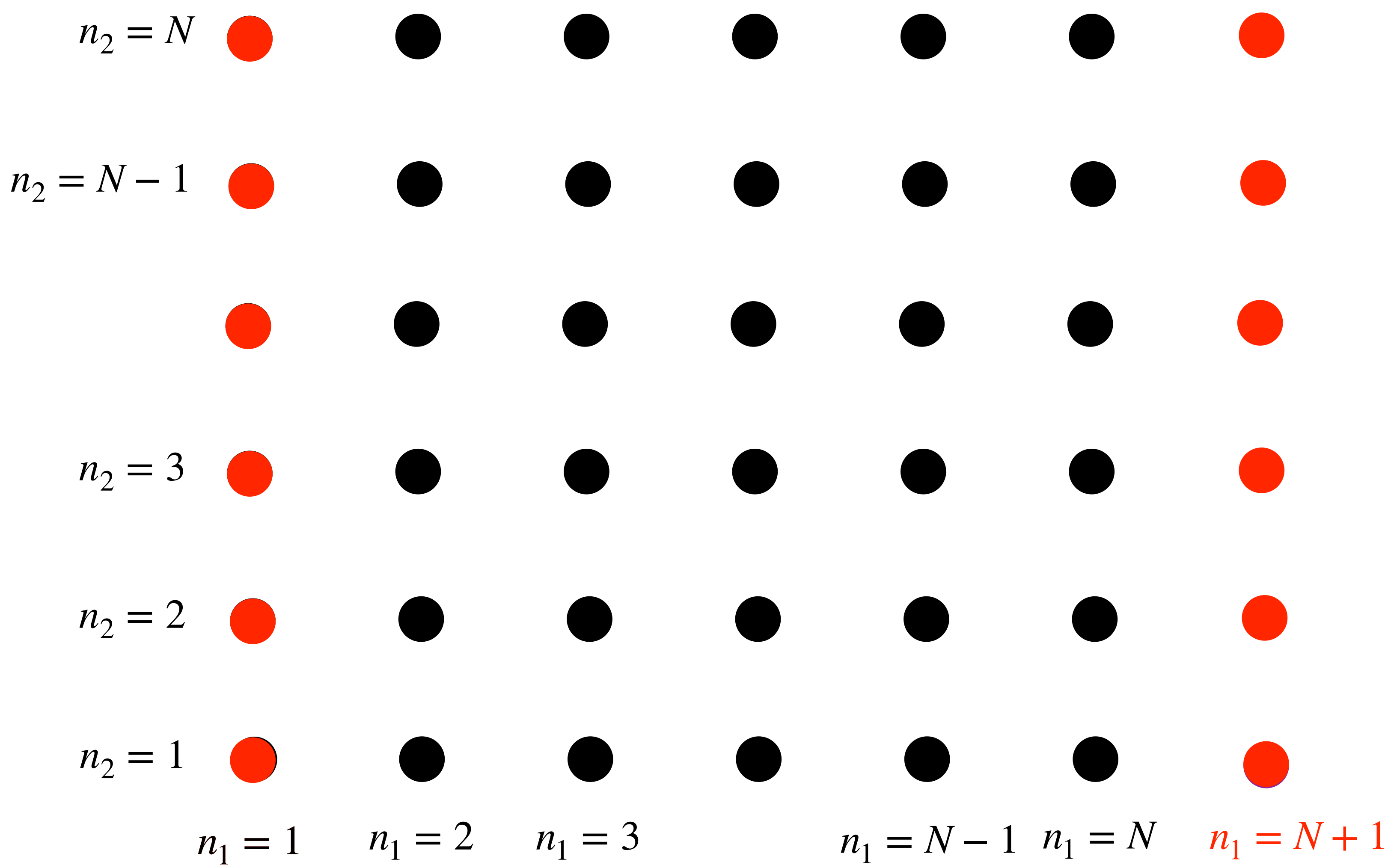
2. Boundary conditions

Periodic Boundary Conditions: 2D

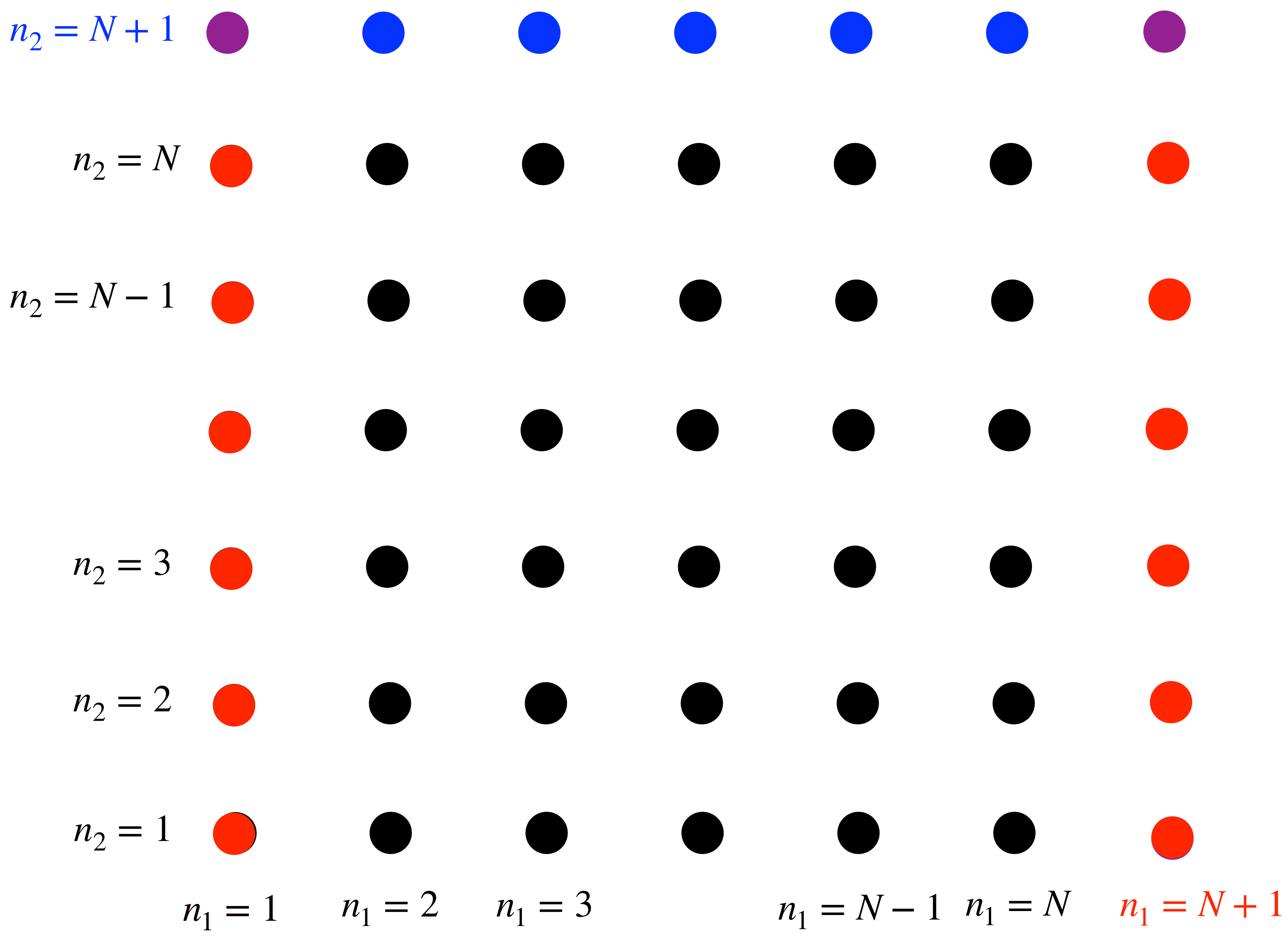


2. Boundary conditions

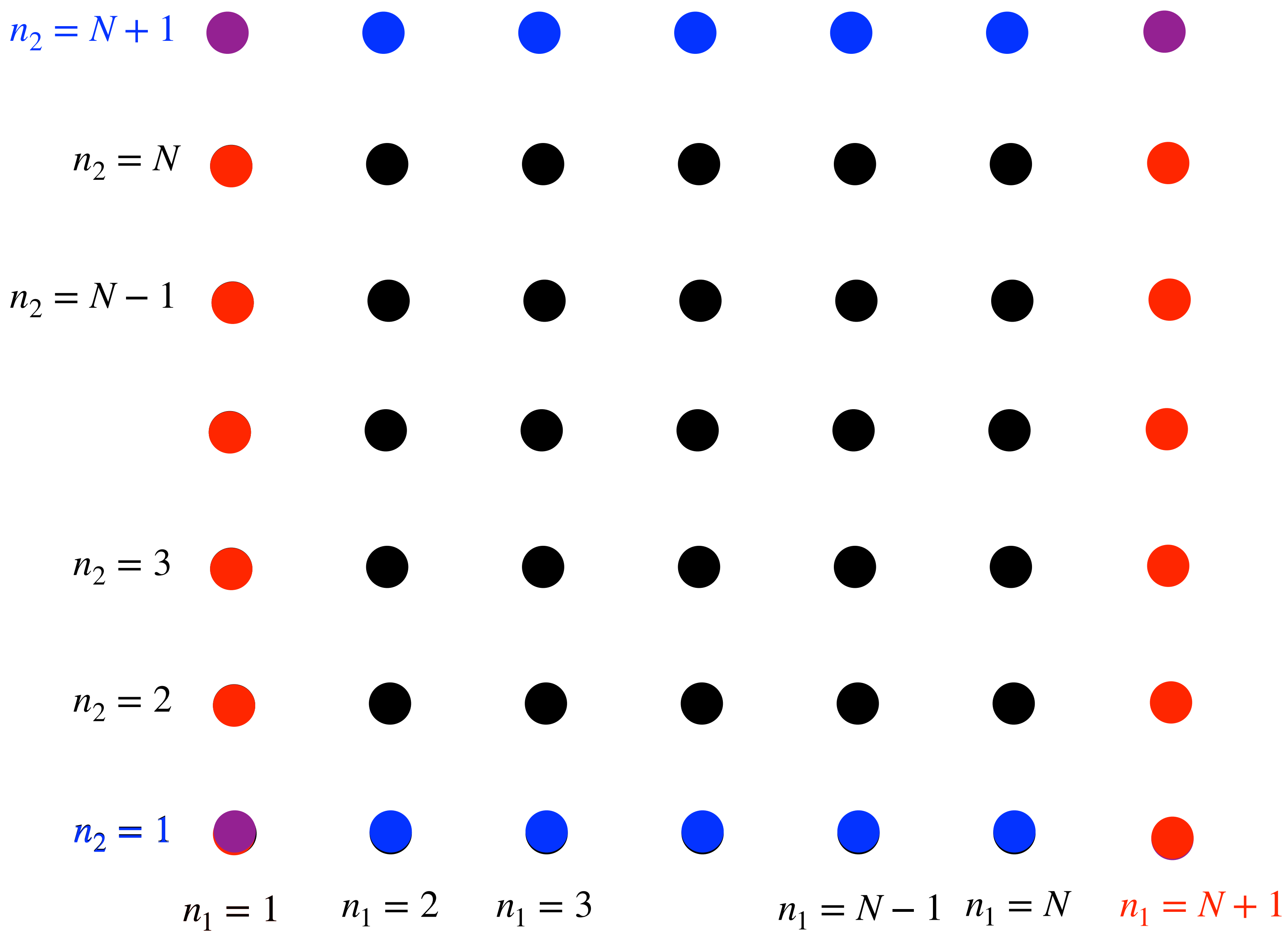
Periodic Boundary Conditions: 2D



Periodic Boundary Conditions: 2D

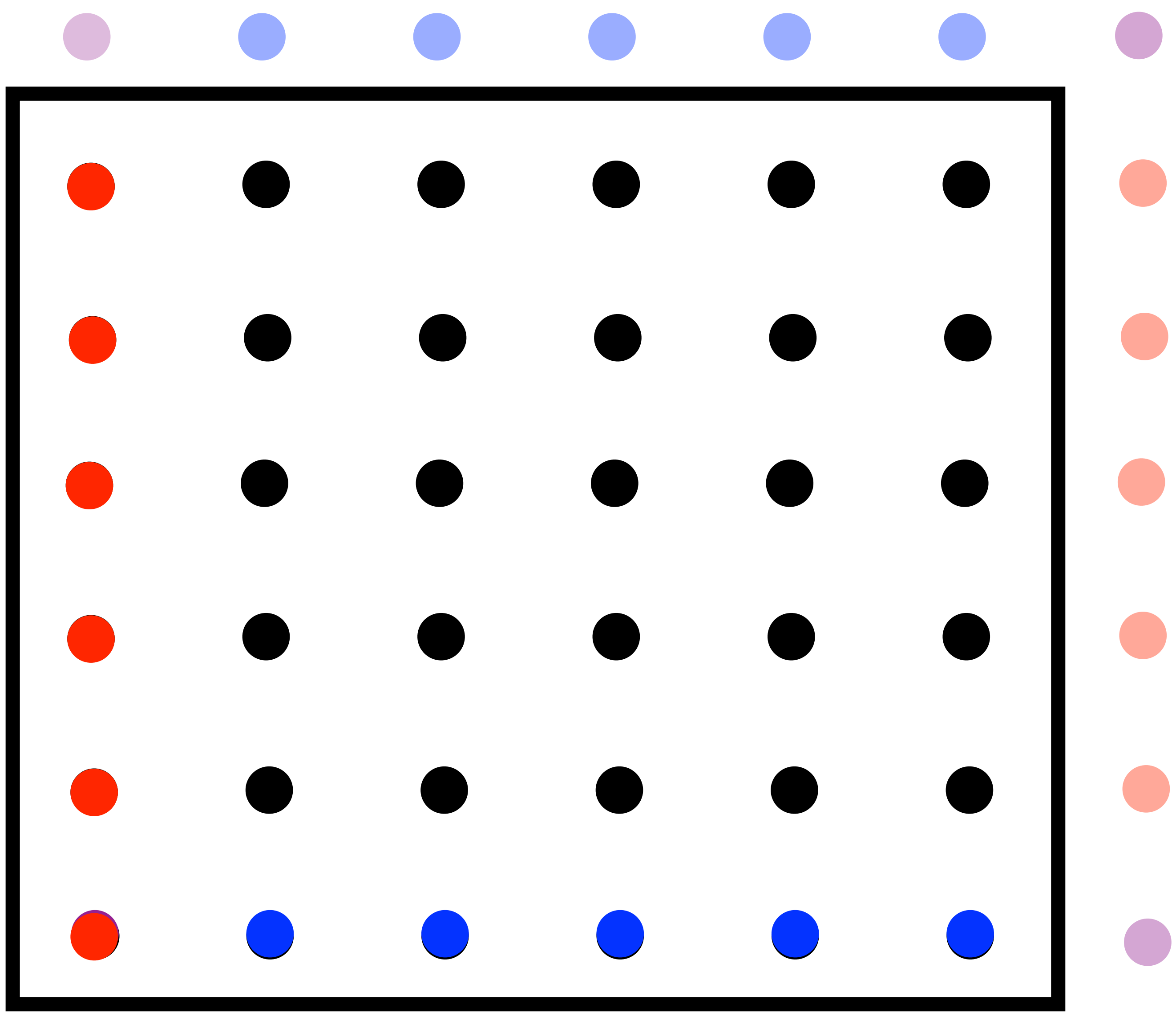


Periodic Boundary Conditions: 2D



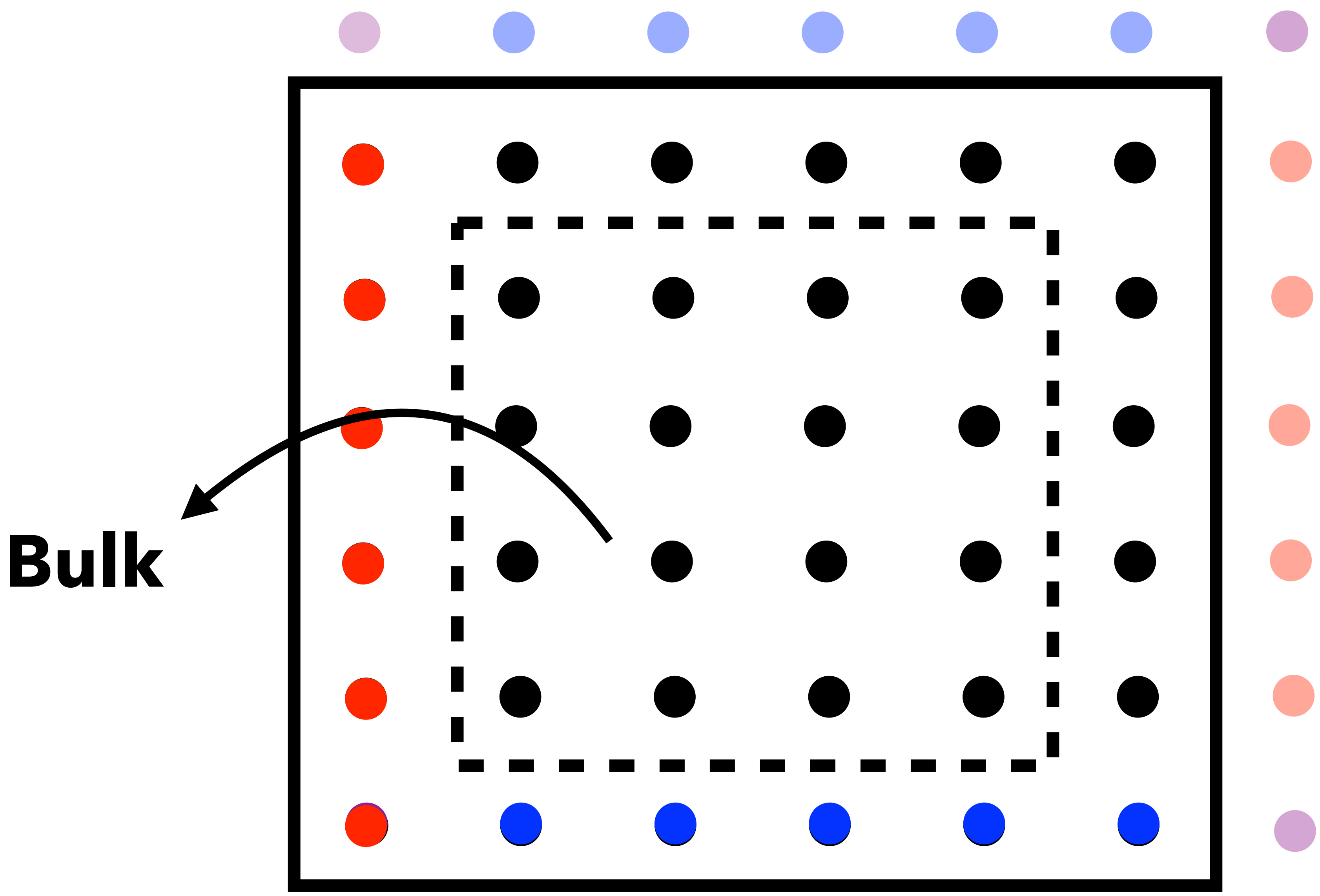
2. Boundary conditions

Periodic Boundary Conditions: 2D

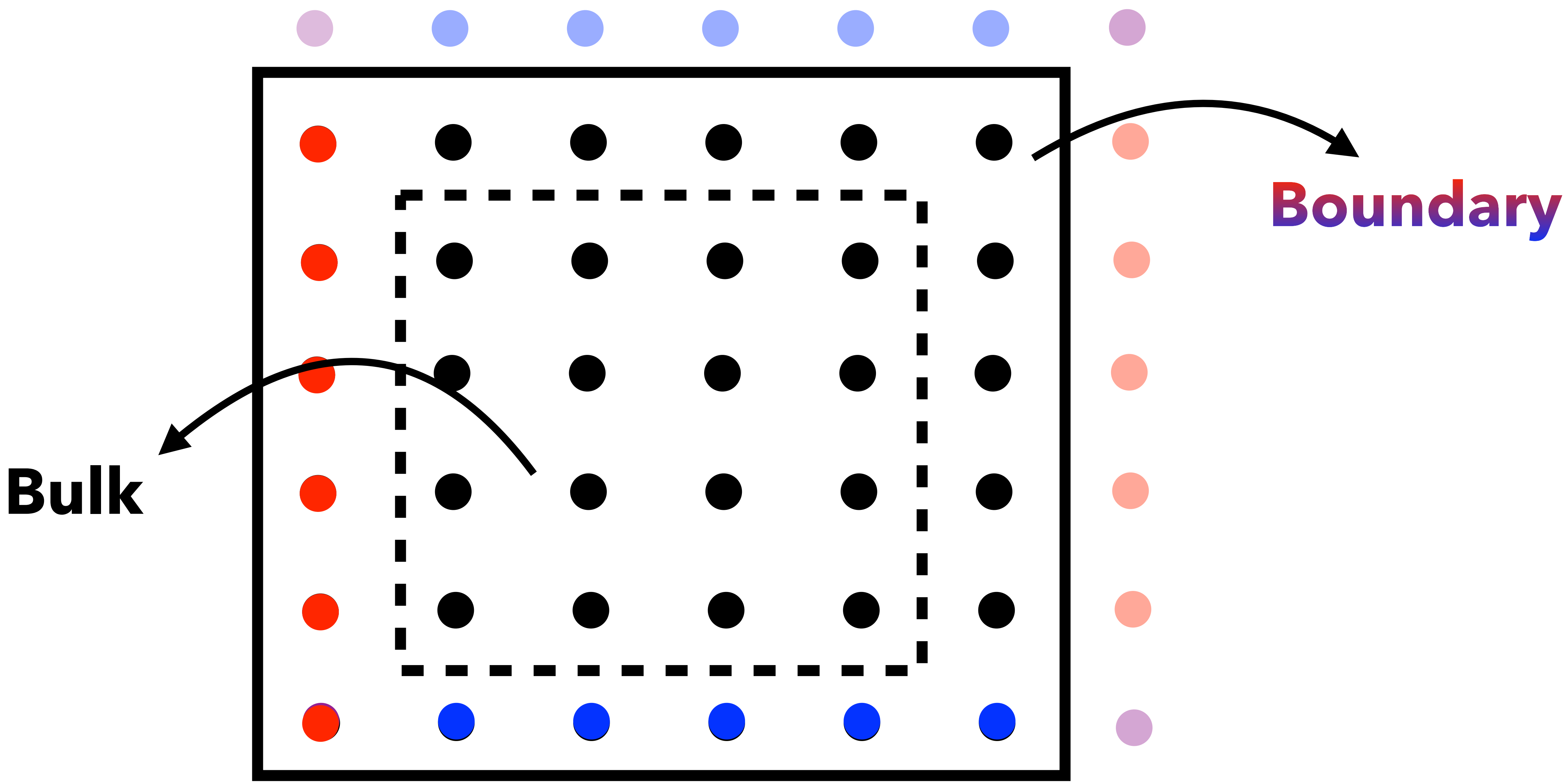


2. Boundary conditions

Periodic Boundary Conditions: 2D

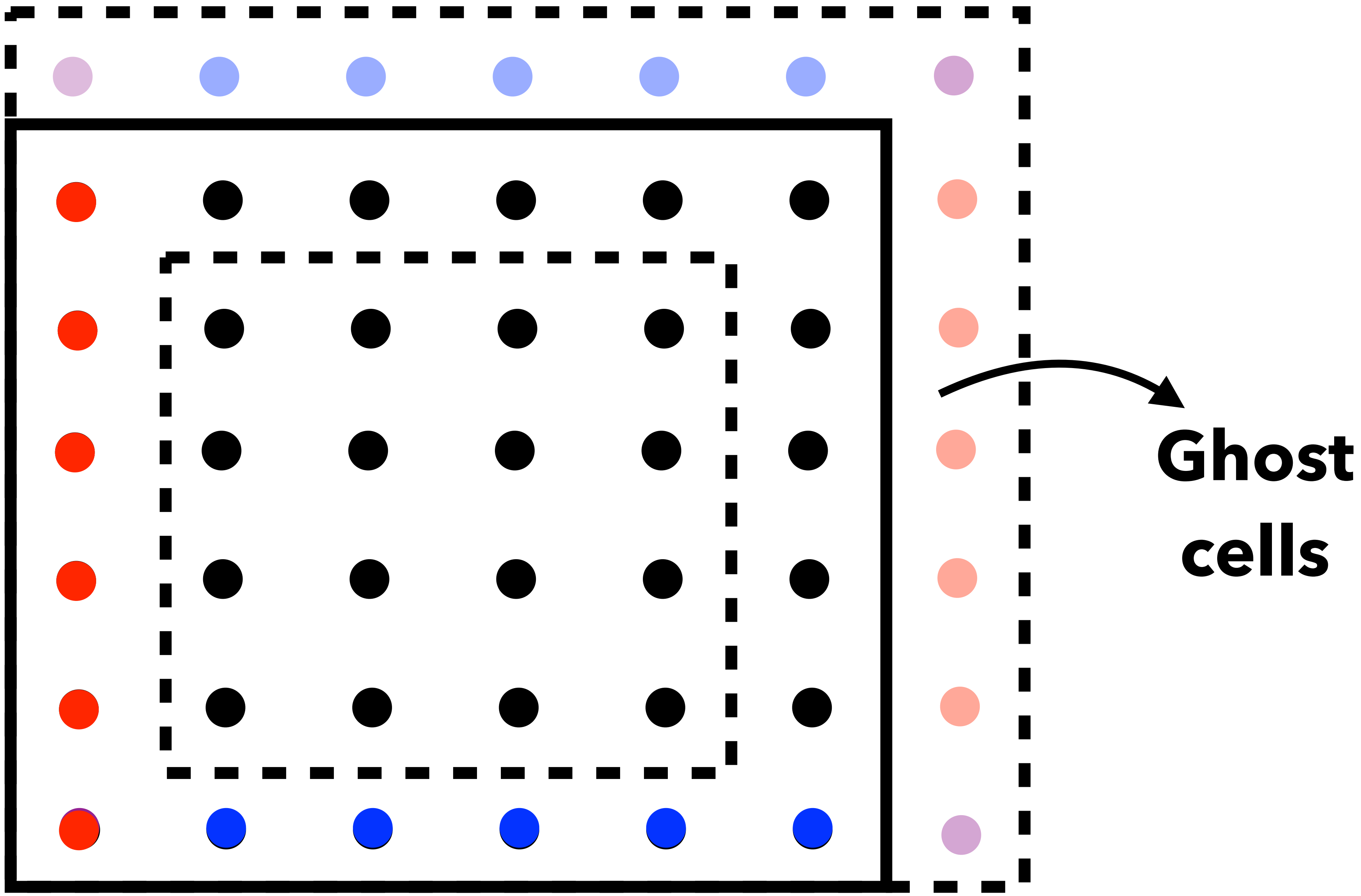


Periodic Boundary Conditions: 2D



2. Boundary conditions

Periodic Boundary Conditions: 2D



2. Boundary conditions

Periodic Boundary Conditions: 2D

$$\left.\begin{array}{c} \{\mathbf{n} \equiv (n_1, n_2)\}\ ,\ \{f(\mathbf{n})\}\ , \\ n_i = 1,\, 2,\, \ldots.,\, N;\ \ i = 1{,}2 \\ (N^2 \text{ entries}) \end{array}\right\}$$

2. Boundary conditions

Periodic Boundary Conditions: 2D

$$\left.\begin{array}{c} \{\mathbf{n} \equiv (n_1, n_2)\} \ , \ \{f(\mathbf{n})\} \ , \\ n_i = 1,\, 2,\, \ldots.,\, N; \ \ i = 1{,}2 \\ (N^2 \text{ entries}) \end{array}\right\} \quad \xLeftrightarrow[\textbf{Lattice}]{\textbf{Periodic}} \quad \boxed{f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})}$$

$\hat{i} \equiv$ **unit vector in *i*-direction**
(i = 1,2)

2. Boundary conditions

Periodic Boundary Conditions: 2D

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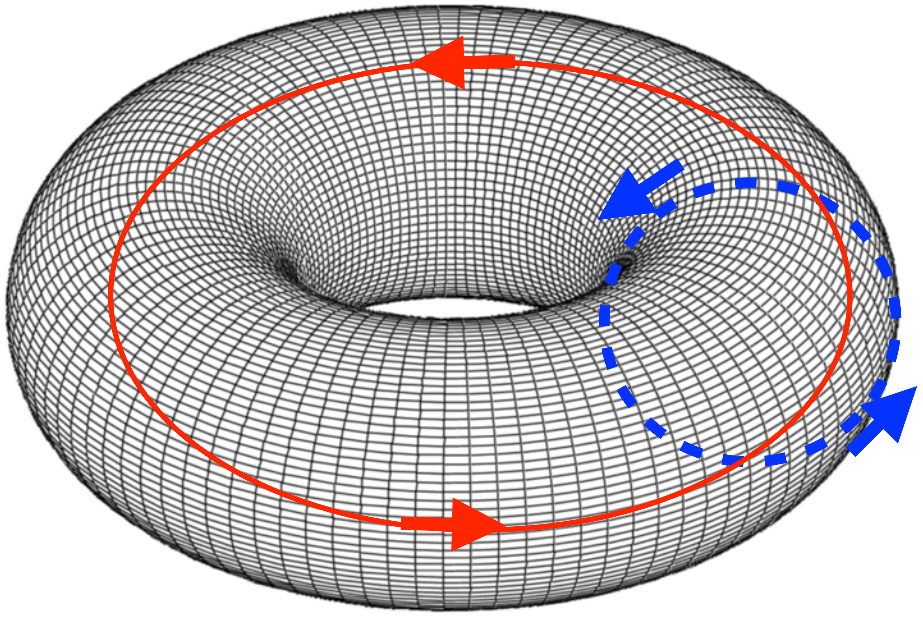
Torus (T2)



2. Boundary conditions

Periodic Boundary Conditions: 3D

$$\left.\begin{array}{c} \{\mathbf{n} \equiv (n_1, n_2, n_3)\}\ ,\ \{f(\mathbf{n})\}\ , \\ n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1,2,3 \\ (N^3 \text{ entries}) \end{array}\right\}$$

2. Boundary conditions

Periodic Boundary Conditions: 3D

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2. Boundary conditions

Periodic Boundary Conditions: 3D

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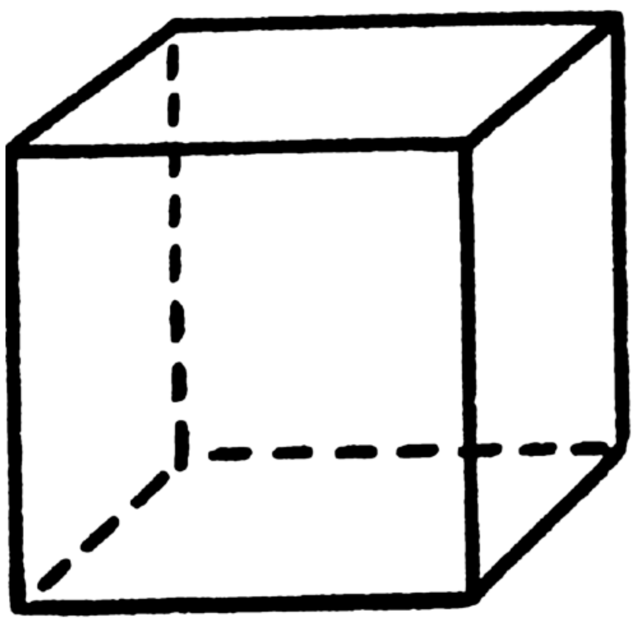
Periodic $\Longleftrightarrow$ **Lattice**

$$\boxed{f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})}$$

$\hat{i} \equiv$ **unit vector in *i*-direction**
(i = 1,2,3)

Torus (T3)



2. Boundary conditions

Periodic Boundary Conditions: 3D

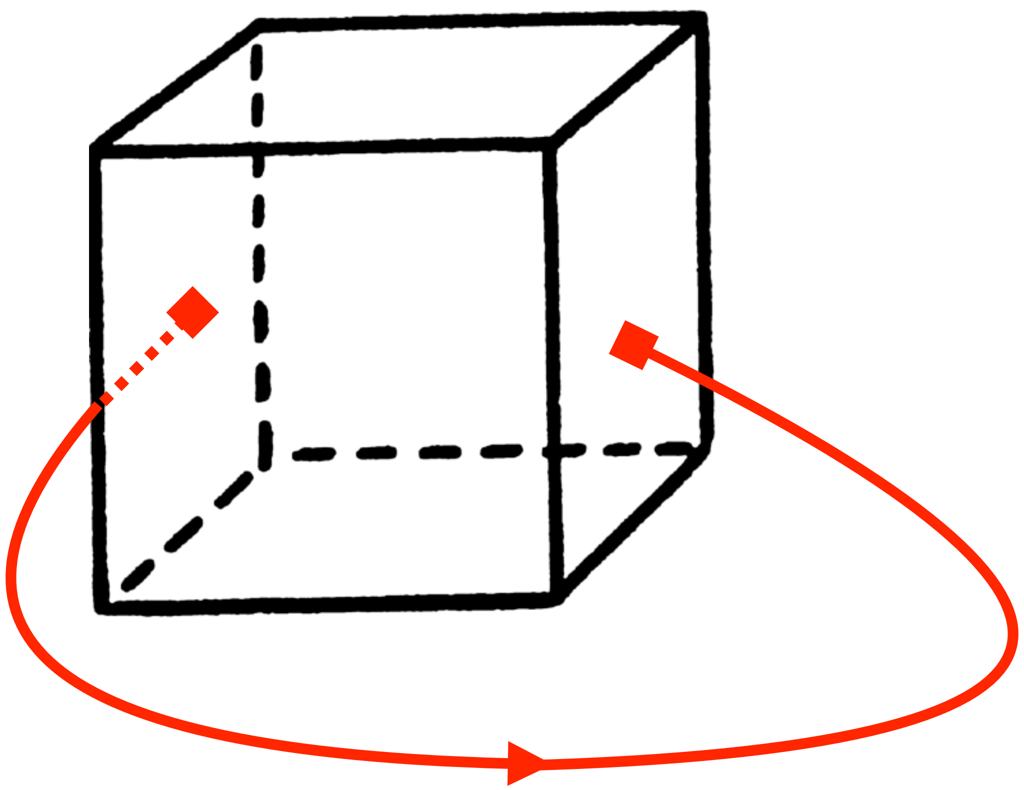
$$\left.\begin{array}{c}\{\mathbf{n} \equiv (n_1, n_2, n_3)\}\ ,\ \{f(\mathbf{n})\}\ , \\ n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1{,}2{,}3 \\ (N^3\ \text{entries})\end{array}\right\}$$

Periodic ⟺ **Lattice**

$$\boxed{f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})}$$

$\hat{i} \equiv$ **unit vector in *i*-direction** (i = 1,2,3)

Torus (T3)



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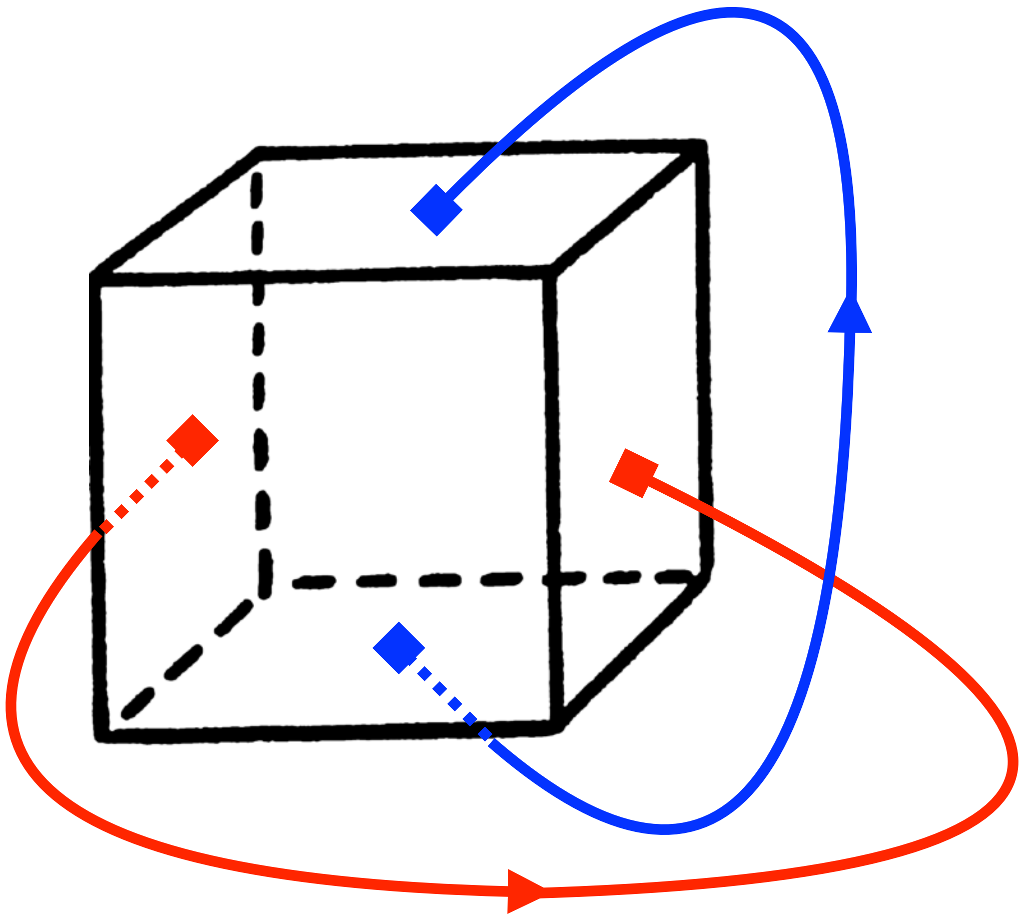

2. Boundary conditions

Periodic Boundary Conditions: 3D

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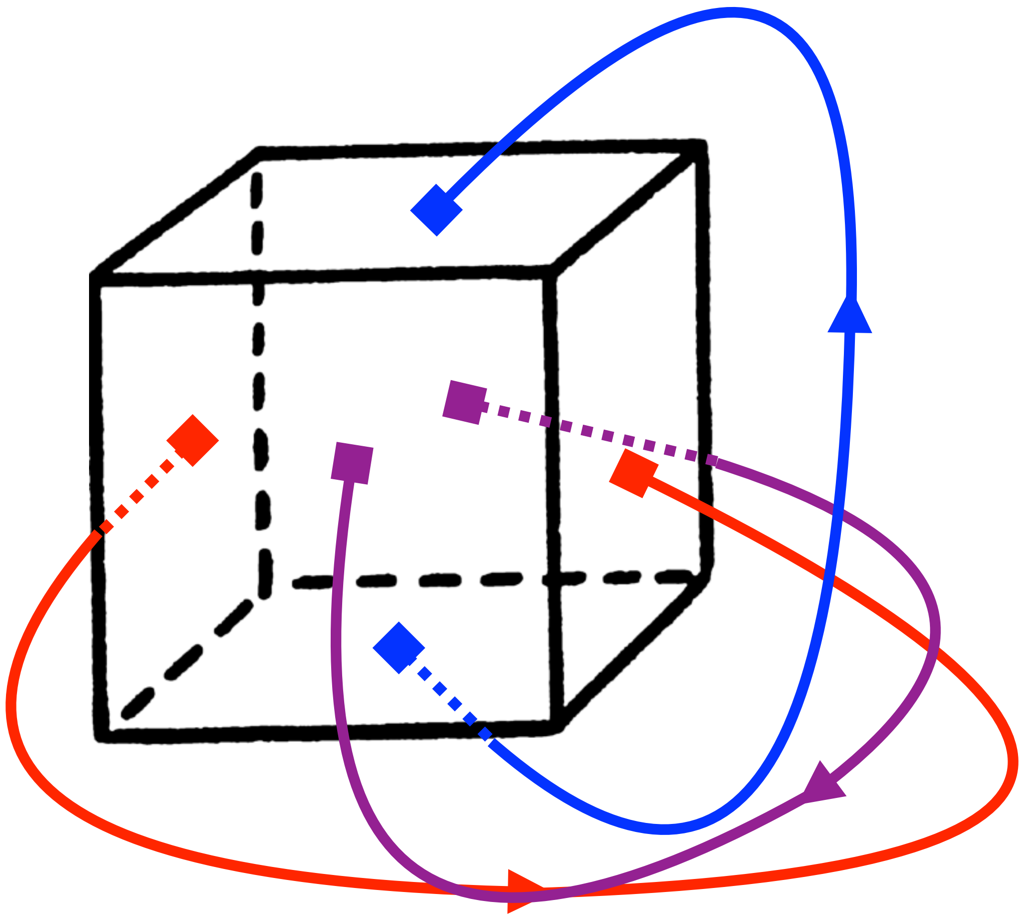

2. Boundary conditions

Periodic Boundary Conditions: d-D

$$\left.\begin{array}{c}\{\mathbf{n} \equiv (n_1, n_2, \ldots, n_d)\}\ ,\ \{f(\mathbf{n})\}\ , \\ n_i = 1,\ 2,\ \ldots,\ N;\ \ i = 1,2,...,d \\ (N^d \text{ entries})\end{array}\right\}$$

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(i = 1,2,…,d)

2. Boundary conditions

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(i = 1,2,…,d)

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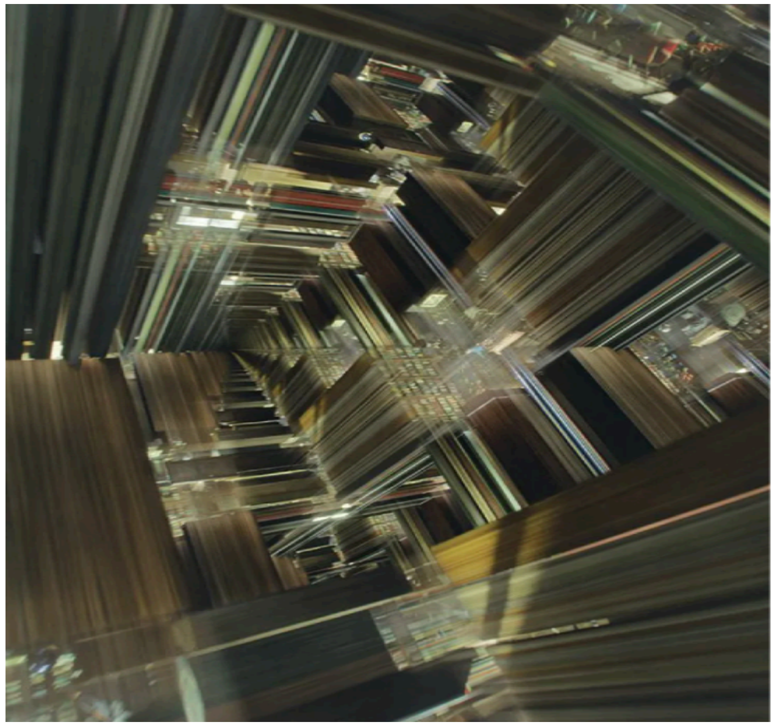

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3. Fourier lattice

Real Lattice

$$\left.\begin{array}{l} \{\mathbf{n} \equiv (n_1, n_2, n_3)\}\ ,\ \{f_j(\mathbf{n})\} \\ n_i = 1,\ 2,\ \ldots.,\ N;\ \ i = 1{,}2{,}3 \\ (N^3\ \text{ sites}) \quad [\ j = 1,\ 2\ ,\ldots,\ \#\ ] \end{array}\right\}$$

Periodic

Lattice

$$\boxed{f_j(\mathbf{n} + N\hat{i}\,) \equiv f_j(\mathbf{n})}$$

$\hat{i} \equiv$ **unit vector in i-direction**

(i = 1,2,3)

3. Fourier lattice

Real Lattice

$$\{\mathbf{n} \equiv (n_1, n_2, n_3)\} \;,\; \{f_j(\mathbf{n})\}$$

$$n_i = 0, 1, \ldots\ldots, N-1\,;\; i = 1,2,3$$

$(N^3 \text{ sites})$ [j = 1, 2 ,..., #]

Periodic $\Longleftrightarrow$ **Lattice**

$$\boxed{f_j(\mathbf{n} + N\hat{i}) \equiv f_j(\mathbf{n})}$$

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(i = 1,2,3)

3. Fourier lattice

Real Lattice

$\{\mathbf{n} \equiv (n_1, n_2, n_3)\}$, $\{f_j(\mathbf{n})\}$

$n_i = 0, 1, \ldots., N-1$; $i = 1,2,3$

$(N^3$ sites) [j = 1, 2 ,…, #]

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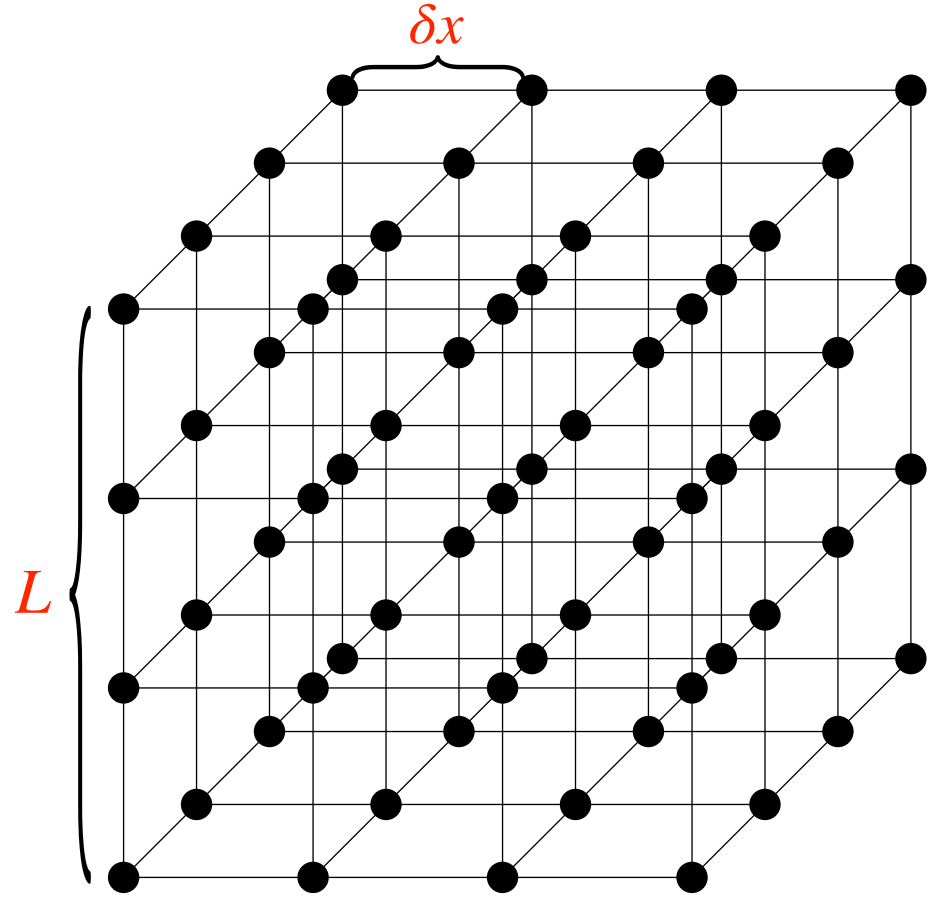

N : number of points/dimension

$L = N \cdot \delta x$: length side

$\delta x \equiv \dfrac{L}{N}$: lattice spacing

3. Fourier lattice

Definition of **discrete Fourier Transform**:

$$\boxed{f(\mathbf{n}) \equiv \frac{1}{N^3} \sum_{\tilde{\mathbf{n}}} e^{+i\frac{2\pi}{N}\tilde{\mathbf{n}}\mathbf{n}} f(\tilde{\mathbf{n}})}$$

3. Fourier lattice

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3. Fourier lattice

Definition of **discrete Fourier Transform**:

$$\left(\sum_{\mathbf{n}} e^{i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} = N^3 \delta_{\mathbf{0},\tilde{\mathbf{n}}} \right)$$

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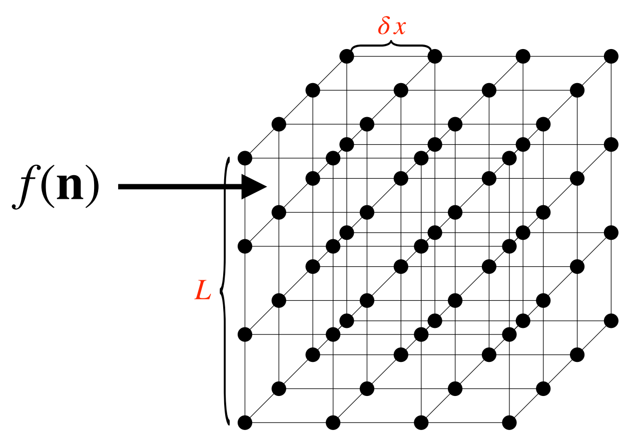

Periodicity condition: $f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$

$\hat{i} \equiv$ unit vector in n_i-direction

Discrete Position: $\begin{cases} \mathbf{n} = (n_1, n_2, n_3) \\ n_i = 0,1,\ldots,N-1 \end{cases}$

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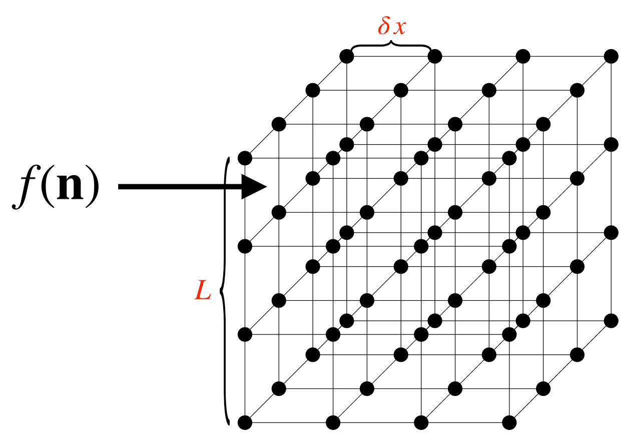

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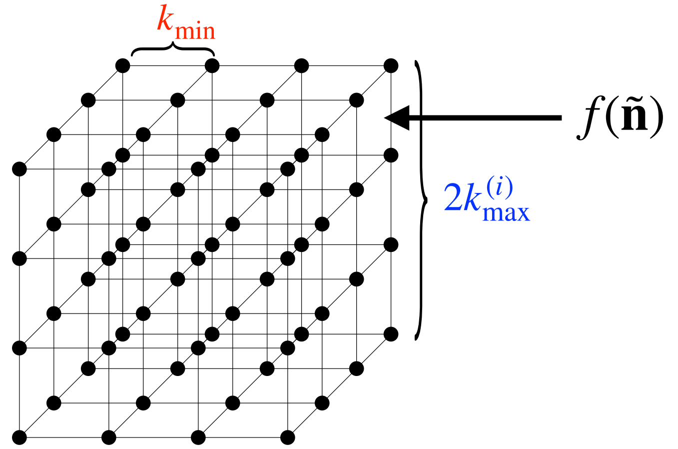
Fourier Lattice



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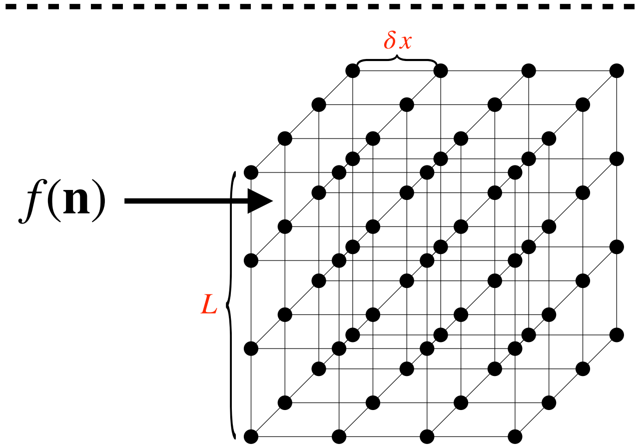

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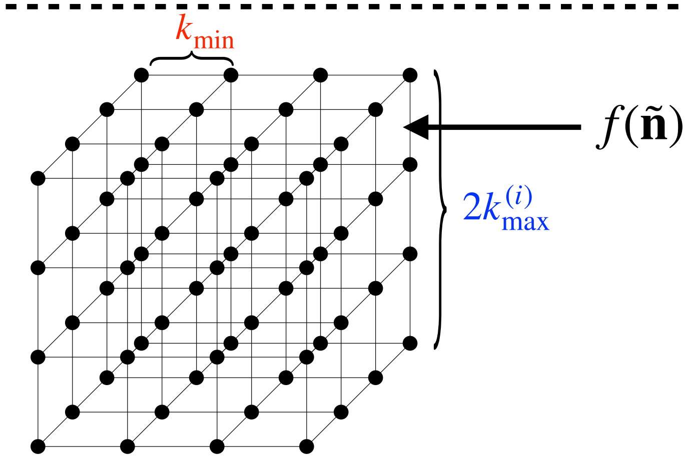
Fourier Lattice



Discrete Momentum: $\begin{cases} \tilde{\mathbf{n}} = (\tilde{n}_1, \tilde{n}_2, \tilde{n}_3) \\ \tilde{n}_i = -\frac{N}{2}+1, \ldots, -1, 0, 1, \ldots, \frac{N}{2} \end{cases}$

3. Fourier lattice

Definition of **discrete Fourier Transform**: $\left(\sum_{\mathbf{n}} e^{i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} = N^3\delta_{\mathbf{0},\tilde{\mathbf{n}}}\right)$

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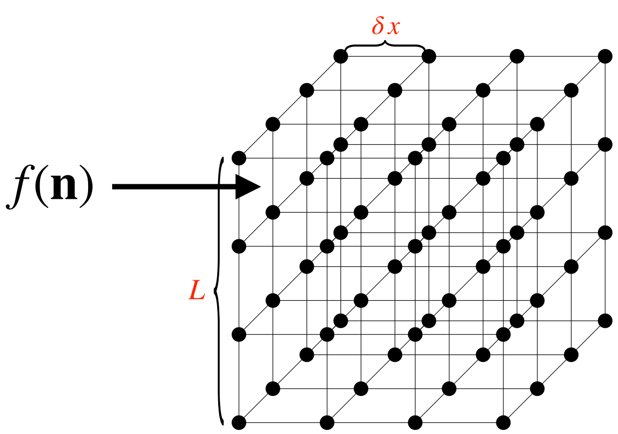

Periodicity condition: $f(\mathbf{n} + N\hat{\imath}) \equiv f(\mathbf{n})$

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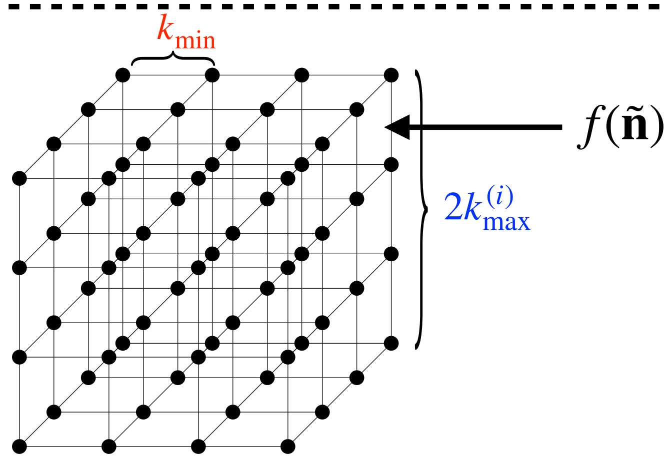
Fourier Lattice



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3. Fourier lattice

Physical momenta captured by the Fourier lattice:

$$\mathbf{k} = \frac{2\pi}{L}\tilde{\mathbf{n}}$$

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$$\Rightarrow \quad (k_1, k_2, k_3) = \frac{2\pi}{L}(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$$

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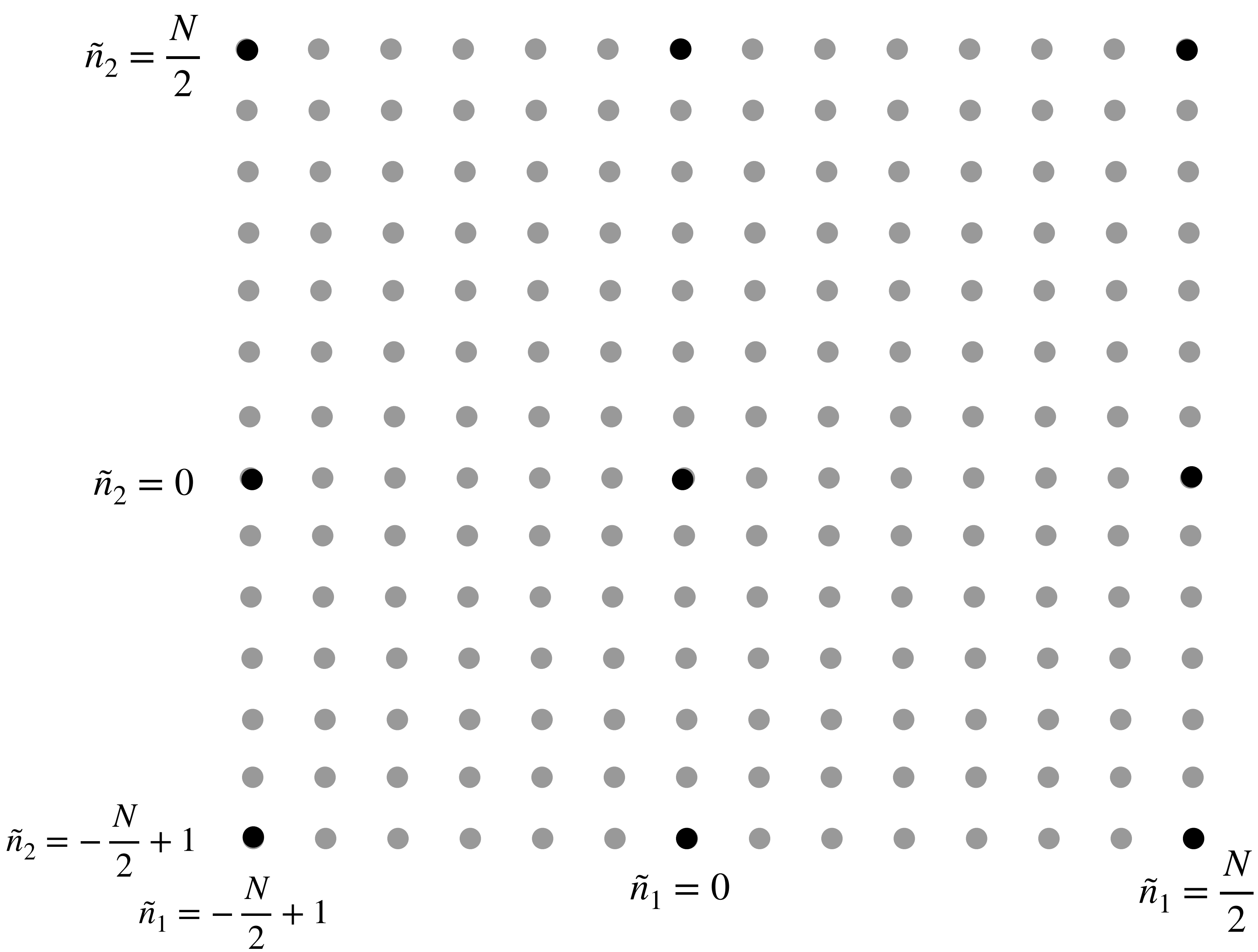
[Slice $\tilde{n}_3 = 0$]



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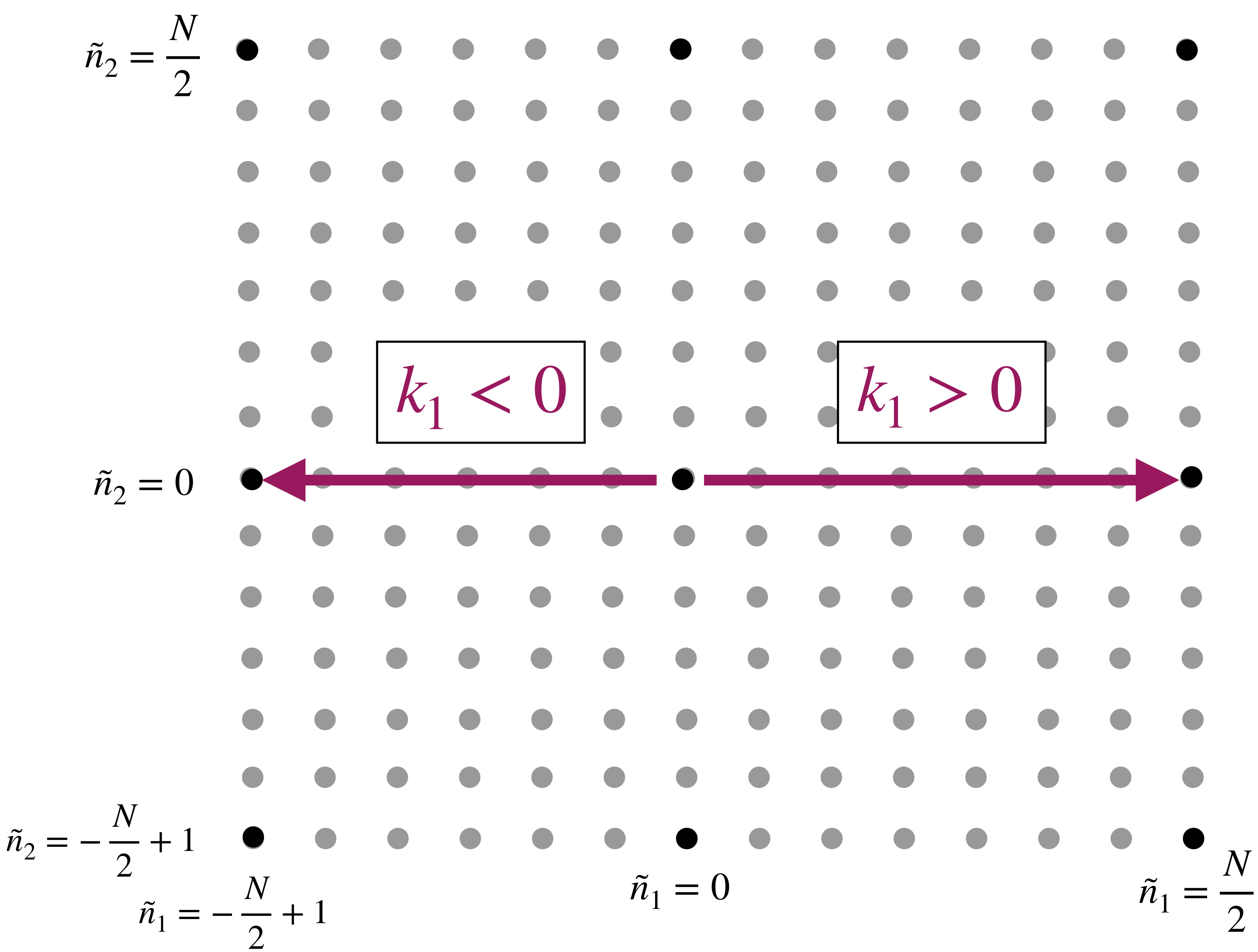

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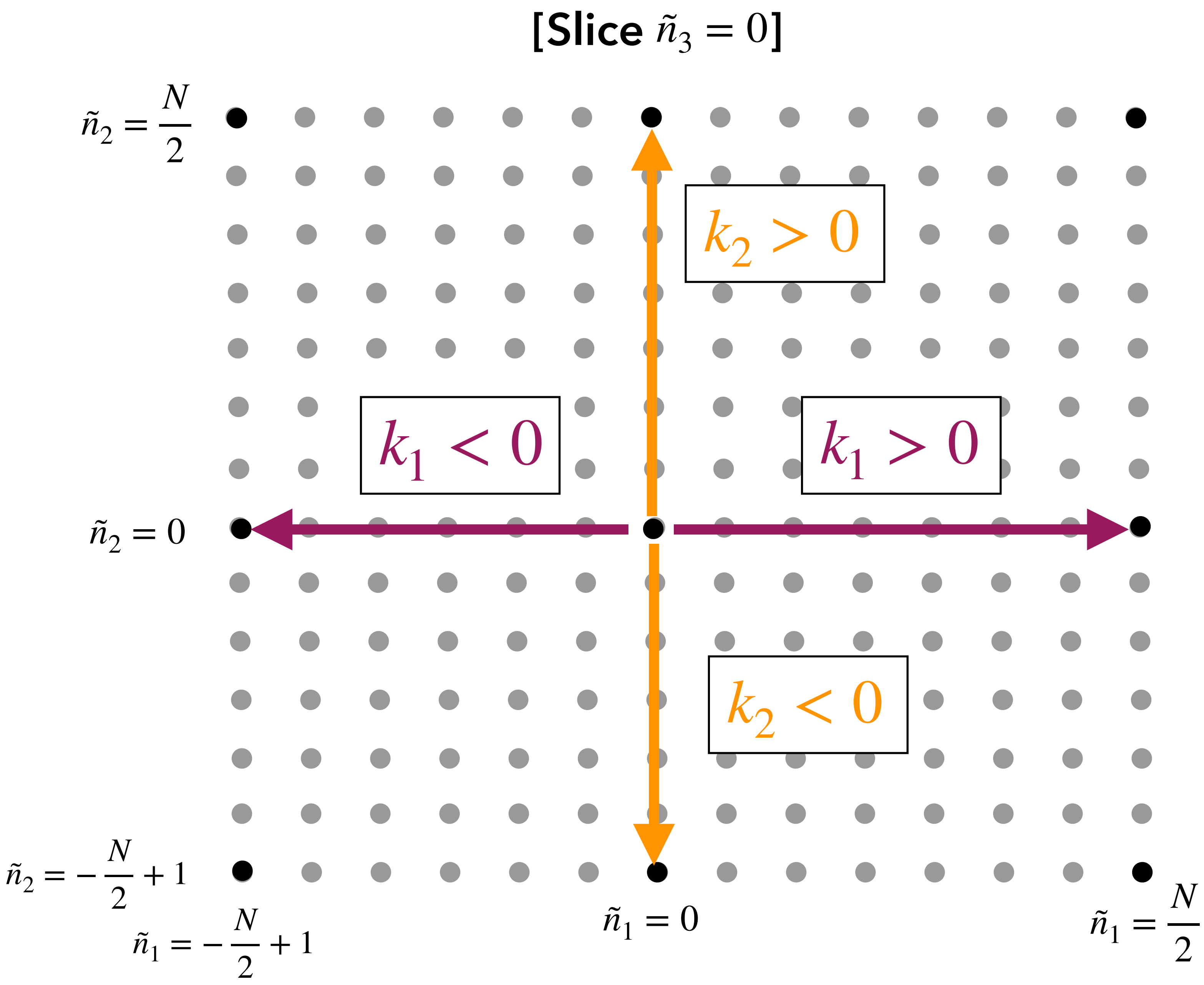

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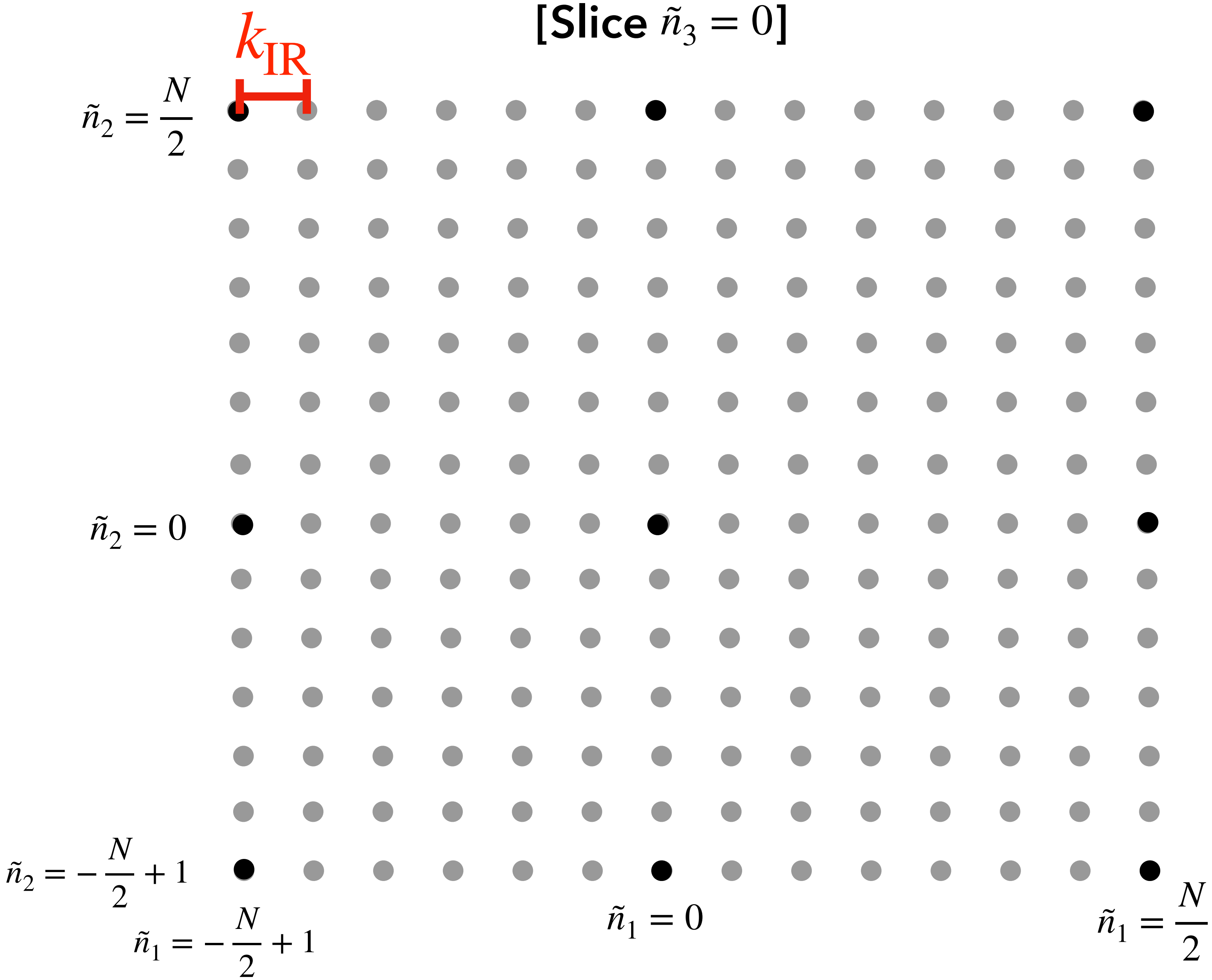

Infrared momentum:
Minimum momentum captured by the lattice.

$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

3. Fourier lattice

Physical momenta captured by the Fourier lattice: $\mathbf{k} = \frac{2\pi}{L}\tilde{\mathbf{n}}$

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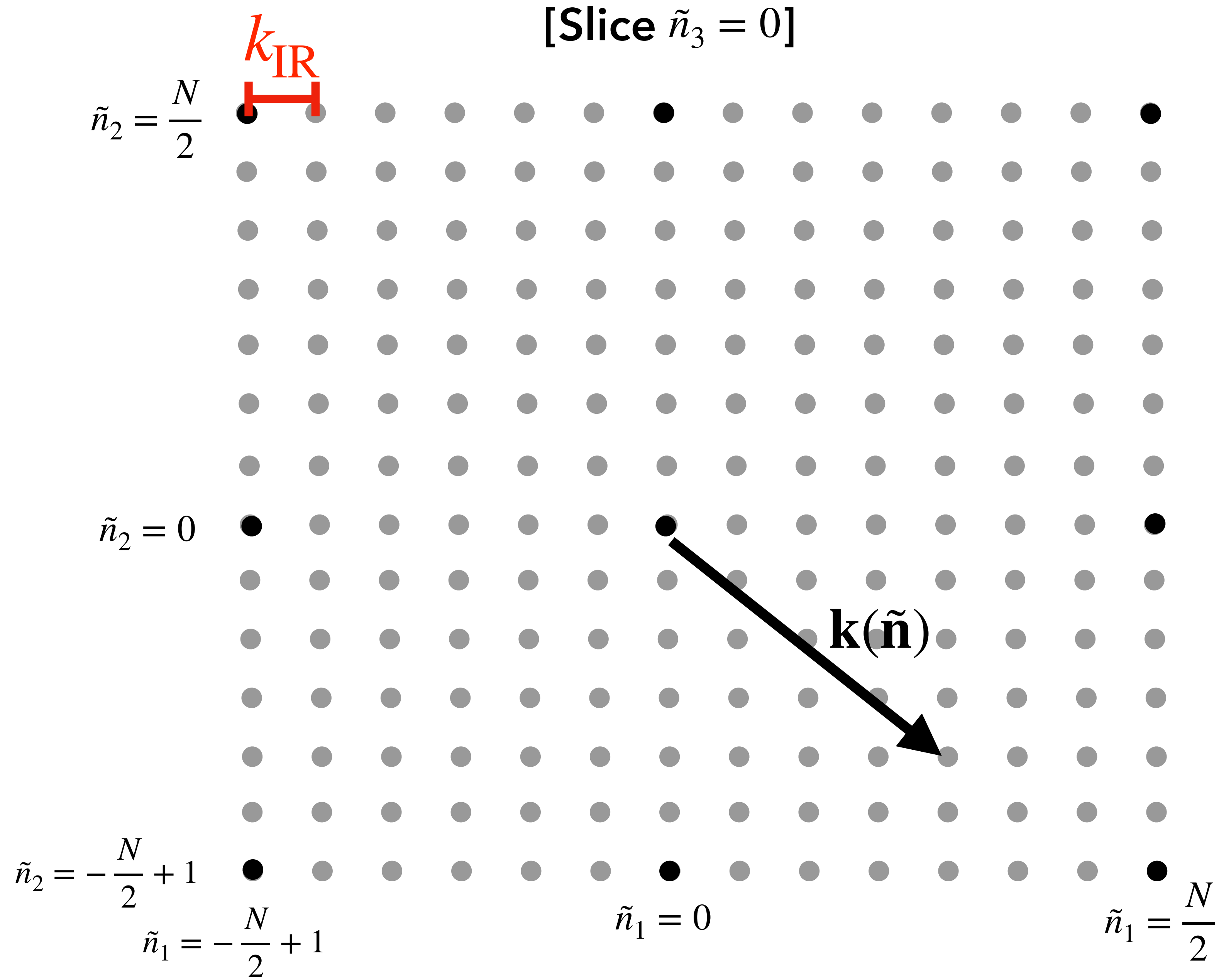

Infrared momentum: Minimum momentum captured by the lattice.

$$k_{\rm IR} \equiv \frac{2\pi}{L} \longrightarrow \boxed{\mathbf{k}(\tilde{\mathbf{n}}) = \tilde{\mathbf{n}}\, k_{\rm IR}}$$

Linear Momentum

3. Fourier lattice

Physical momenta captured by the Fourier lattice: $\mathbf{k} = \frac{2\pi}{L}\tilde{\mathbf{n}}$

$$(k_1, k_2, k_3) = \frac{2\pi}{L}(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$$

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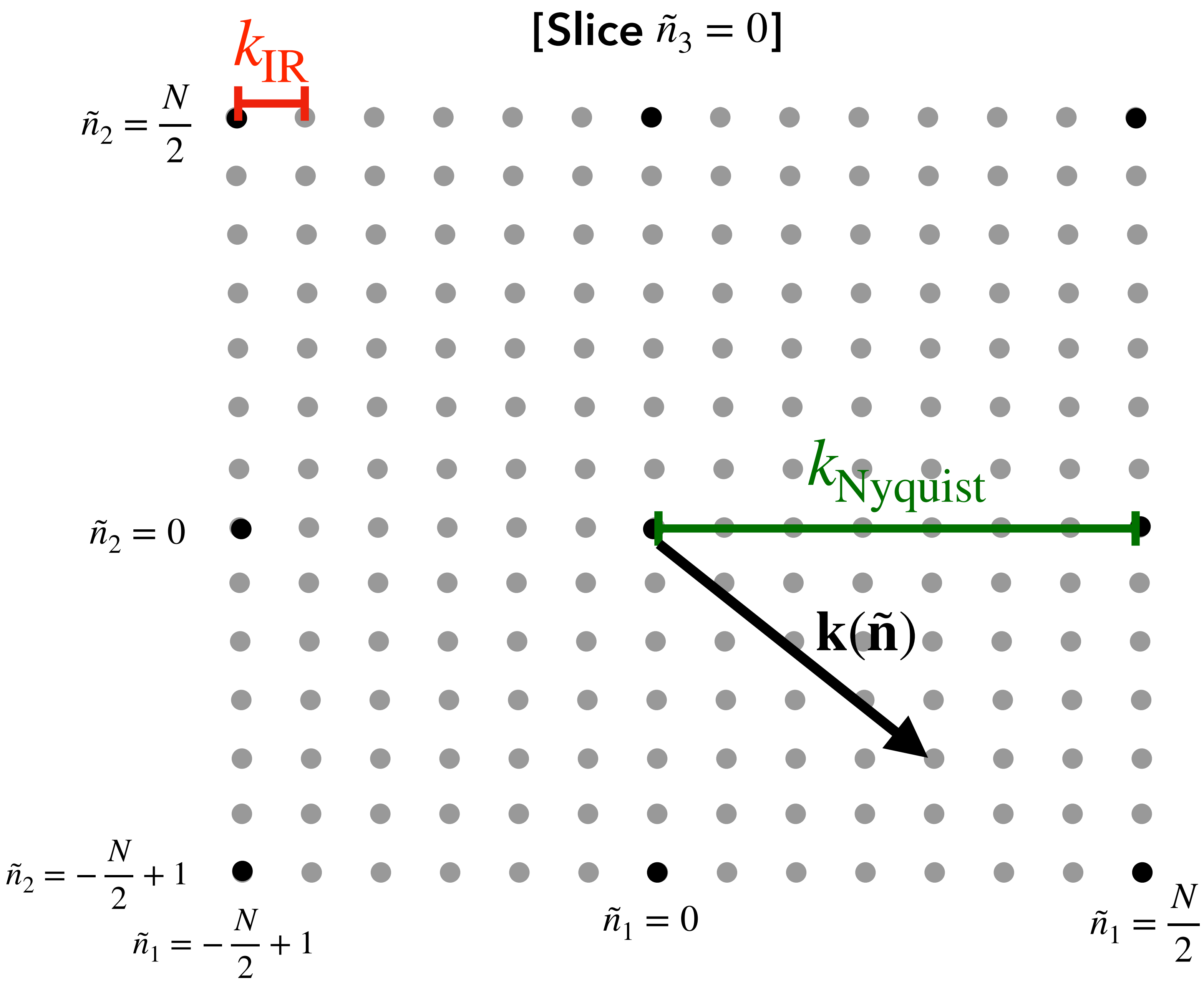

Infrared momentum: Minimum momentum captured by the lattice.

$$k_{\rm IR} \equiv \frac{2\pi}{L} \longrightarrow \boxed{\mathbf{k}(\tilde{\mathbf{n}}) = \tilde{\mathbf{n}}\, k_{\rm IR}}$$

Linear Momentum

Nyquist momentum:

$$k_{\rm Nyquist} \equiv \frac{N}{2} k_{\rm IR} = \frac{\pi}{dx}$$

3. Fourier lattice

Physical momenta captured by the Fourier lattice: $\mathbf{k} = \frac{2\pi}{L}\tilde{\mathbf{n}}$

$$(k_1, k_2, k_3) = \frac{2\pi}{L}(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3)$$

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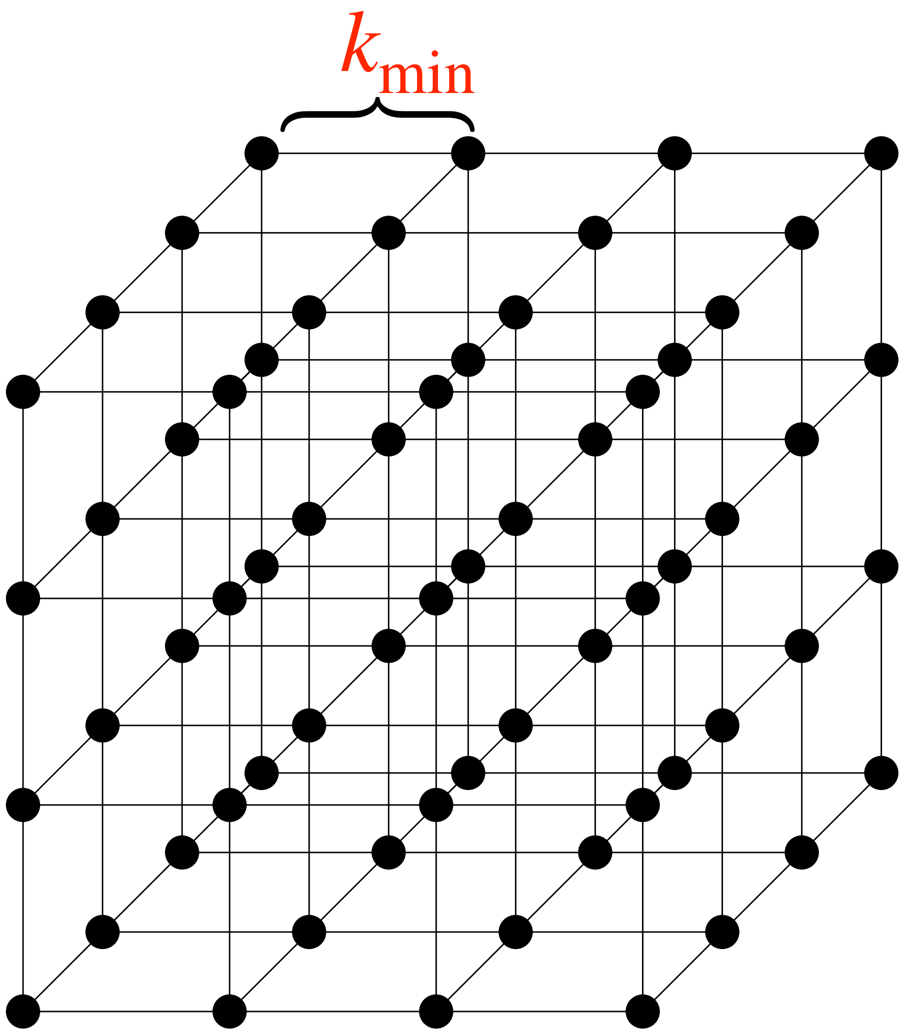

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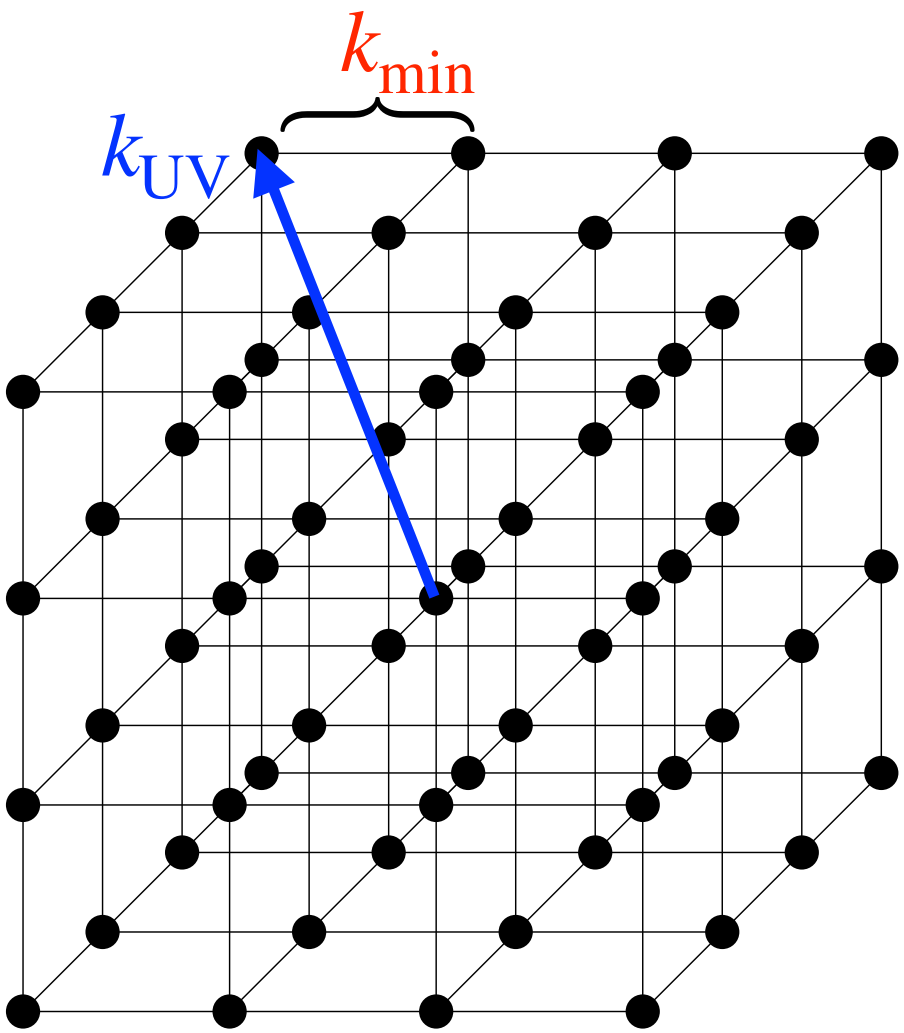

Ultraviolet momentum: Maximum momentum captured by the lattice.

3. Fourier lattice

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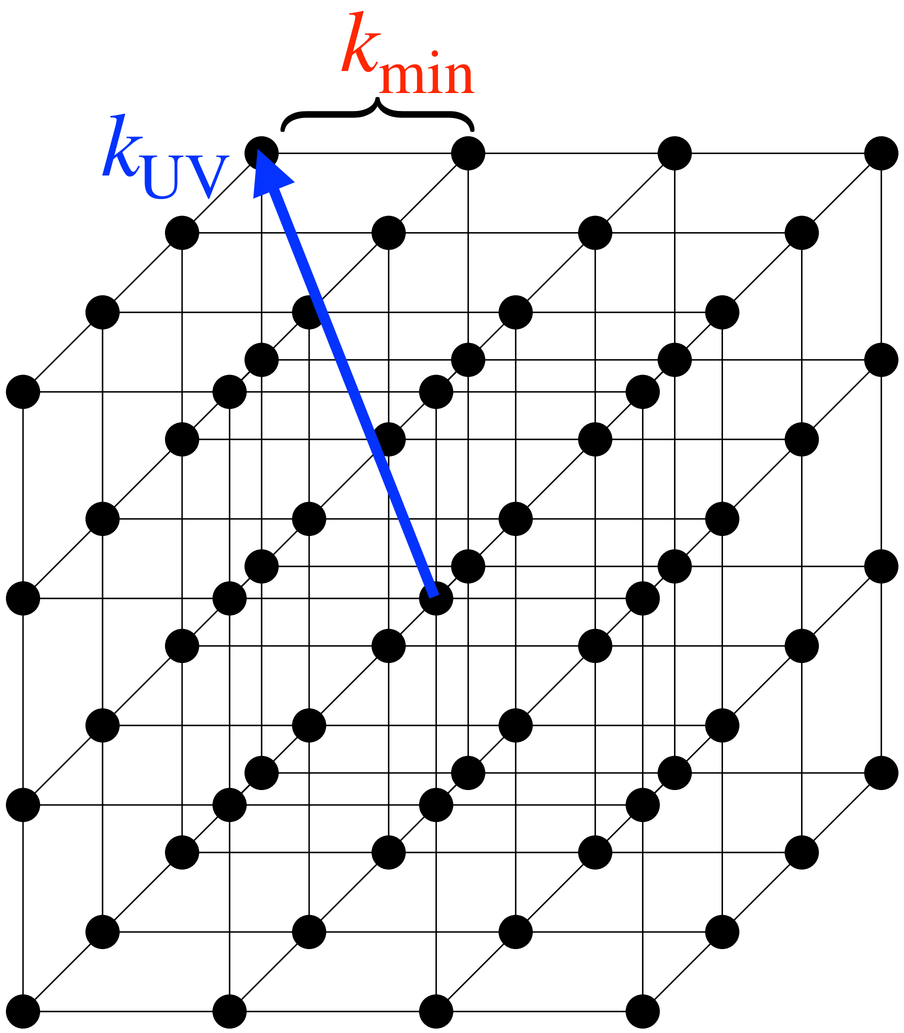

Ultraviolet momentum: Maximum momentum captured by the lattice.

$$k_{\rm UV} \equiv \sqrt{d}\,\frac{N}{2}k_{\rm IR} = \sqrt{d}\,\frac{\pi}{dx}$$

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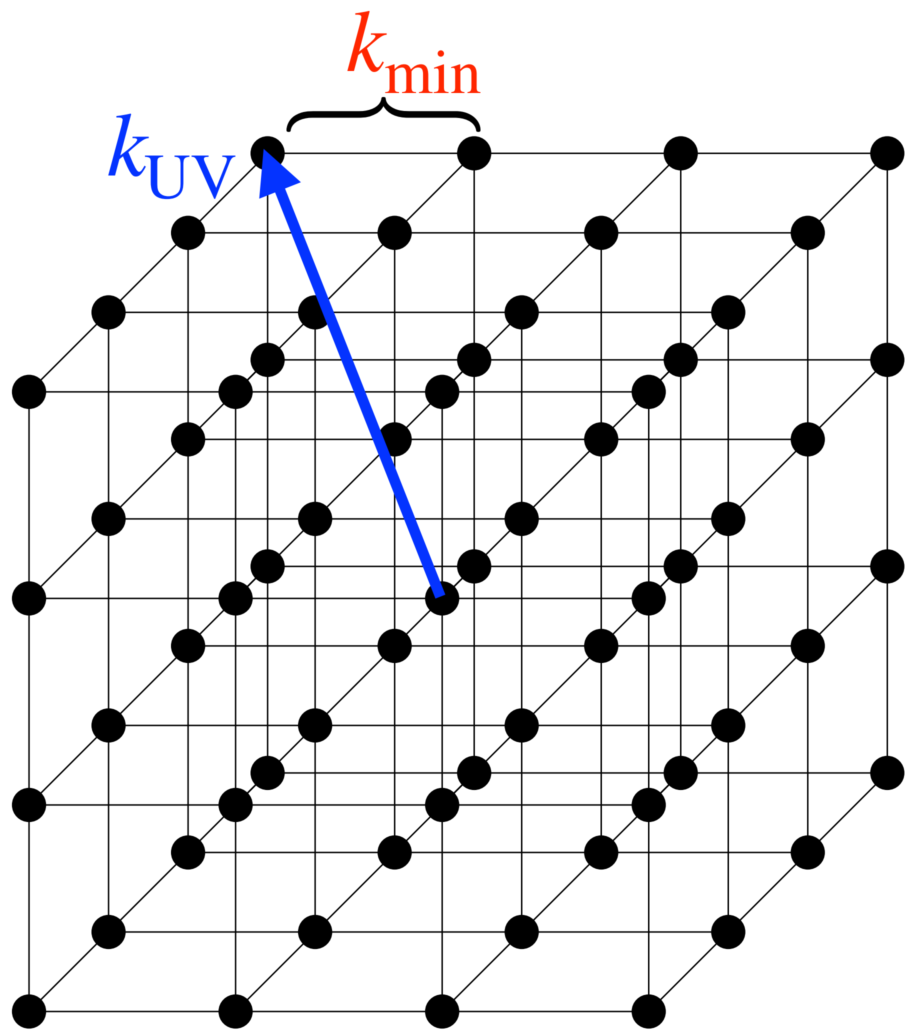


Ultraviolet momentum: Maximum momentum captured by the lattice.

$$k_{\rm UV} \equiv \sqrt{d}\frac{N}{2}k_{\rm IR} = \sqrt{d}\frac{\pi}{dx}$$

IR (min) and UV (max) momenta for d=3:

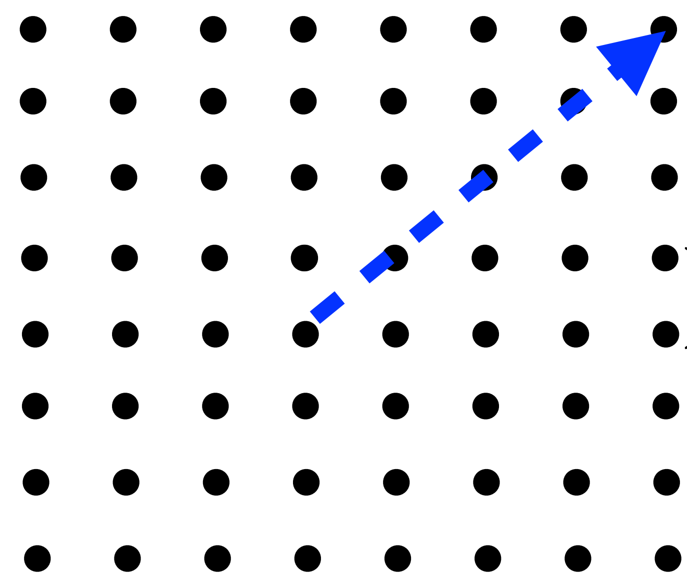
$$k_{\rm IR} = \frac{2\pi}{L} \qquad k_{\rm UV} = \frac{\sqrt{3}}{2}Nk_{\rm IR}$$

3. Fourier lattice

Fourier Lattice

$$\left\{ \begin{array}{l} \tilde{n}_i = -\frac{N}{2}+1, \ldots, \; -1, \; 0, \; 1, \ldots, \; \frac{N}{2} \\ f(\tilde{\mathbf{n}} + N\hat{i}) \equiv f(\tilde{\mathbf{n}}) \qquad (i = 1,2,\ldots,d) \end{array} \right\}$$

$$k_{\rm UV} \equiv \sqrt{d}\,\frac{N}{2}k_{\rm IR} = \sqrt{d}\,\frac{\pi}{dx}$$



$$k_{\rm IR} \equiv \frac{2\pi}{L}$$

3. Fourier lattice

Real Lattice

$$n_i = 1,2,\ldots,N \quad (i = 1,2,\ldots,d)$$

$$f(\mathbf{n} + N\hat{i}) \equiv f(\mathbf{n})$$

Fourier Lattice

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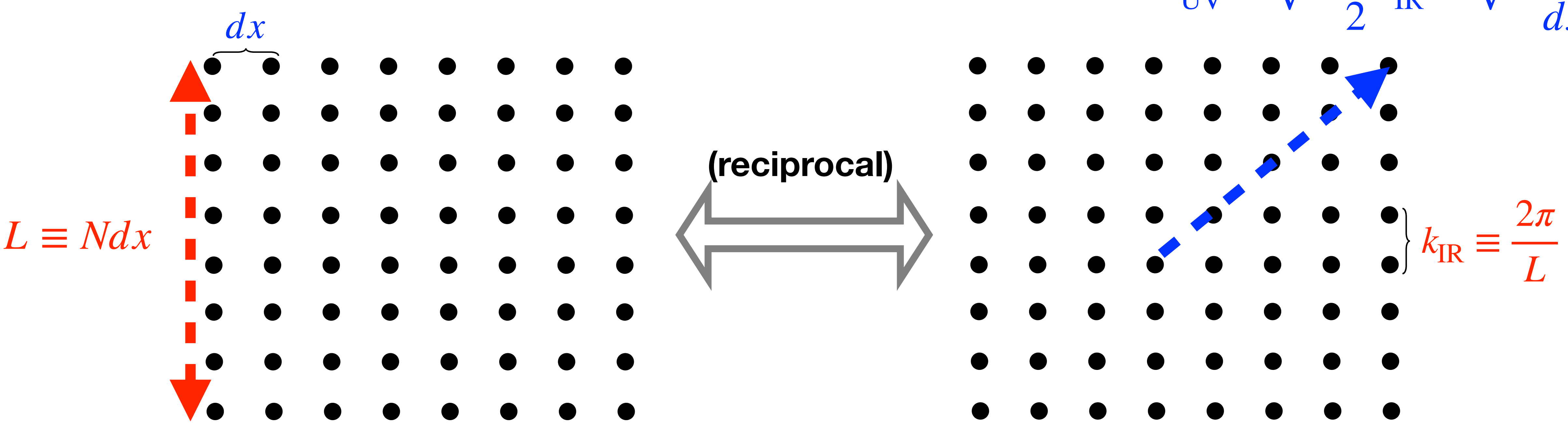

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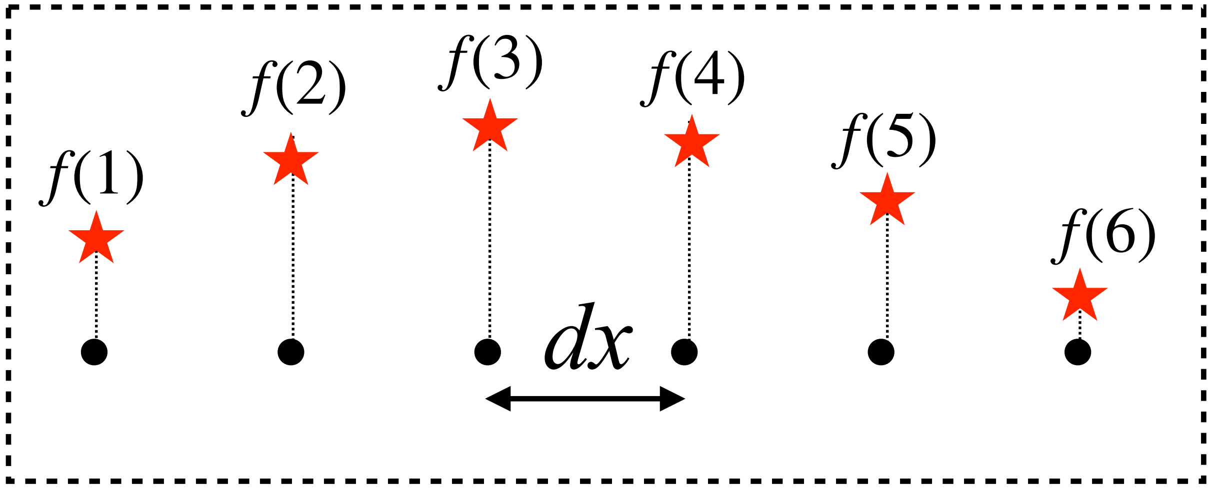
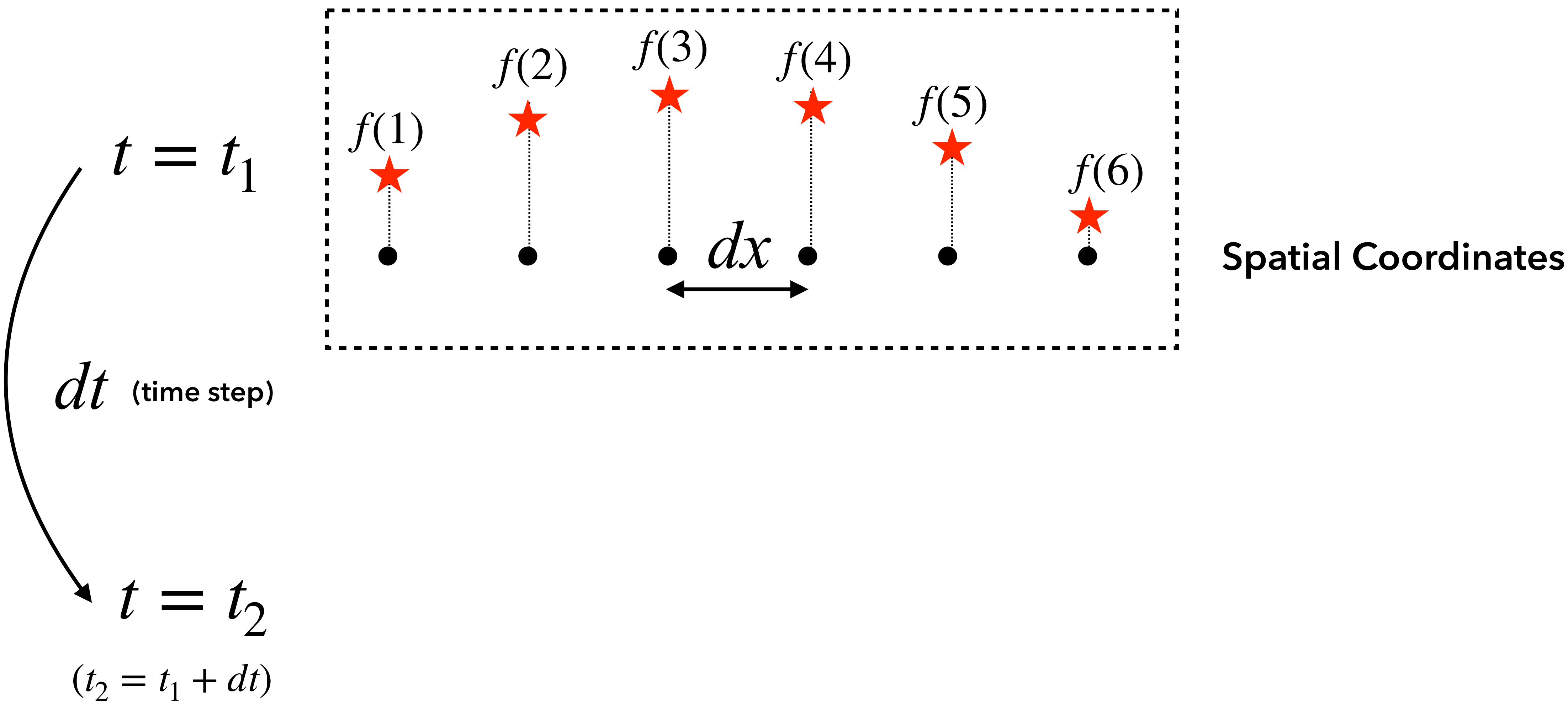
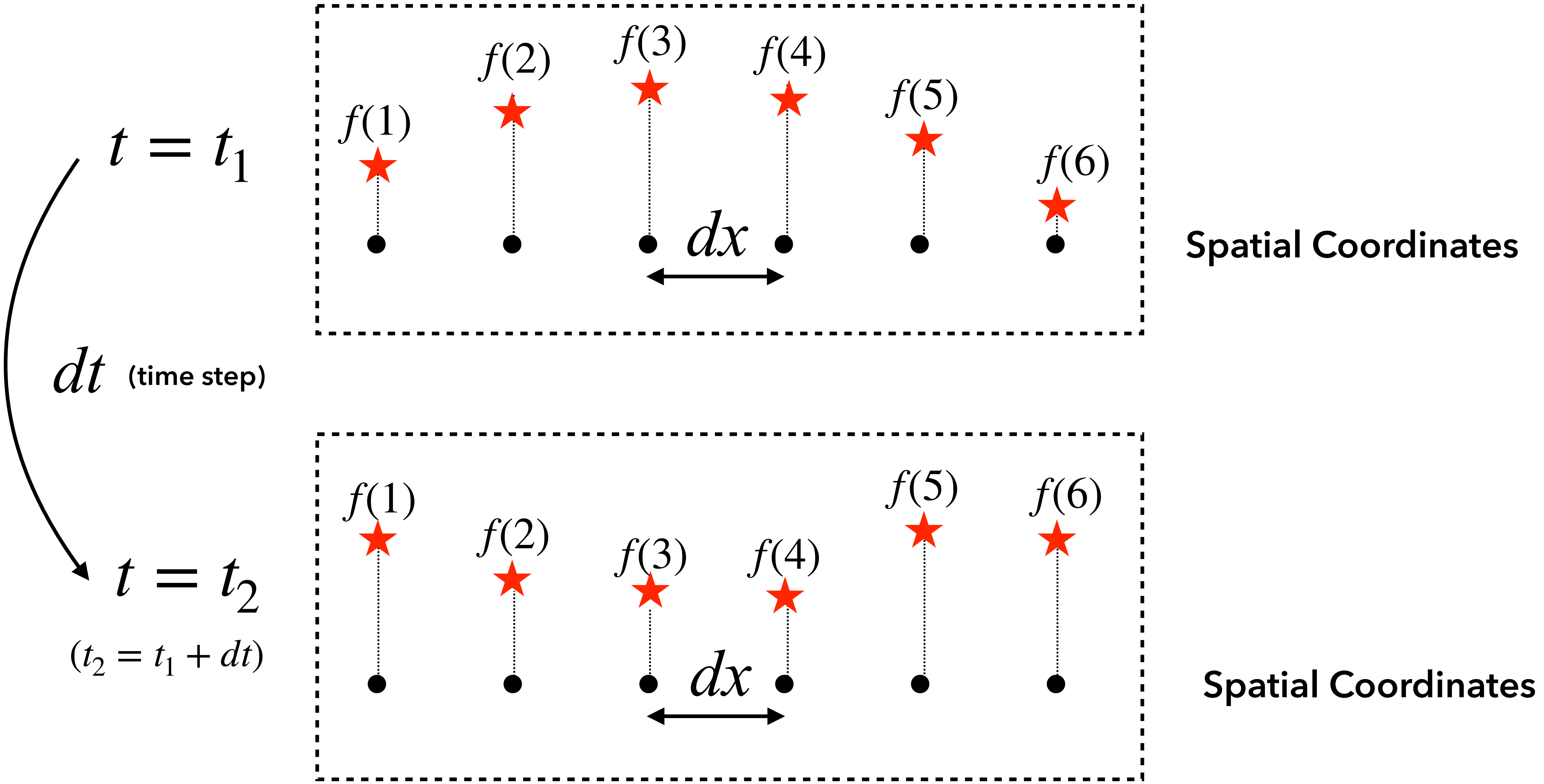
$t = t_1$



Spatial Coordinates

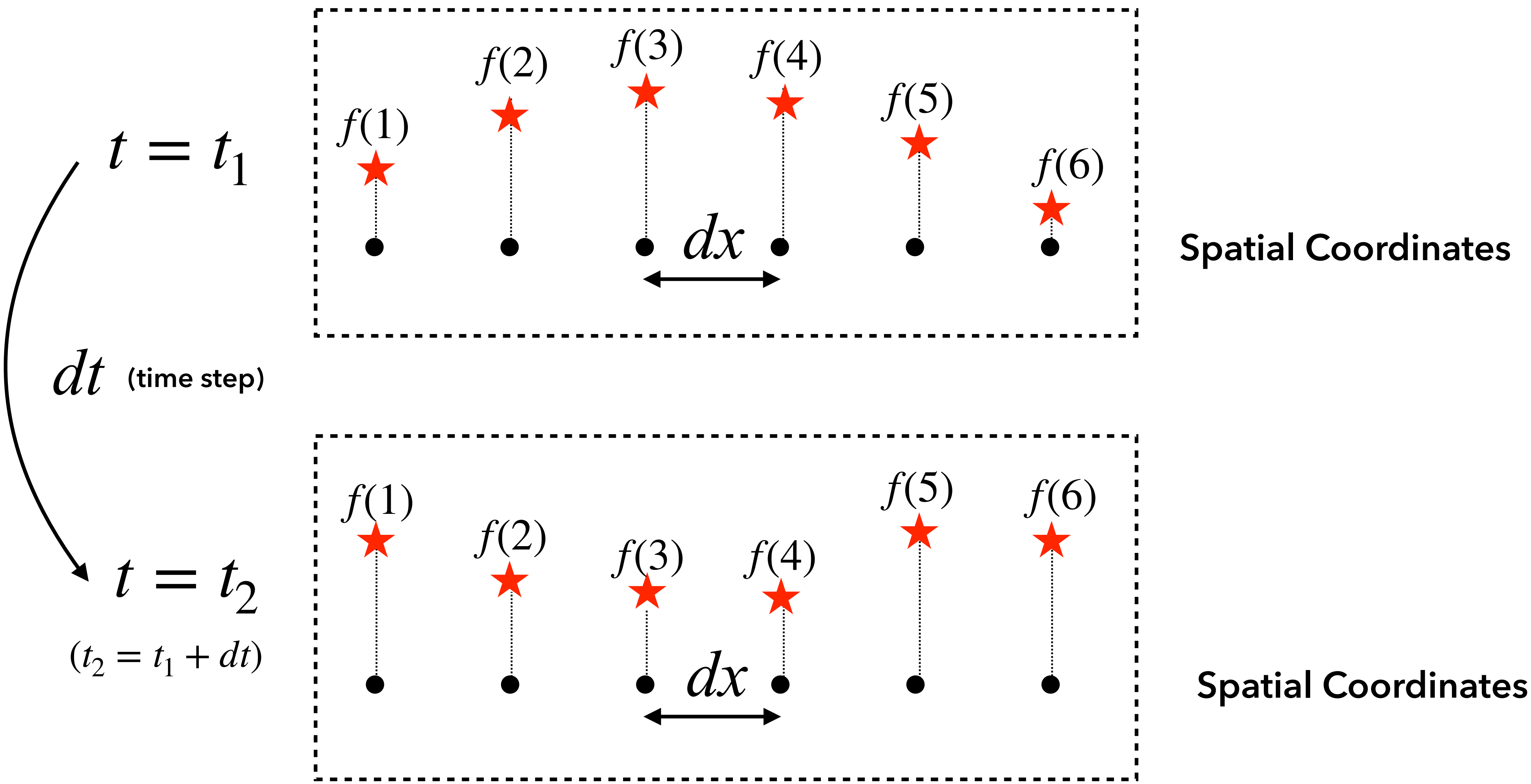
4. Lattice derivatives





$t = t_1$
dt (time step)
$t = t_2$
$(t_2 = t_1 + dt)$
$f(1)$
$f(2)$
$f(3)$
$f(4)$
$f(5)$
$f(6)$
dx
Spatial Coordinates
$f(1)$
$f(2)$
$f(3)$
$f(4)$
$f(5)$
$f(6)$
dx
Spatial Coordinates

4. Lattice derivatives



New notation: $\boxed{n \equiv (n_0, \mathbf{n}) = (n_0, n_1, n_2, n_3)}$

$n_0 = 1,\ 2,\ 3,\ \ldots\ ,\ \#\ \text{(time steps)}$

$n_i = 0,\ 1, \ldots,\ N - 1\ ;\ i = 1,2,3$

4. Lattice derivatives

Neutral derivative:

$$\boxed{[\nabla_\mu^{(0)} f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu}}$$

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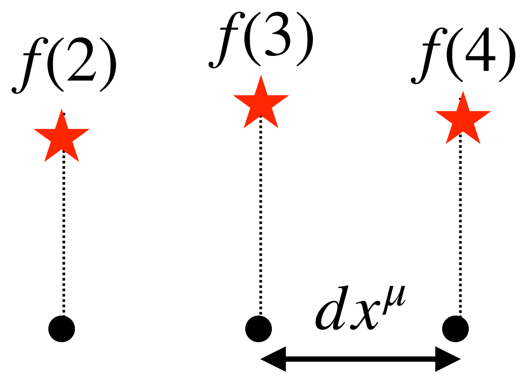
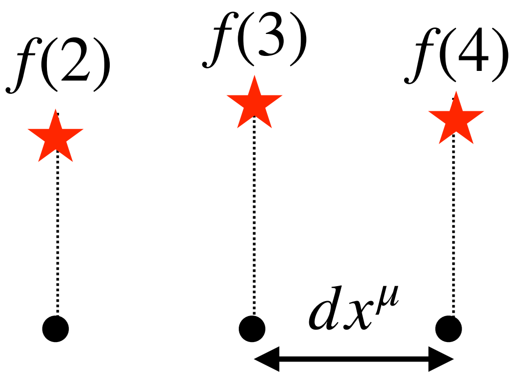


$$[\nabla_\mu^{(0)} f](3) \equiv \frac{f(4) - f(2)}{2dx^\mu}$$

4. Lattice derivatives

Neutral derivative:

$$\boxed{[\nabla^{(0)}_\mu f](n) = \frac{f(n+\hat\mu) - f(n-\hat\mu)}{2dx^\mu}} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x\equiv \mathbf{n}dx+n_0dt} + \mathcal{O}(dx_\mu^2)$$



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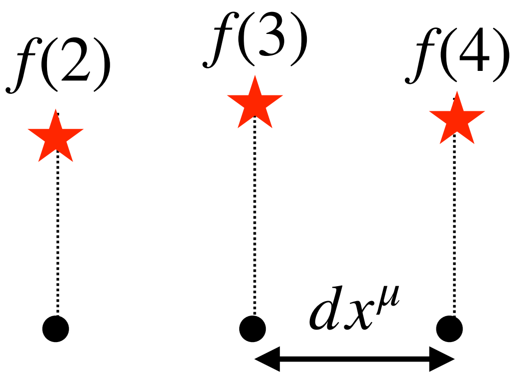
$dx\ ,\, dt$



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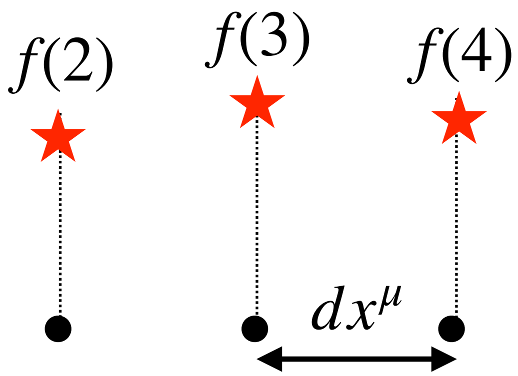
$dx\ ,dt$



$$[\nabla^{(0)}_\mu f](3) \equiv \frac{f(4) - f(2)}{2dx^\mu}$$

Charged derivative:

(+): Forward ; (-): Backward

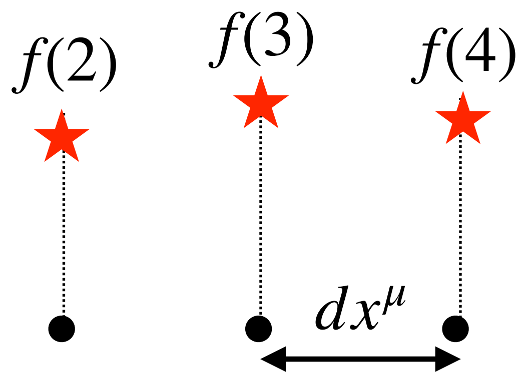
$$\boxed{[\nabla^{\pm}_\mu f] = \frac{\pm f(n\pm\hat{\mu}) \mp f(n)}{dx^\mu}}$$

4. Lattice derivatives

Neutral derivative:

$$\boxed{[\nabla^{(0)}_\mu f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu}} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu^2)$$

$dx\ ,dt$



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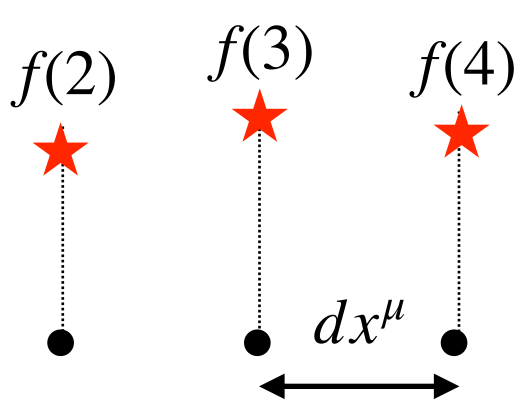


$$[\nabla^{(+)}_\mu f](3.0) \equiv \frac{f(4) - f(3)}{dx^\mu}$$

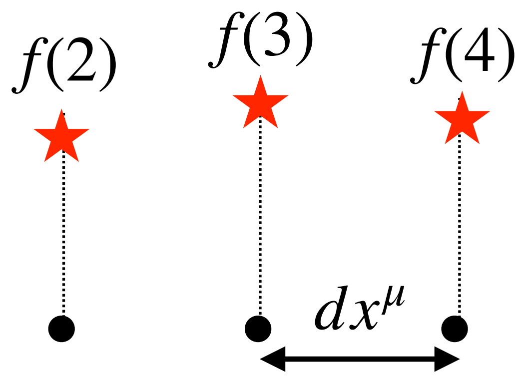
$$[\nabla^{(+)}_\mu f](3.5) \equiv \frac{f(4) - f(3)}{dx^\mu}$$

4. Lattice derivatives

Neutral derivative:

$$\boxed{[\nabla^{(0)}_\mu f](n) = \frac{f(n+\hat\mu) - f(n-\hat\mu)}{2dx^\mu}} \longrightarrow \partial_i \mathtt{f}(x)\Big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu^2)$$

$dx\ ,\ dt$

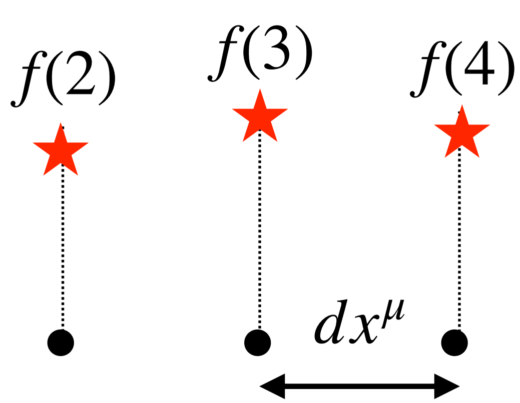


$$[\nabla^{(0)}_\mu f](3) \equiv \frac{f(4) - f(2)}{2dx^\mu}$$

Charged derivative:

(+): Forward ; (-): Backward

$$\boxed{[\nabla^{\pm}_\mu f] = \frac{\pm f(n \pm \hat\mu) \mp f(n)}{dx^\mu}} \longrightarrow \begin{cases} \partial_i \mathtt{f}(x)\Big|_{x \equiv \mathbf{n}dx + n_0 dt} + \mathcal{O}(dx_\mu) \\ \partial_i \mathtt{f}(x)\Big|_{x \equiv (n \pm \hat\mu/2)dx^\mu} + \mathcal{O}(dx_\mu^2) \end{cases}$$



$$[\nabla^{(+)}_\mu f](3.0) \equiv \frac{f(4) - f(3)}{dx^\mu} \longrightarrow \partial_\mu \mathtt{f}(3) + \mathcal{O}(dx^\mu)$$

$$[\nabla^{(+)}_\mu f](3.5) \equiv \frac{f(4) - f(3)}{dx^\mu} \longrightarrow \partial_\mu \mathtt{f}(3.5) + \mathcal{O}(dx_\mu^2)$$

4. Lattice derivatives

Arbitrary derivative:

$$\left[\nabla_i f\right](l)$$

4. Lattice derivatives

Arbitrary derivative:

$$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m)$$

Linear combination of weights: $D_i(l,m)$

(real-valued function of two lattice coordinates)

4. Lattice derivatives

Arbitrary derivative:

$$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m) = \sum_m D_i(l-m) f(m)$$

Linear combination of weights: $D_i(l,m)$

(real-valued function of two lattice coordinates)

Invariant under translations

Arbitrary derivative:

$$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m) = \sum_m D_i(l-m) f(m) = \sum_{m'} D_i(m') f(l-m')$$

Invariant under translations

shift

Linear combination of weights: $D_i(l,m)$

(real-valued function of two lattice coordinates)

4. Lattice derivatives

Arbitrary derivative:

$$\left[\nabla_i f\right](l) = \sum_m D_i(l,m) f(m) = \sum_m D_i(l-m) f(m) = \sum_{m'} D_i(m') f(l-m')$$

Invariant under translations

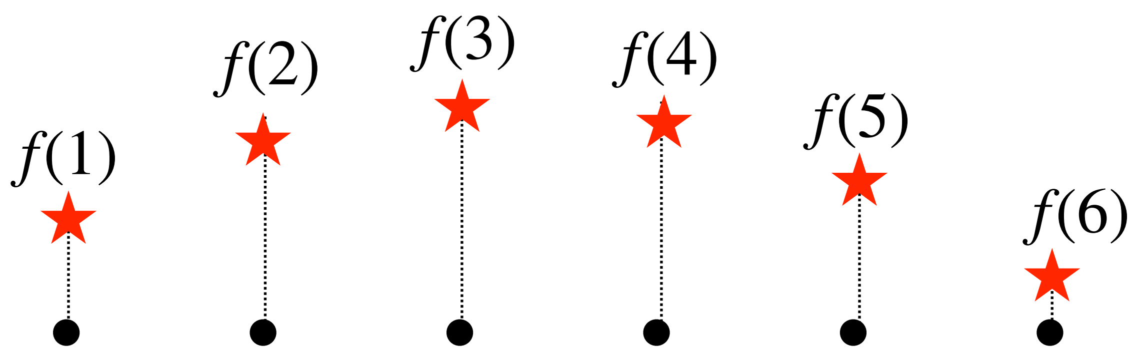
shift

Linear combination of weights: $D_i(l,m)$

(real-valued function of two lattice coordinates)

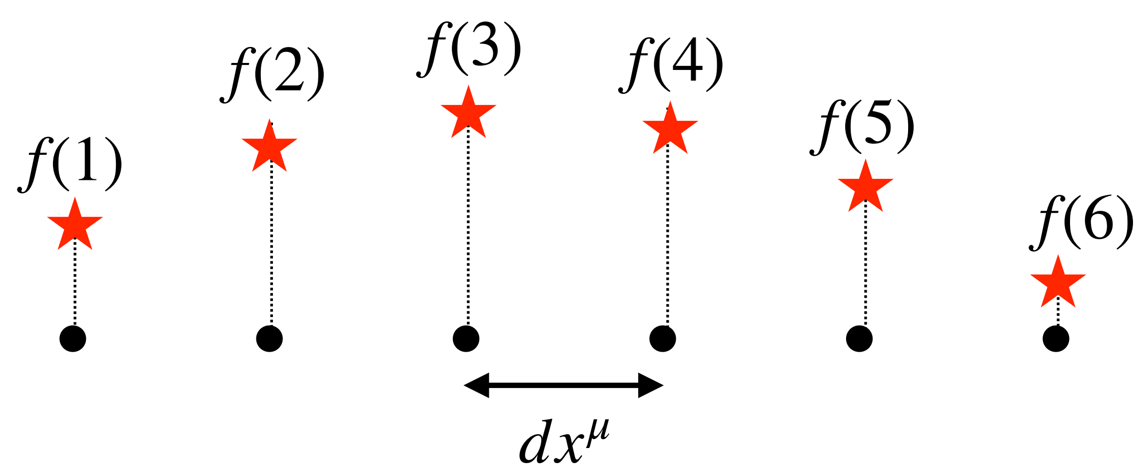
Neutral: $D_i^0(\mathbf{m}') = \dfrac{\delta_{\mathbf{m}',-\hat{\imath}} - \delta_{\mathbf{m}',\hat{\imath}}}{2dx}$, **Charged:** $\begin{cases} D_i^{\pm}(\mathbf{m}') = \dfrac{\pm\delta_{\mathbf{m}',\mp\hat{\imath}/2} \mp \delta_{\mathbf{m}',\pm\hat{\imath}/2}}{dx}, & \text{if } \mathbf{l} = \mathbf{n} + \dfrac{\hat{\imath}}{2}, \\ D_i^{\pm}(\mathbf{m}') = \dfrac{\pm\delta_{\mathbf{m}',\mp\hat{\imath}} \mp \delta_{\mathbf{m}',\mathbf{0}}}{dx}, & \text{if } \mathbf{l} = \mathbf{n}, \end{cases}$

"Goodness" of Lattice representation of continuum fields } controlled by {



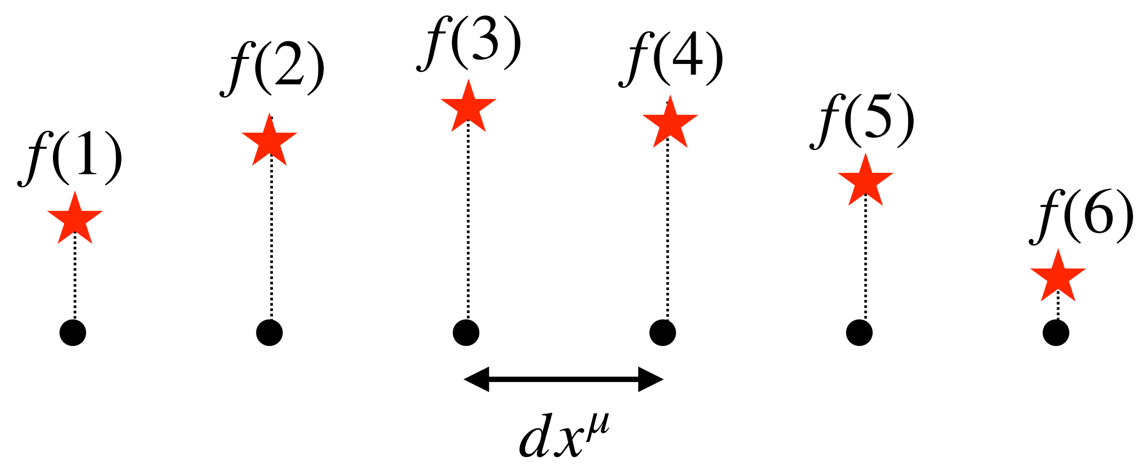
4. Lattice derivatives

"Goodness" of Lattice representation of continuum fields $\Big\}$ controlled by $\Big\{$ **Smallness** of dx^μ



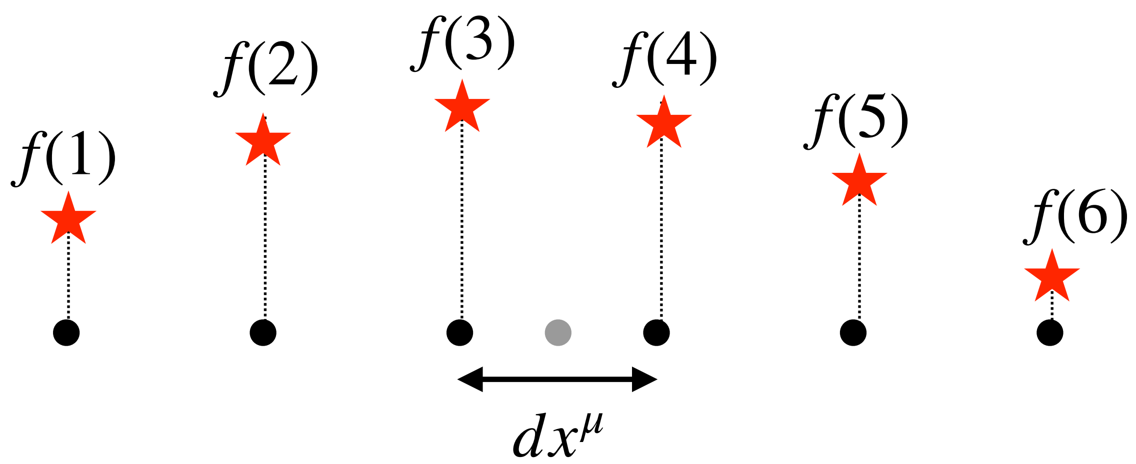
4. Lattice derivatives

"Goodness" of Lattice representation of continuum fields controlled by
- **Smallness** of dx^μ
- **Lattice Derivative** operator ∇_μ



4. Lattice derivatives

"Goodness" of Lattice representation of continuum fields } controlled by {
- **Smallness** of dx^μ
- **Lattice Derivative** operator ∇_μ
- **Location** where operator '*lives*' $n_\mu \pm a_\mu$



Lattice Derivative

4. Lattice derivatives

Lattice Derivative → Lattice Momentum

4. Lattice derivatives

Lattice Derivative → **Lattice Momentum**

Lattice derivative:

$$[\nabla_i f](\mathbf{n}) = \sum_{\mathbf{m}'} D_i(\mathbf{m}') f(\mathbf{n} - \mathbf{m}')$$

4. Lattice derivatives

Lattice Derivative → Lattice Momentum

$$\left(f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n}) \right)$$

Lattice derivative:

$$[\nabla_i f](\mathbf{n}) = \sum_{\mathbf{m}'} D_i(\mathbf{m}') f(\mathbf{n} - \mathbf{m}')$$

Fourier transform of a Lattice Derivative:

$$[\nabla_i f](\tilde{\mathbf{n}}) = \sum_{\mathbf{n}} e^{-\frac{2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}} \sum_{\mathbf{m}} D_i(\mathbf{n} - \mathbf{m}) f(\mathbf{m})$$

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$$= \sum_{\mathbf{n}'} e^{\frac{-2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'} D_i(\mathbf{n}') \sum_{\mathbf{m}} e^{\frac{-2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{m}} f(\mathbf{m})$$

4. Lattice derivatives

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$$= \underbrace{\sum_{\mathbf{n}'} e^{\frac{-2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{n}'} D_i(\mathbf{n}')}_{\equiv\ -ik_{\mathrm{Lat},i}(\tilde{\mathbf{n}})} \underbrace{\sum_{\mathbf{m}} e^{\frac{-2\pi i}{N}\tilde{\mathbf{n}}\cdot\mathbf{m}} f(\mathbf{m})}_{f(\tilde{\mathbf{n}})}$$

4. Lattice derivatives

Lattice Derivative → Lattice Momentum

Lattice derivative: $$[\nabla_i f](\mathbf{n}) = \sum_{\mathbf{m}'} D_i(\mathbf{m}') f(\mathbf{n}-\mathbf{m}')$$ $$\left(f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n}) \right)$$

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Lattice Momentum:

$$\boxed{\mathbf{k}_{\mathrm{Lat}}(\tilde{\mathbf{n}}) = i \sum_{\mathbf{n}'} e^{\frac{-2\pi i}{N}\tilde{\mathbf{n}}\mathbf{n}'} \overrightarrow{D}(\mathbf{n}')}$$

4. Lattice derivatives

Lattice Derivative → Lattice Momentum

Lattice derivative:

$$[\nabla_i f](\mathbf{n}) = \sum_{\mathbf{m}'} D_i(\mathbf{m}') f(\mathbf{n}-\mathbf{m}') \qquad \left(f(\tilde{\mathbf{n}}) \equiv \sum_{\mathbf{n}} e^{-i\frac{2\pi}{N}\mathbf{n}\tilde{\mathbf{n}}} f(\mathbf{n}) \right)$$

Fourier transform of a Lattice Derivative:

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$$\equiv -ik_{\mathrm{Lat},i}(\tilde{\mathbf{n}}) \, f(\tilde{\mathbf{n}})$$

Lattice Momentum:

$$\boxed{\mathbf{k}_{\mathrm{Lat}}(\tilde{\mathbf{n}}) = i \sum_{\mathbf{n}'} e^{\frac{-2\pi i}{N}\tilde{\mathbf{n}}\mathbf{n}'} \overrightarrow{D}(\mathbf{n}')} \longrightarrow$$

weights define Lattice momentum

4. Lattice derivatives

Lattice Momentum:

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weights define Lattice momentum

4. Lattice derivatives

Lattice Momentum:

$$\boxed{\mathbf{k}_{\rm Lat}(\tilde{\mathbf{n}}) = i\sum_{\mathbf{n}'} e^{\frac{-2\pi i}{N}\tilde{\mathbf{n}}\mathbf{n}'}\overrightarrow{D}(\mathbf{n}')}$$

weights define Lattice momentum

Neutral derivative: $k^0_{\rm Lat,i} = \dfrac{\sin(2\pi\tilde{n}_i/N)}{dx}$

$$[\nabla^{(0)}_\mu f](n) = \frac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu}$$

4. Lattice derivatives

Lattice Momentum:

$$\boxed{\mathbf{k}_{\rm Lat}(\tilde{\mathbf{n}}) = i\sum_{\mathbf{n}'} e^{\frac{-2\pi i}{N}\tilde{\mathbf{n}}\mathbf{n}'}\overrightarrow{D}(\mathbf{n}')}$$

$\longrightarrow$ **weights** define Lattice momentum

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$[\nabla^{(0)}_\mu f](n) = \dfrac{f(n+\hat{\mu}) - f(n-\hat{\mu})}{2dx^\mu}$

Charged derivative:
$$\begin{cases} k^+_{\rm Lat,i} = k^-_{\rm Lat,i} = 2\dfrac{\sin(\pi\tilde{n}_i/N)}{dx} & \text{if} \quad \mathbf{l} = \mathbf{n} \pm \dfrac{\hat{i}}{2} \\ k^\pm_{\rm Lat,i} = \dfrac{\sin(2\pi\tilde{n}_i/N)}{dx} \mp i\dfrac{1-\cos(2\pi\tilde{n}_i/N)}{dx} & \text{if} \quad \mathbf{l} = \mathbf{n} \end{cases}$$

$[\nabla^\pm_\mu f] = \dfrac{\pm f(n\pm\hat{\mu}) \mp f(n)}{dx^\mu}$

4. Lattice derivatives

Lineal: $k^{(\mathrm{Lin})}_{\mathrm{Lat},i} = \tilde{n}_i \frac{2\pi}{N dx}\,,$

Charged: $k^{+}_{\mathrm{Lat},i} = k^{-}_{\mathrm{Lat},i} = 2\frac{\sin(\pi \tilde{n}_i / N)}{dx}\,,$

(if $\mathbf{l} = \mathbf{n} \pm \frac{\hat{\imath}}{2}$)

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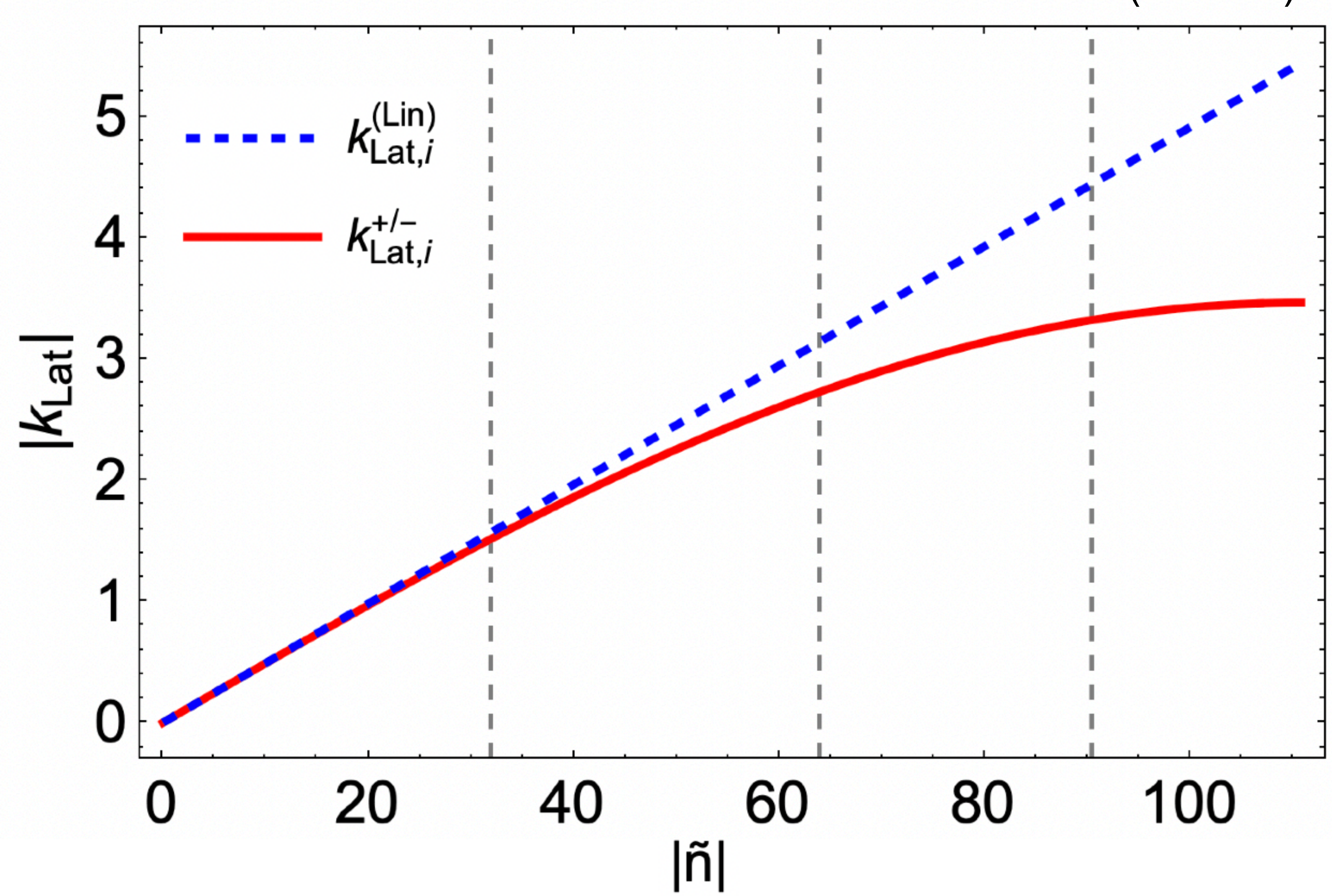
(if $\mathbf{l} = \mathbf{n} \pm \frac{\hat{\imath}}{2}$)

(N =128)



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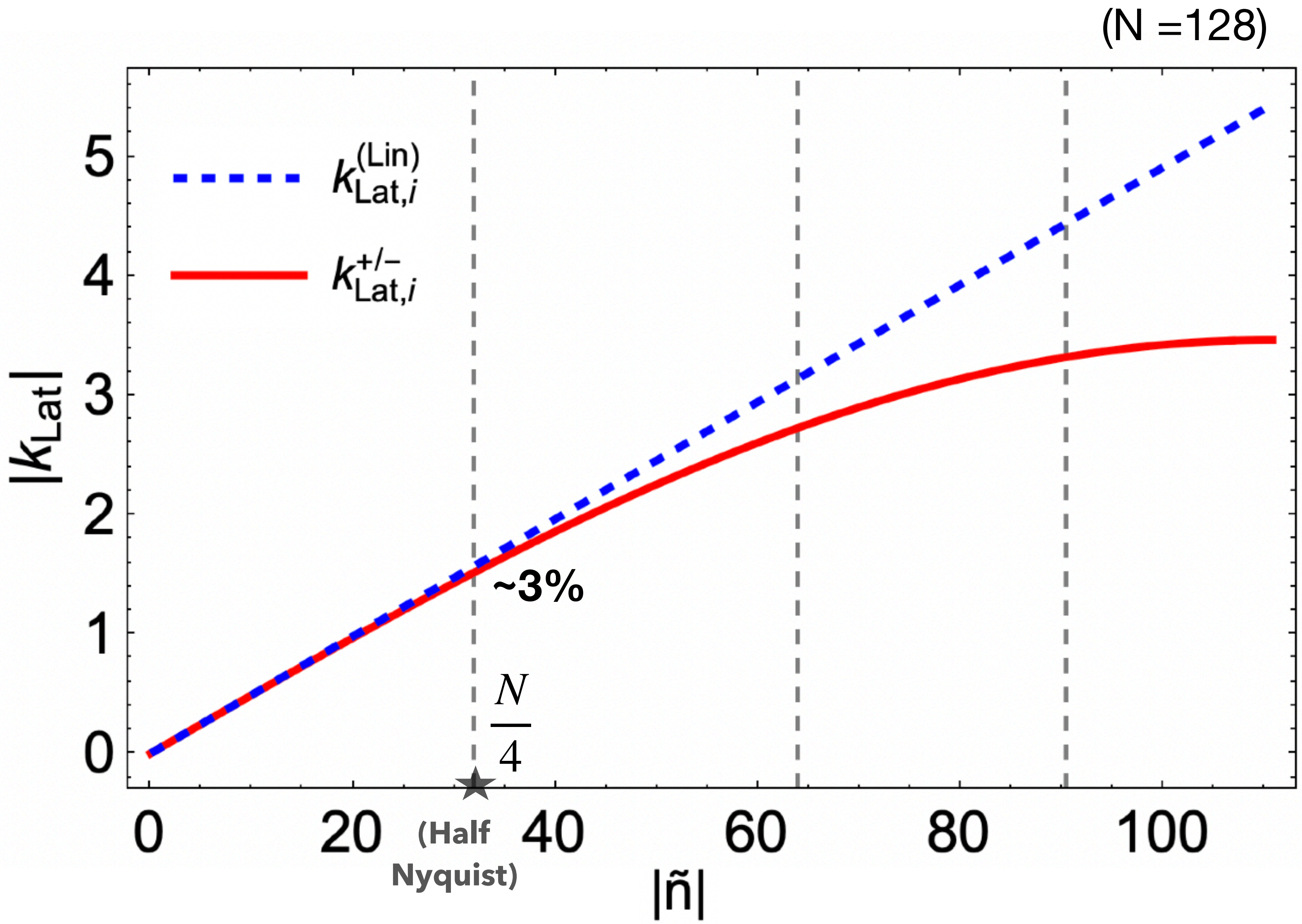

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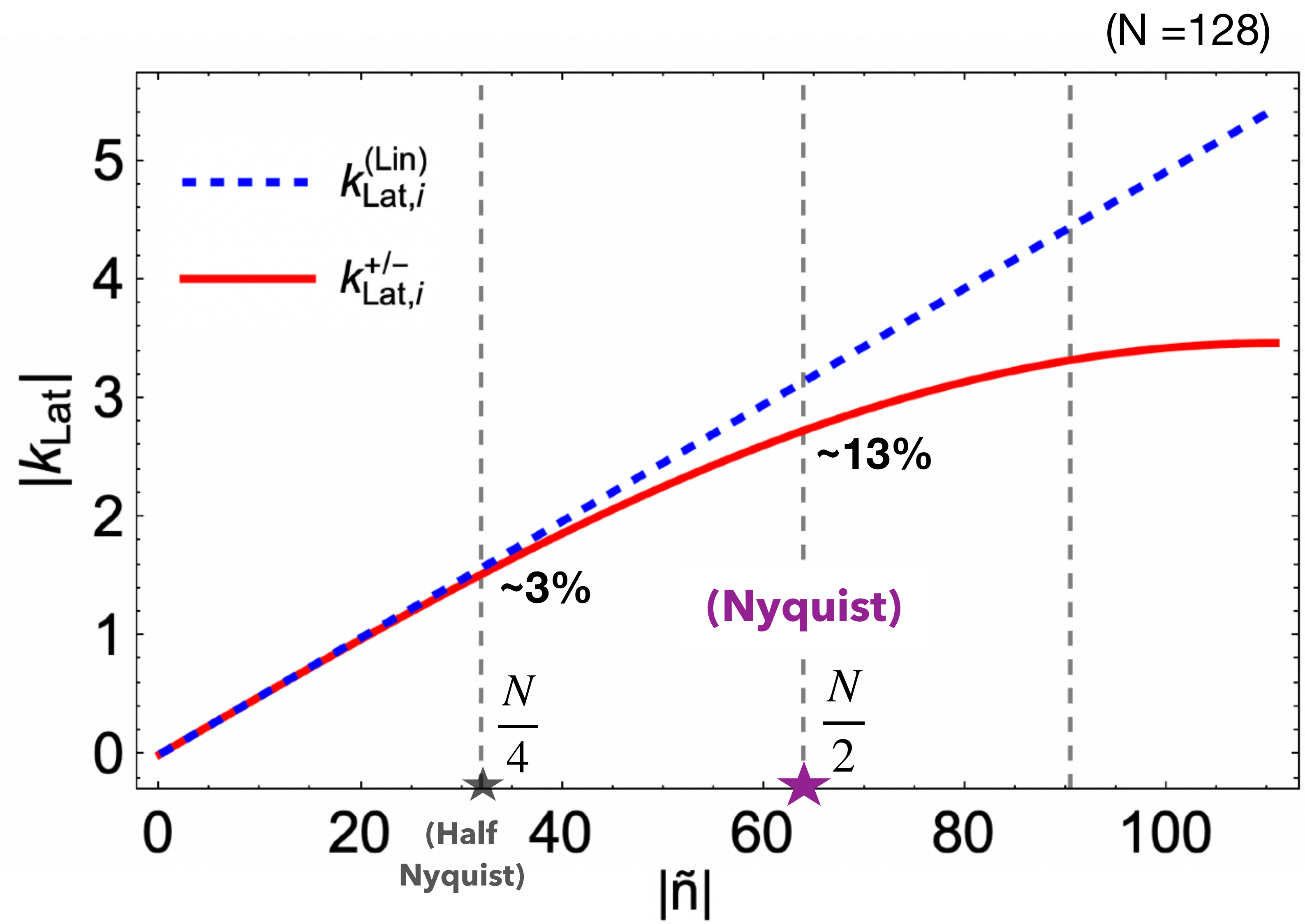

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Lineal: $k^{(\mathrm{Lin})}_{\mathrm{Lat},i} = \tilde{n}_i \dfrac{2\pi}{N dx}\,,$

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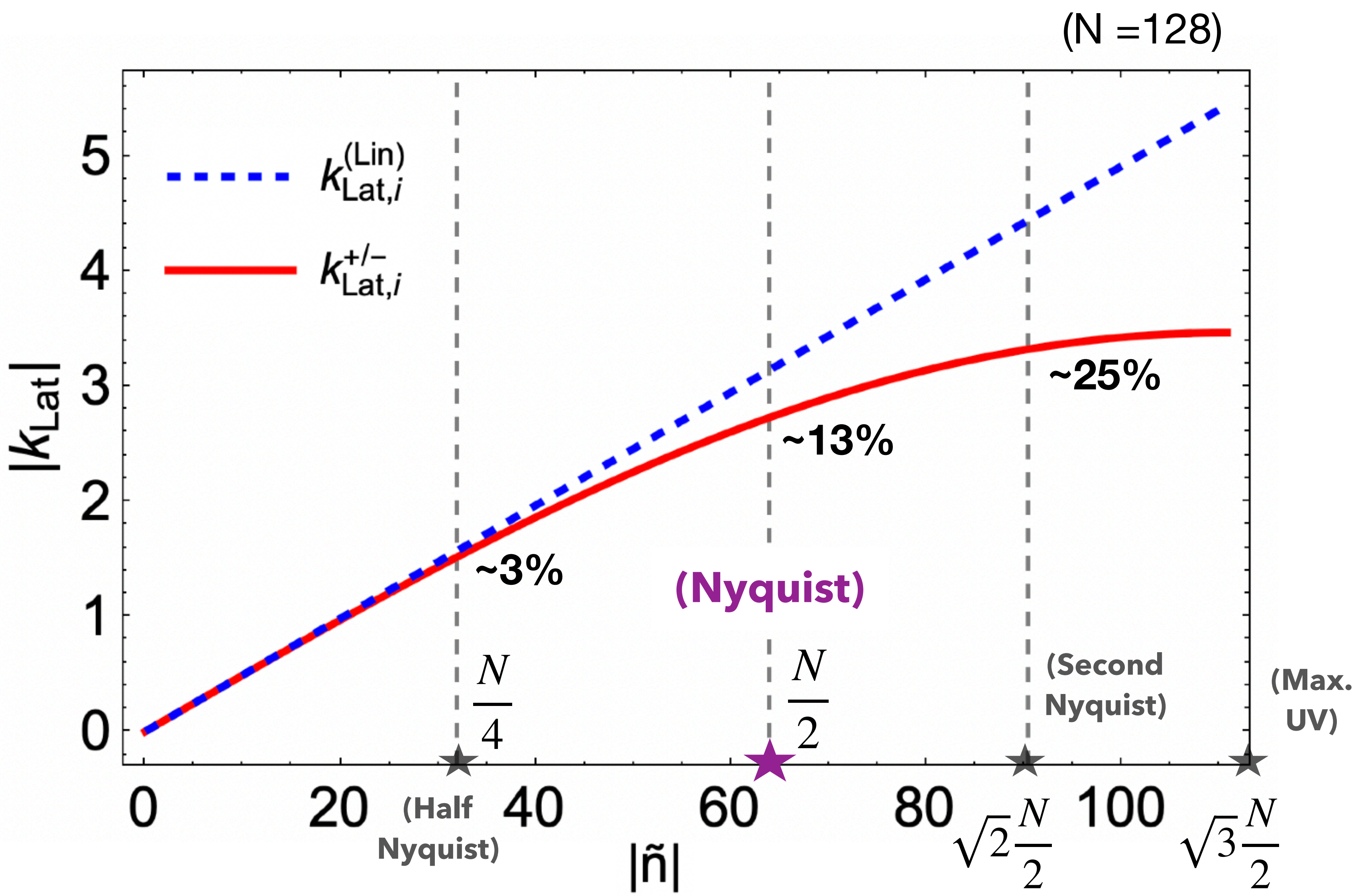

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5. Discrete power spectrum

Fourier transform of a continuous function:

$$\mathtt{f}(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\mathtt{f}_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}} \quad \longleftrightarrow \quad \mathtt{f}_{\mathbf{k}}(t) = \int d^3\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathtt{f}(\mathbf{x},t)$$

5. Discrete power spectrum

Fourier transform of a continuous function:

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Power spectrum (continuum):

$$\langle \mathtt{f}^2(\mathbf{x},t)\rangle = \int d\log k\;\Delta_{\mathtt{f}}(k,t)\;, \qquad \Delta_{\mathtt{f}}(k,t) \equiv \frac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k,t)$$

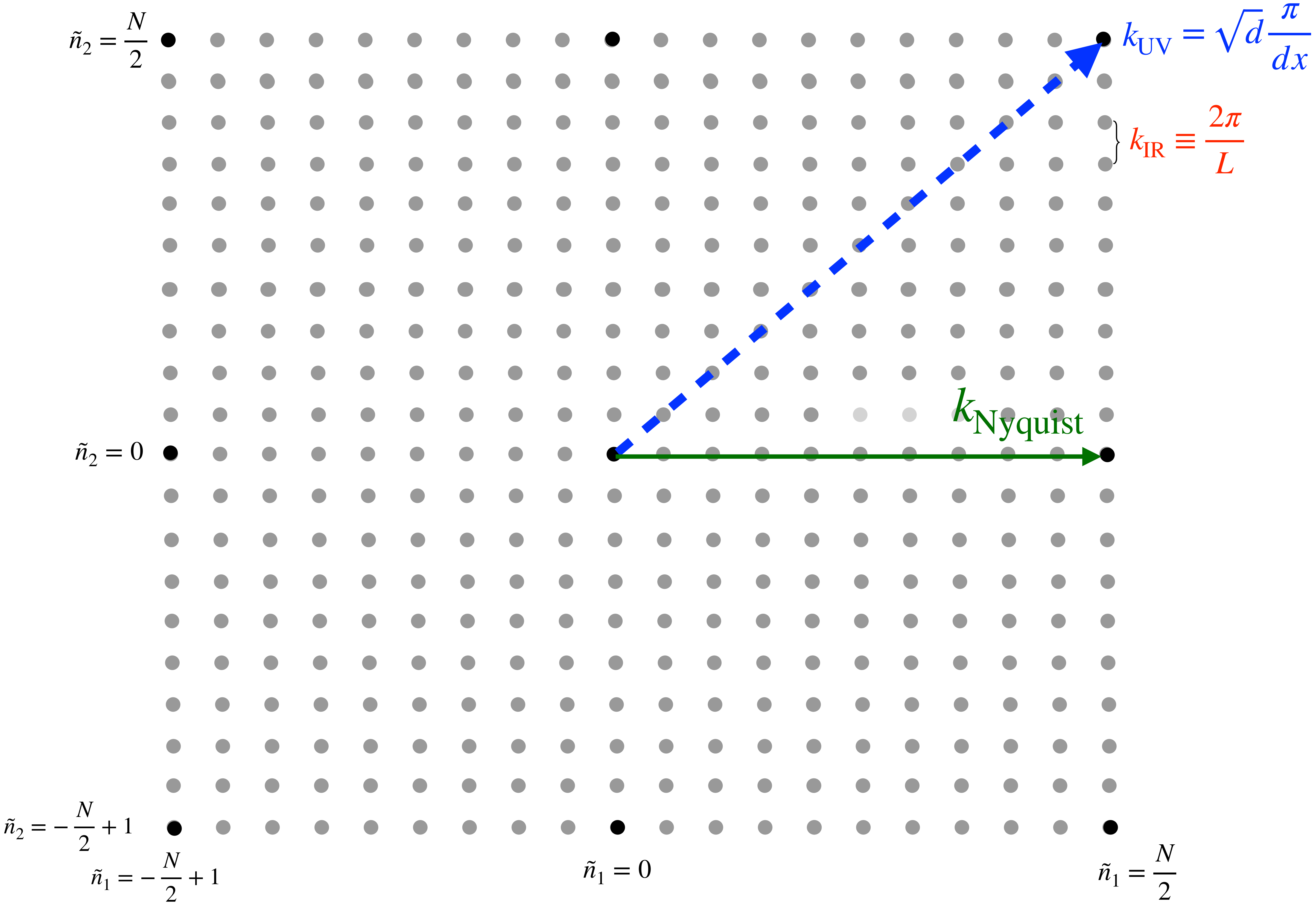
Ensemble Average

Power Spectrum

$$\langle \mathtt{f}_{\mathbf{k}}(t)\mathtt{f}^*_{\mathbf{k}'}(t)\rangle = (2\pi)^3\mathscr{P}_{\mathtt{f}}(k,t)\delta(\mathbf{k}-\mathbf{k}')$$

2-point Funct.

5. Discrete power spectrum



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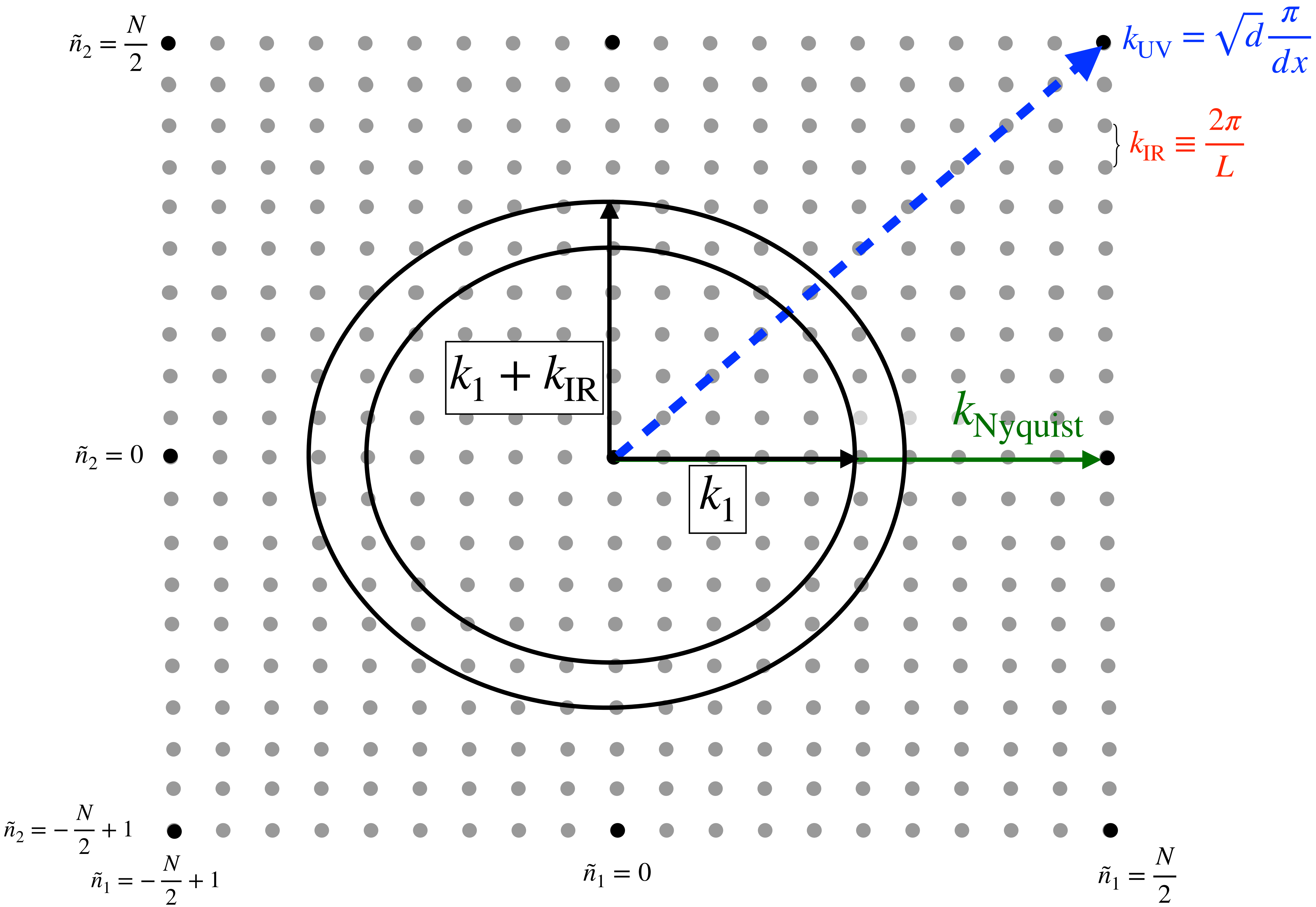
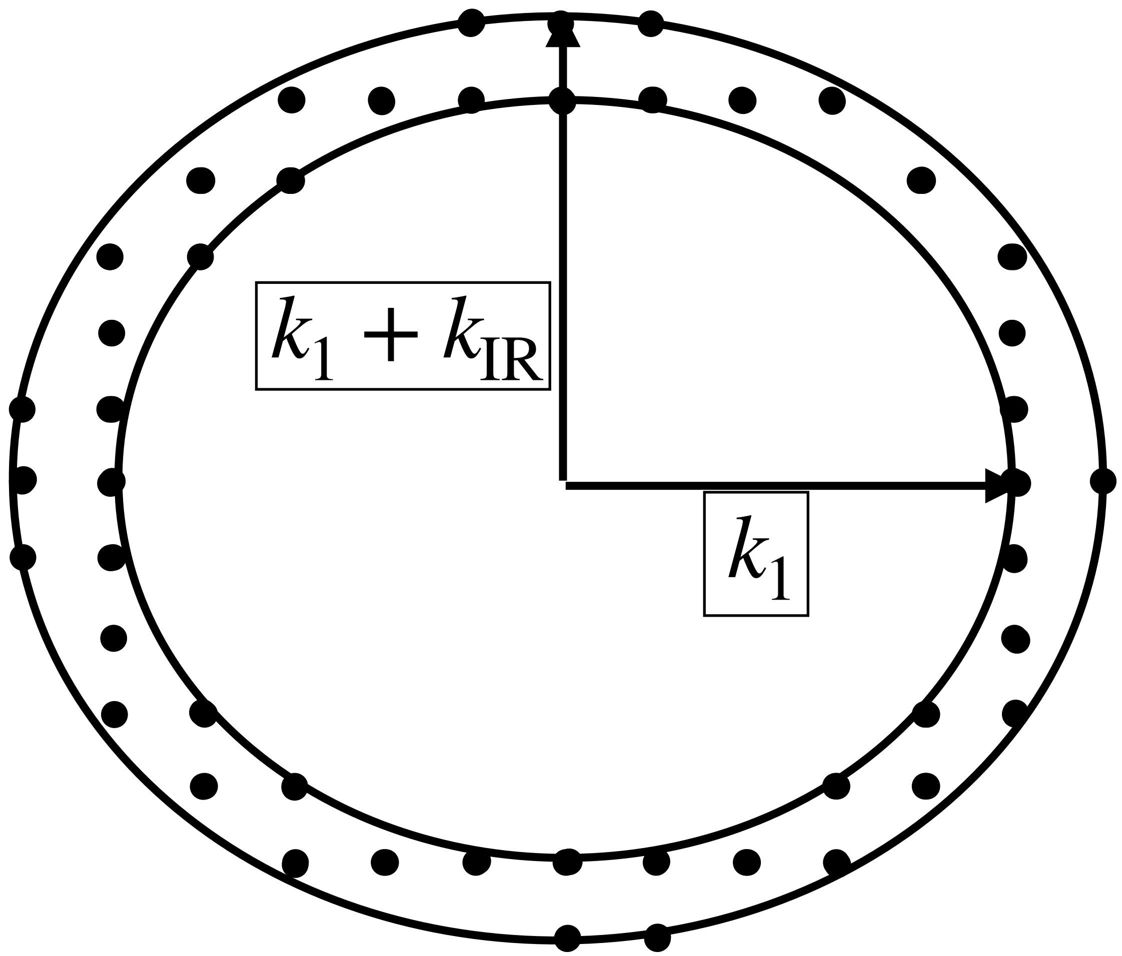
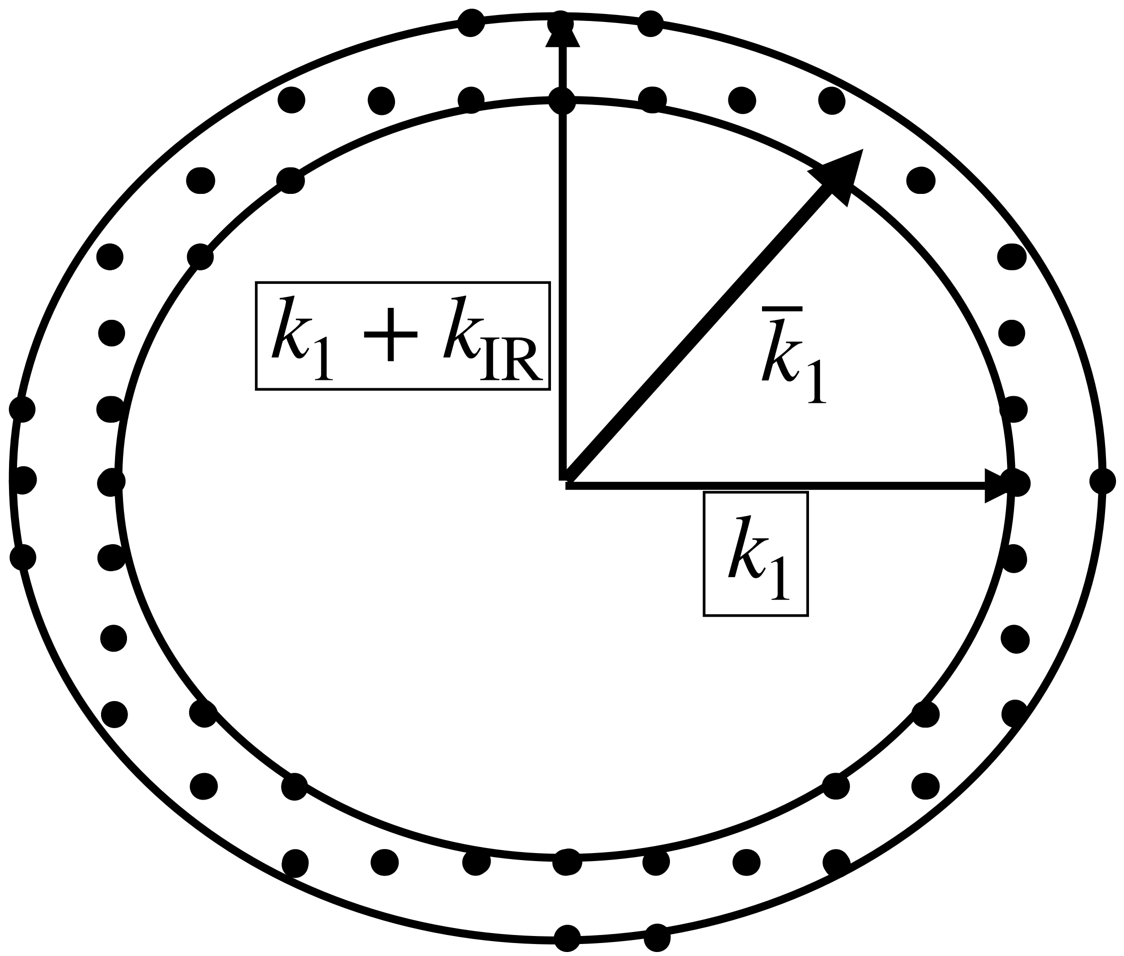


5. Discrete power spectrum



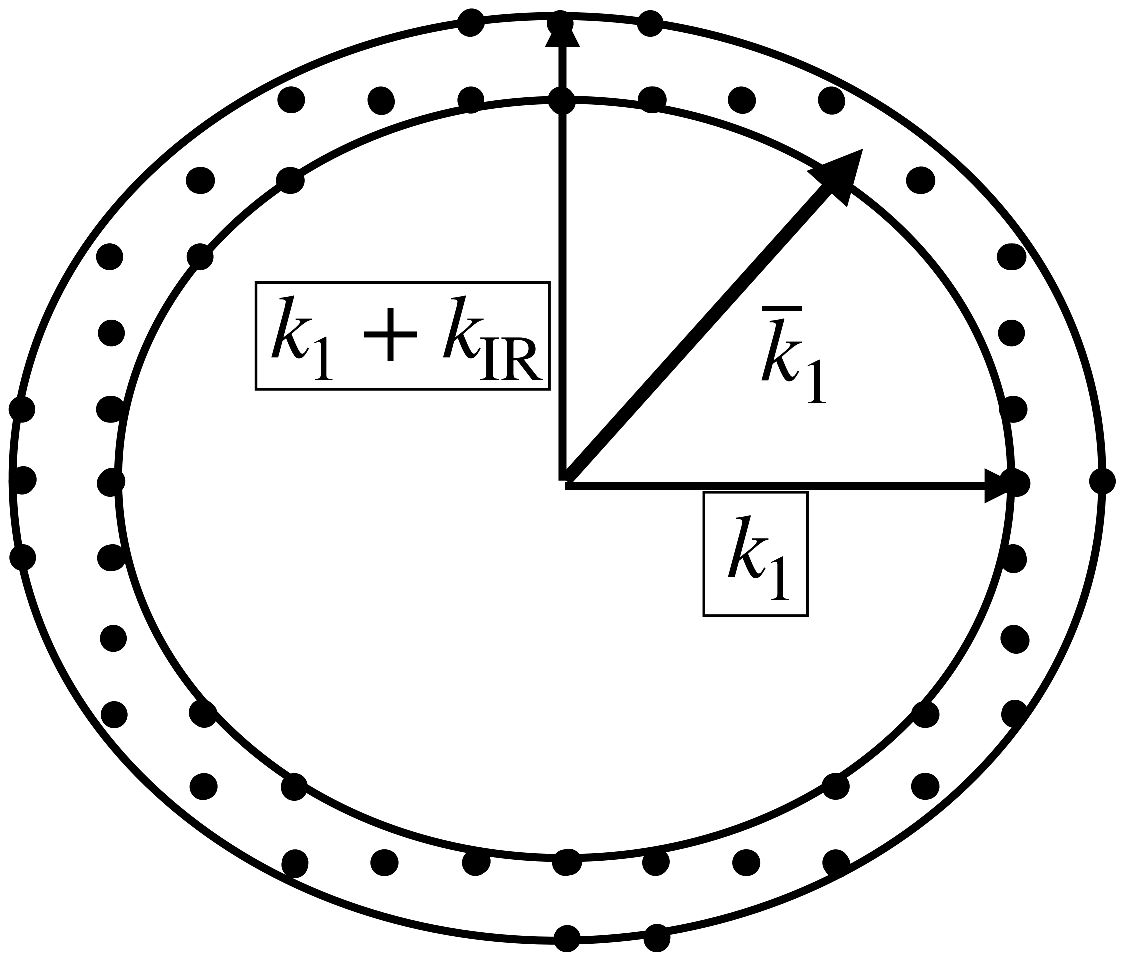
Bin 1: $k(\tilde{\mathbf{n}}) \in [k_1, \overbrace{k_1 + k_{\rm IR}}^{\equiv R_1}]$



Bin 1: $k(\tilde{\mathbf{n}}) \in \overbrace{[k_1, k_1 + k_{\rm IR}]}^{\equiv R_1}$

$\downarrow$

Average momenta: $\bar{k}_1 \simeq n_1 k_{\rm IR}$



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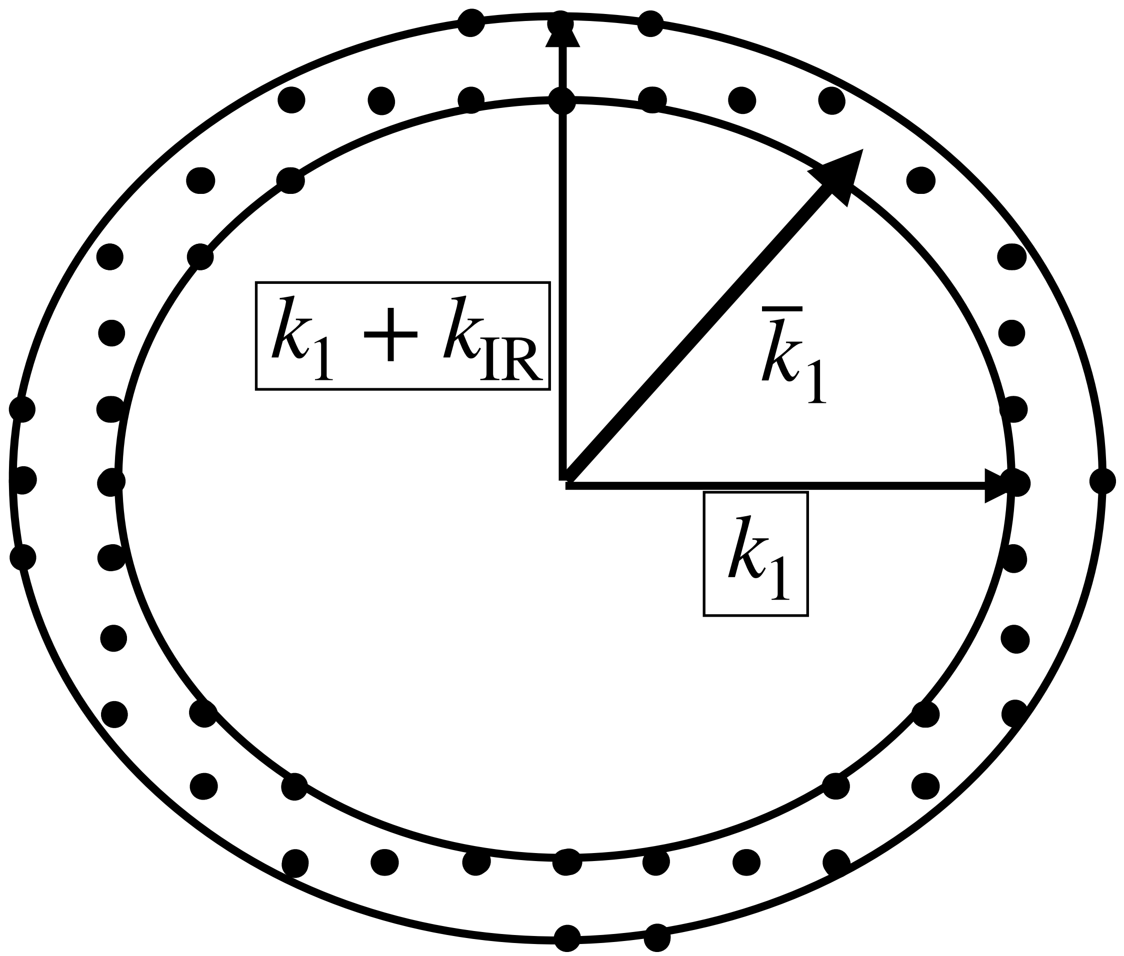
Modes within the shell: $\{\mathtt{f}_{\mathbf{k}}\}$;

5. Discrete power spectrum



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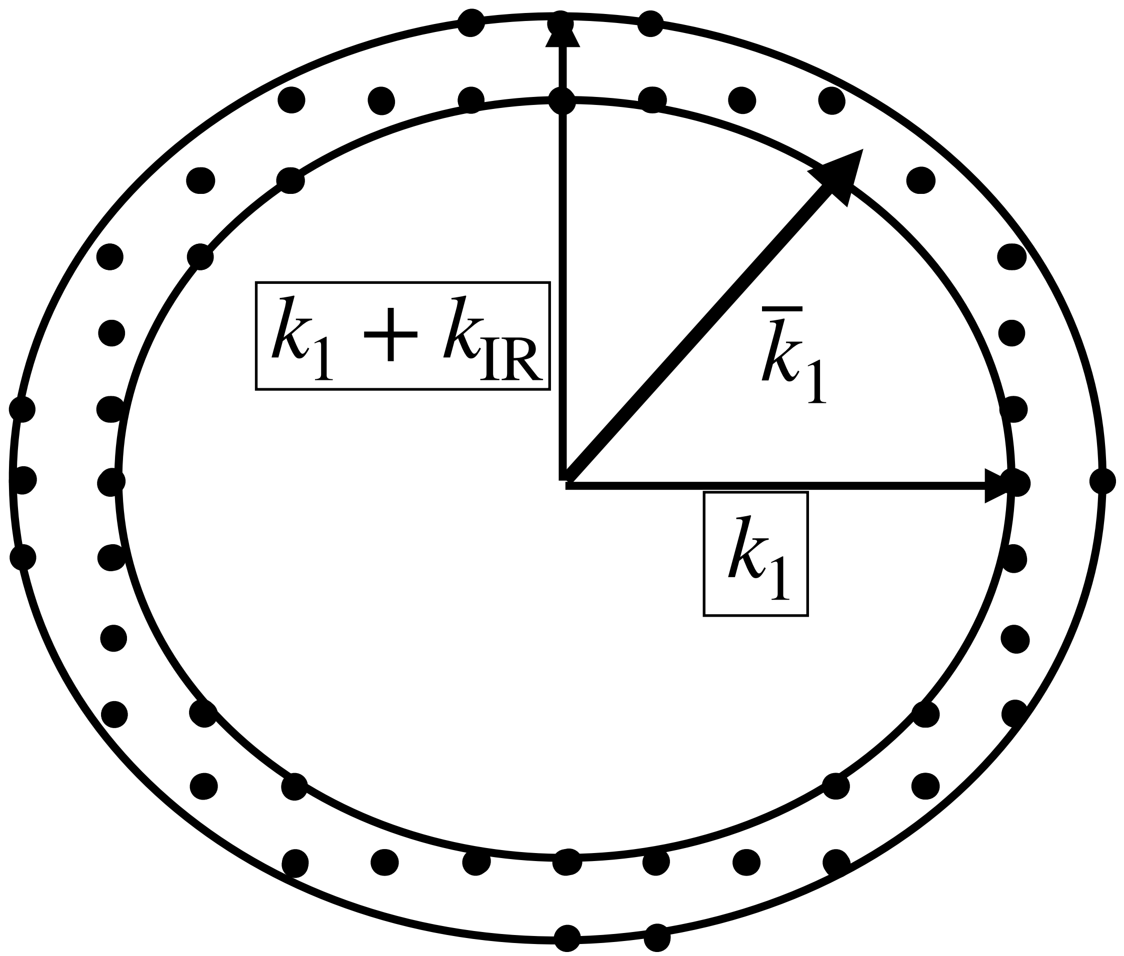
MULTIPLICITY: $\#_1 \simeq \Omega_d n_1^{d-1}$

(d-dimensional solid angle in d spatial dims)

$\Omega_3 = 4\pi$



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(d-dimensional solid angle in d spatial dims)

$\Omega_3 = 4\pi$

$$|\mathtt{f}_{\bar{k}_1}|^2 \equiv \langle |\mathtt{f}_{\mathbf{k}}|^2 \rangle_{R_1}$$

5. Discrete power spectrum

Continuum: $\langle \mathtt{f}^2 \rangle = \int d \log k \ \Delta_{\mathtt{f}}(k) \quad , \quad \Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k) \quad , \quad \langle \mathtt{f}_{\mathbf{k}}\mathtt{f}_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k}-\mathbf{k}')$

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Lattice:

5. Discrete power spectrum

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Lattice:

$$\langle f^2 \rangle_V = \frac{1}{N^3} \sum_{\mathbf{n}} f^2(\mathbf{n})$$

5. Discrete power spectrum

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Lattice:

Lattice FT

$$\langle f^2 \rangle_V = \frac{1}{N^3} \sum_{\mathbf{n}} f^2(\mathbf{n}) = \frac{1}{N^6} \sum_{\tilde{\mathbf{n}}} \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2$$

5. Discrete power spectrum

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k \; \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}}\mathtt{f}_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k}-\mathbf{k}')$

Lattice:

$$\langle f^2 \rangle_V = \frac{1}{N^3}\sum_{\mathbf{n}} f^2(\mathbf{n}) \overset{\text{Lattice FT}}{=} \frac{1}{N^6}\sum_{\tilde{\mathbf{n}}} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \overset{\text{Radial-Angular Decomposition}}{=} \frac{1}{N^6}\sum_{|\tilde{\mathbf{n}}|}\sum_{\tilde{\mathbf{n}}'\in R(\tilde{\mathbf{n}})} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2$$

5. Discrete power spectrum

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k \; \Delta_{\mathtt{f}}(k) \quad , \quad \Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2}\mathscr{P}_{\mathtt{f}}(k) \quad , \quad \langle \mathtt{f}_{\mathbf{k}}\mathtt{f}_{\mathbf{k}'} \rangle = (2\pi)^3 \mathscr{P}_{\mathtt{f}}(k)\delta(\mathbf{k}-\mathbf{k}')$

Lattice:

Lattice FT

Radial-Angular Decomposition

MULTIPLICITY: $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi |\tilde{\mathbf{n}}|^2$

$$\langle f^2 \rangle_V = \frac{1}{N^3}\sum_{\mathbf{n}} f^2(\mathbf{n}) = \frac{1}{N^6}\sum_{\tilde{\mathbf{n}}} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 = \frac{1}{N^6}\sum_{|\tilde{\mathbf{n}}|}\sum_{\tilde{\mathbf{n}}'\in R(\tilde{\mathbf{n}})} \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \simeq \frac{4\pi}{N^6}\sum_{|\tilde{\mathbf{n}}|} |\tilde{\mathbf{n}}|^2 \left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

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$$= \sum_{|\tilde{\mathbf{n}}|} \Delta \log k(\tilde{\mathbf{n}}) \, \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6} \left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

$$\left(\begin{array}{c} k(\tilde{\mathbf{n}}) \equiv k_{\rm IR} |\tilde{\mathbf{n}}| \\ \Delta \log k(\tilde{\mathbf{n}}) \equiv \dfrac{k_{\rm IR}}{k(\tilde{\mathbf{n}})} \end{array} \right)$$

5. Discrete power spectrum

Continuum: $\langle \mathtt{f}^2 \rangle = \int d\log k\ \Delta_{\mathtt{f}}(k)$, $\Delta_{\mathtt{f}}(k) \equiv \frac{k^3}{2\pi^2}\mathcal{P}_{\mathtt{f}}(k)$, $\langle \mathtt{f}_{\mathbf{k}}\mathtt{f}_{\mathbf{k}'} \rangle = (2\pi)^3 \mathcal{P}_{\mathtt{f}}(k)\delta(\mathbf{k}-\mathbf{k}')$

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MULTIPLICITY: $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi |\tilde{\mathbf{n}}|^2$

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$$= \sum_{|\tilde{\mathbf{n}}|} \Delta\log k(\tilde{\mathbf{n}}) \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})} = \sum_{|\tilde{\mathbf{n}}|} \Delta\log k(\tilde{\mathbf{n}})\ \Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|)$$

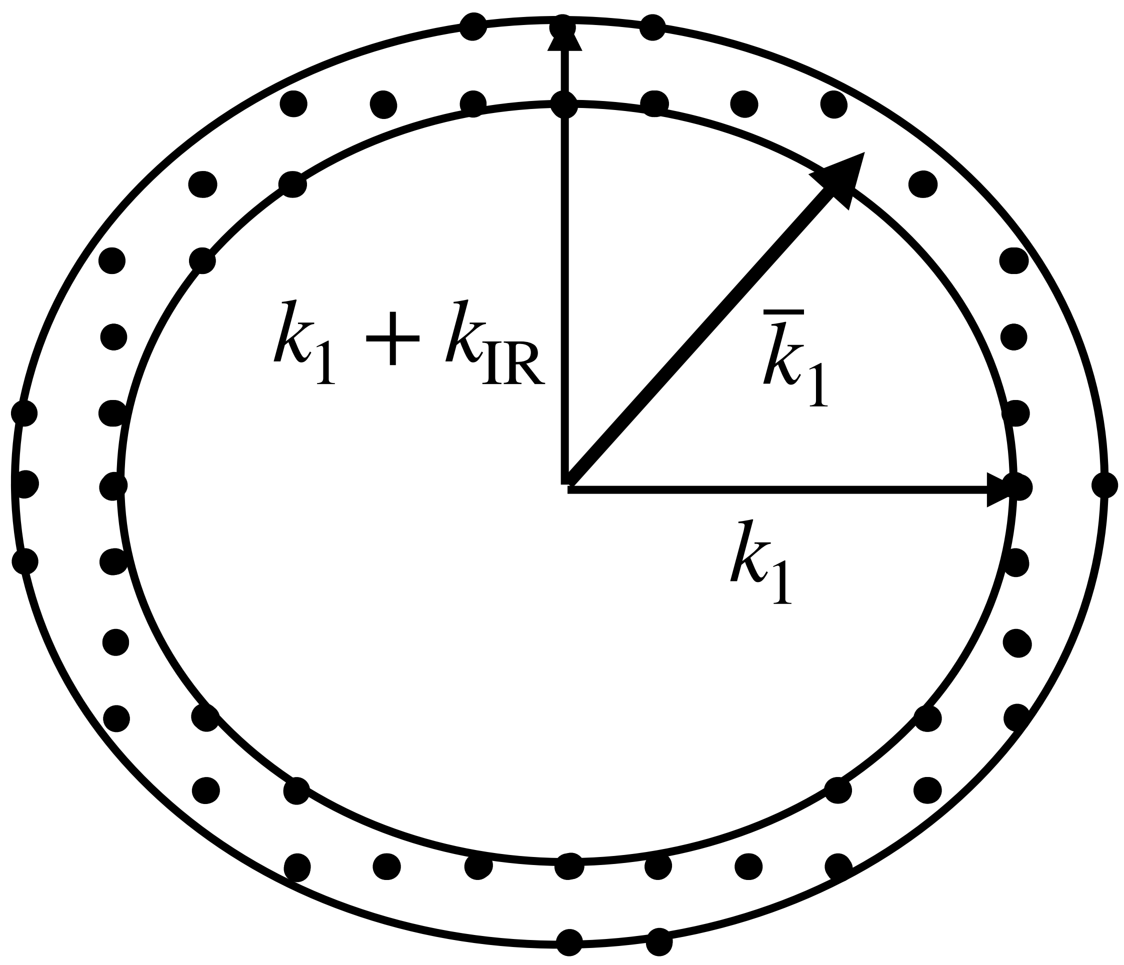
(Summation over bins)

$$\left(\begin{array}{c} k(\tilde{\mathbf{n}}) \equiv k_{\mathrm{IR}}|\tilde{\mathbf{n}}| \\ \Delta\log k(\tilde{\mathbf{n}}) \equiv \dfrac{k_{\mathrm{IR}}}{k(\tilde{\mathbf{n}})} \end{array} \right)$$

Lattice Power Spectrum:

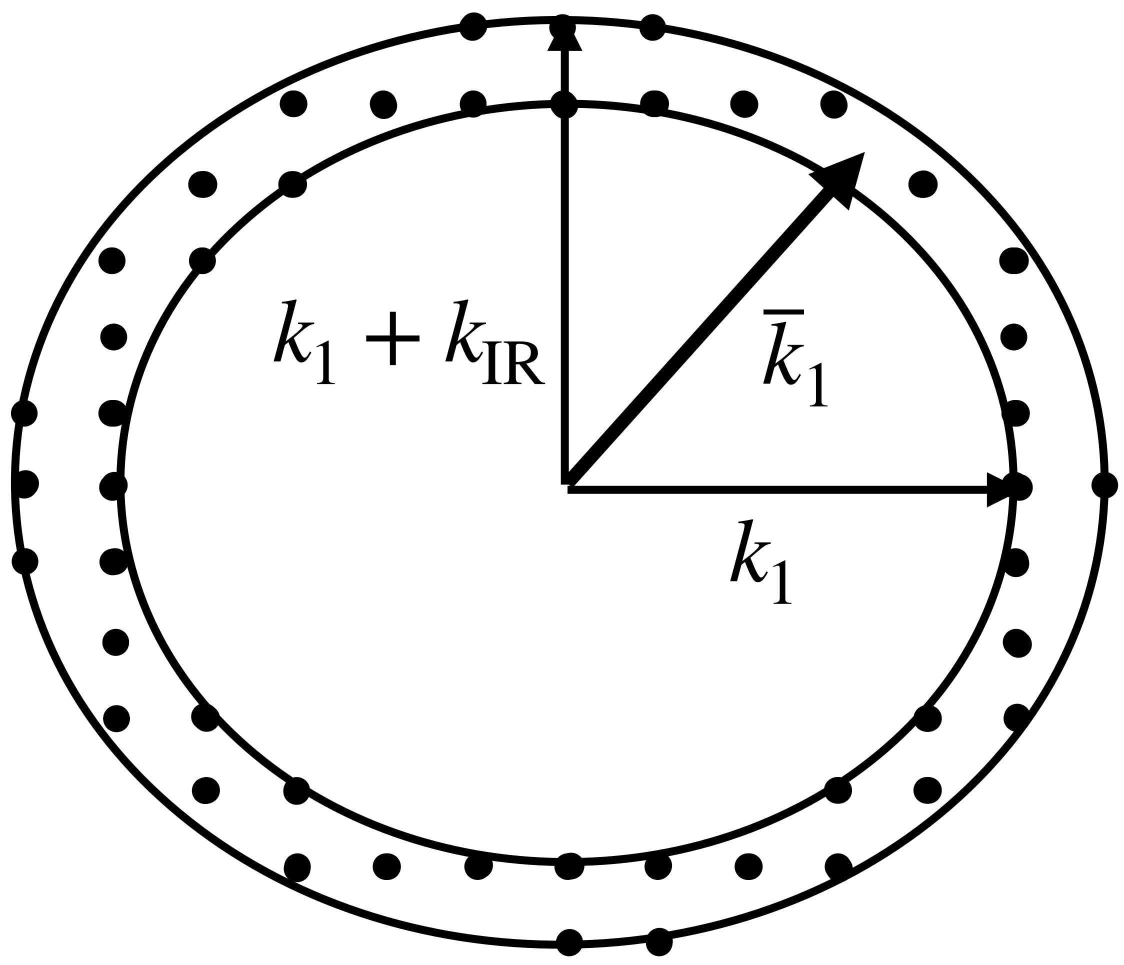
$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

5. Discrete power spectrum



$$\Delta_f^{(\rm L)}(|\tilde{\mathbf{n}}|) \equiv \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

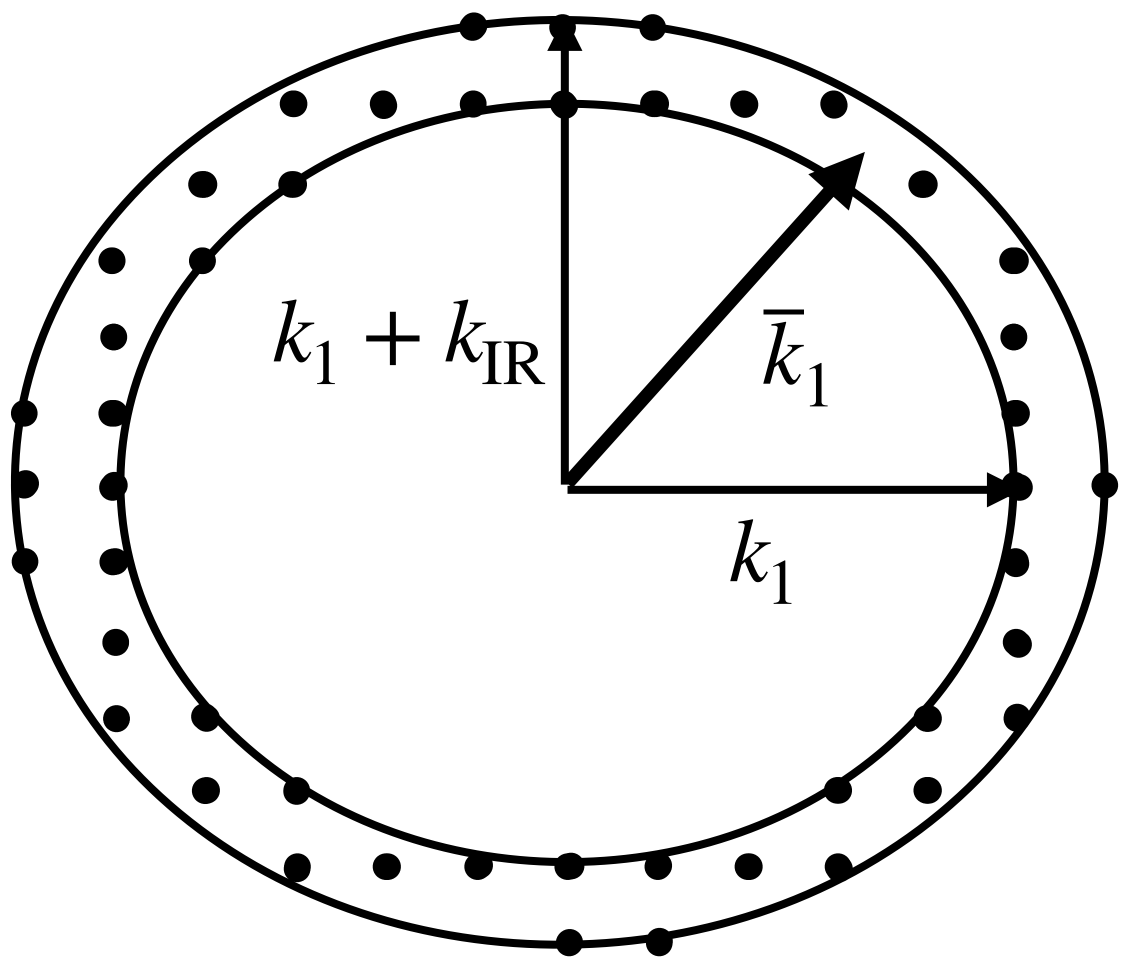
5. Discrete power spectrum



$$\Delta_f^{(\rm L)}(|\tilde{\mathbf{n}}|) \equiv \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2\right\rangle_{R(\tilde{\mathbf{n}})}$$

Based on: $\langle(\cdots)\rangle_{R(\tilde{\mathbf{n}})} \equiv \frac{1}{4\pi|\tilde{\mathbf{n}}|^2}\sum_{\tilde{\mathbf{n}}'\in R(\tilde{\mathbf{n}})}(\cdots)$

Multiplicity $\simeq 4\pi|\tilde{\mathbf{n}}|^2$



$$\Delta_f^{(\rm L)}(|\tilde{\mathbf{n}}|) \equiv \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2\right\rangle_{R(\tilde{\mathbf{n}})}$$

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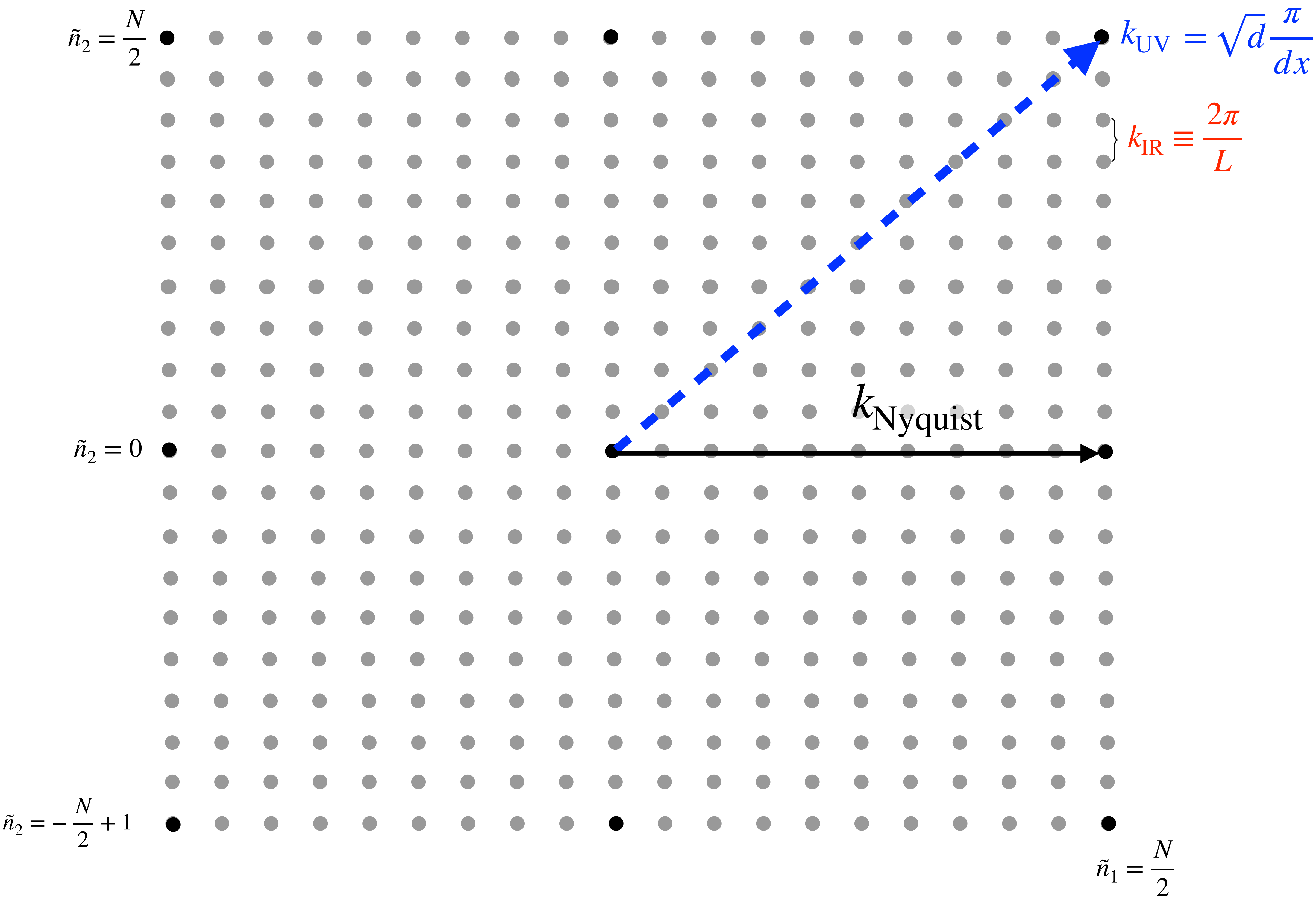
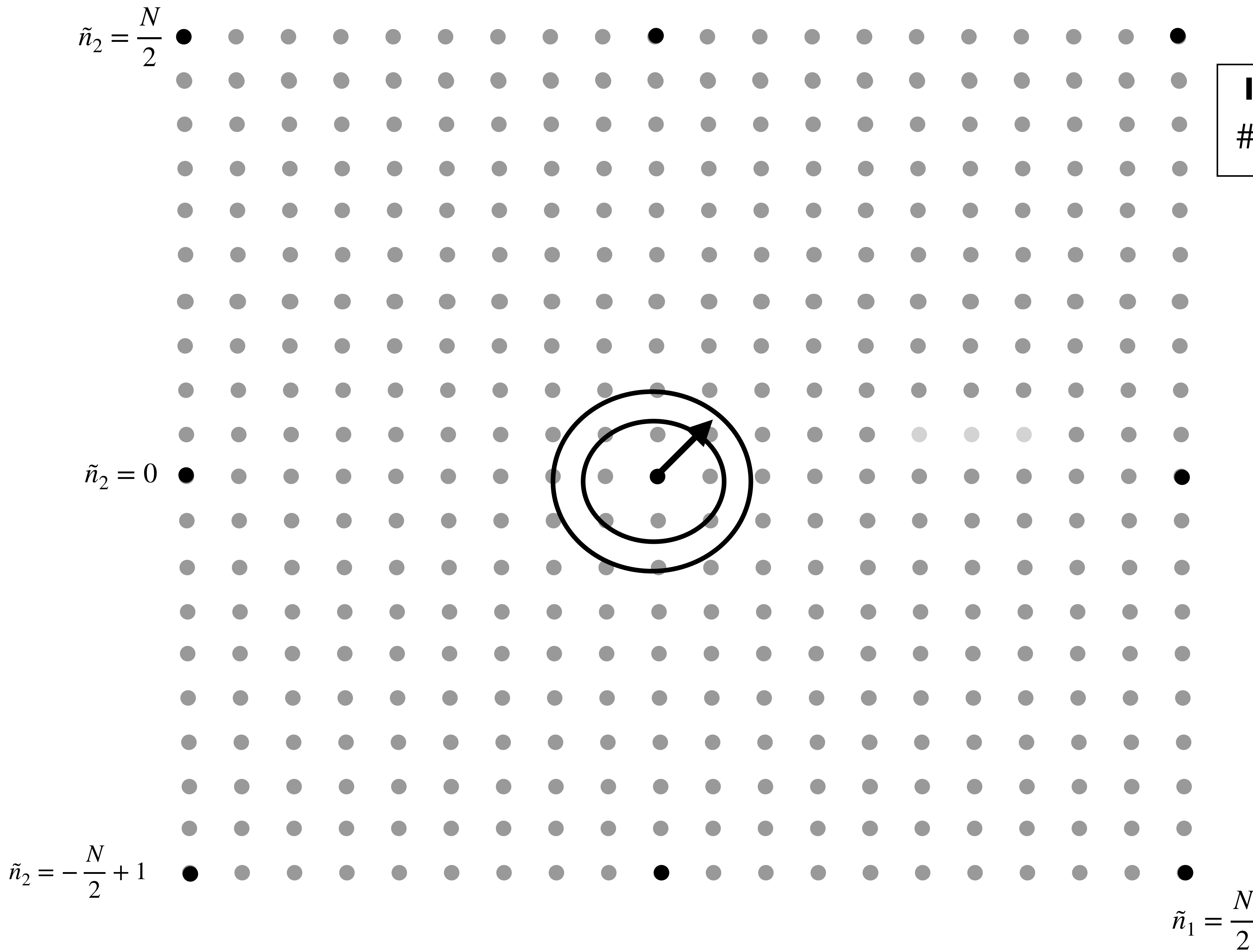
Is multiplicity $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi|\tilde{\mathbf{n}}|^2$ **?**

5. Discrete power spectrum



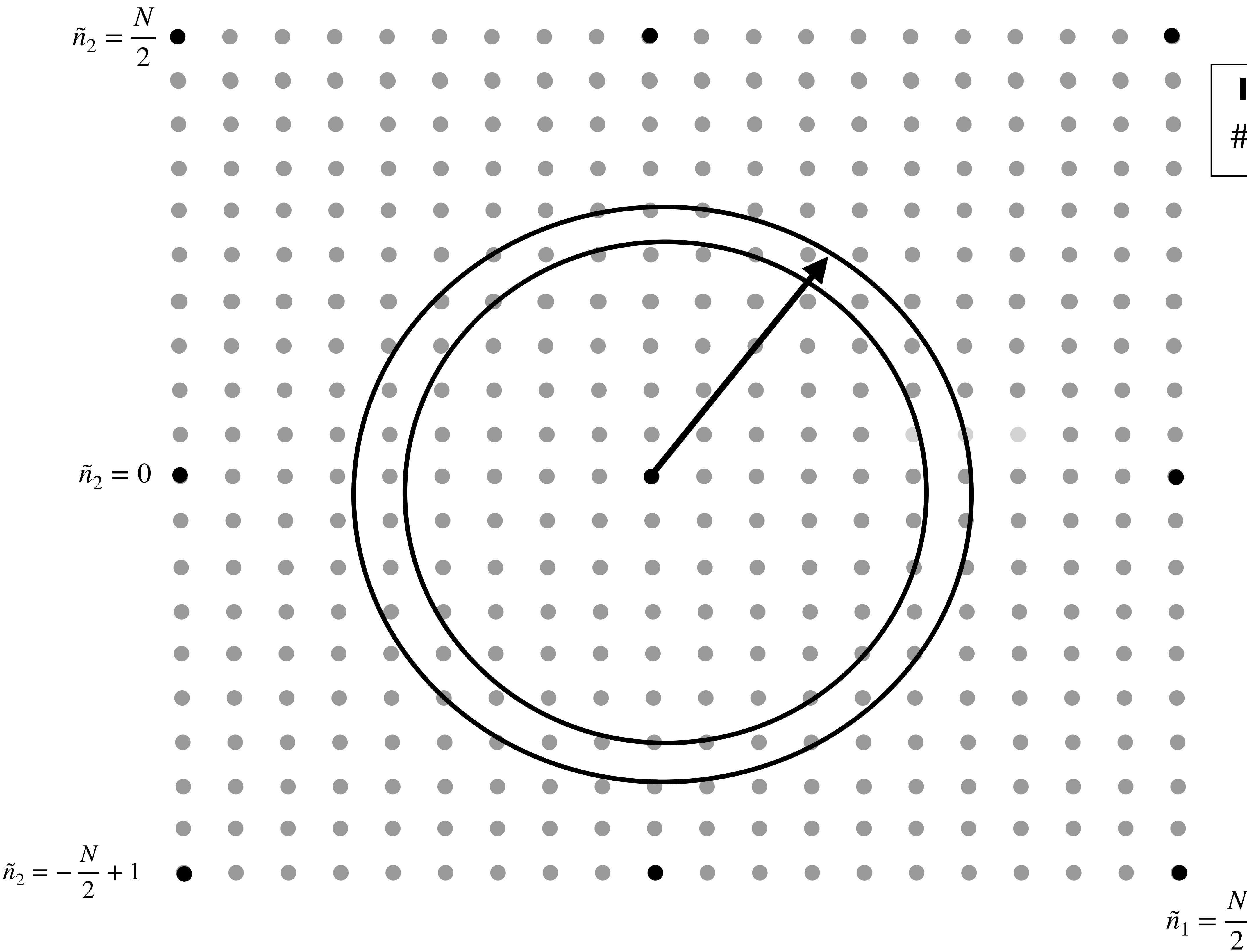
5. Discrete power spectrum



Is multiplicity $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi |\tilde{\mathbf{n}}|^2$?

~ works Ok-ish

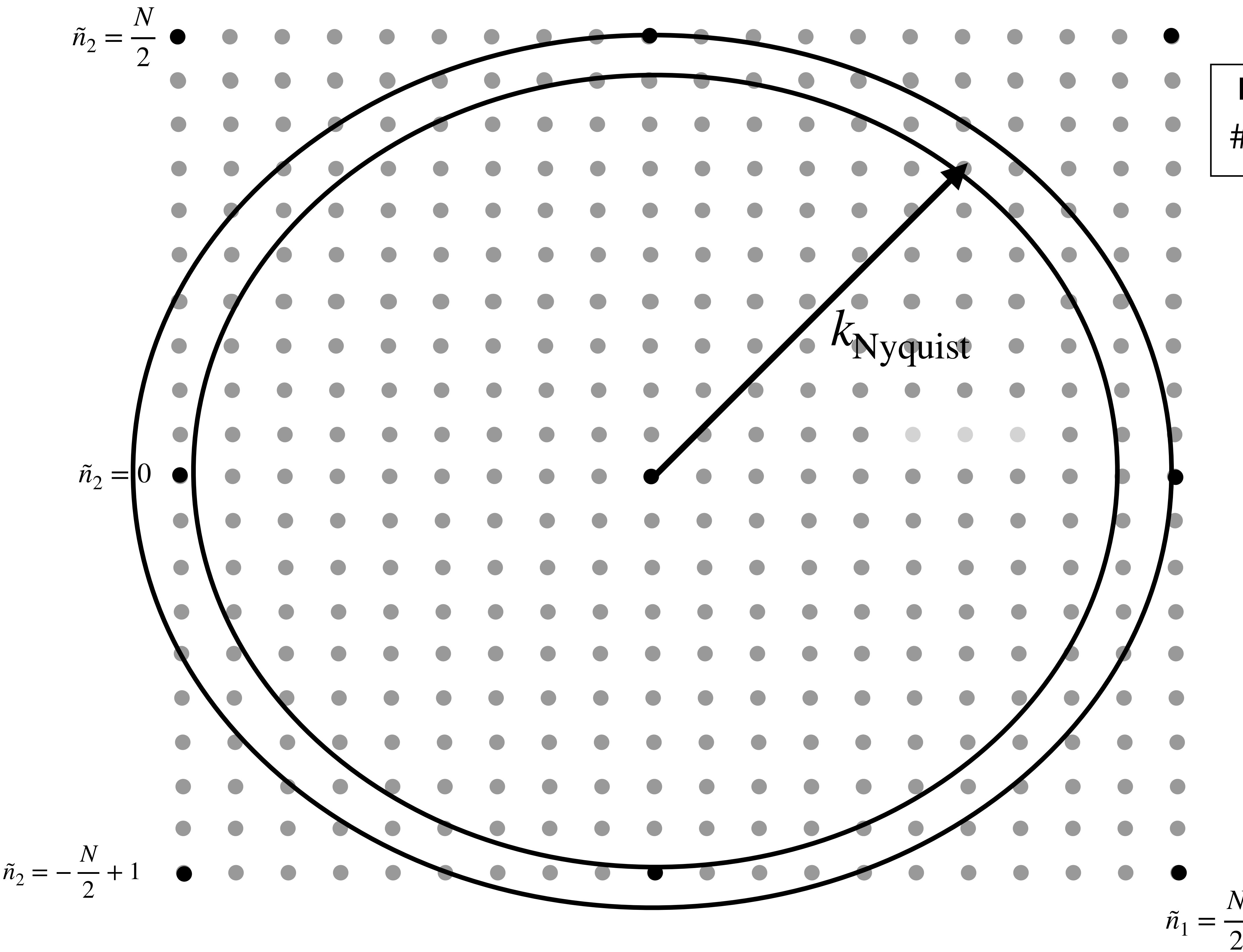
5. Discrete power spectrum



Is multiplicity $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi |\tilde{\mathbf{n}}|^2$**?**

~ Very good

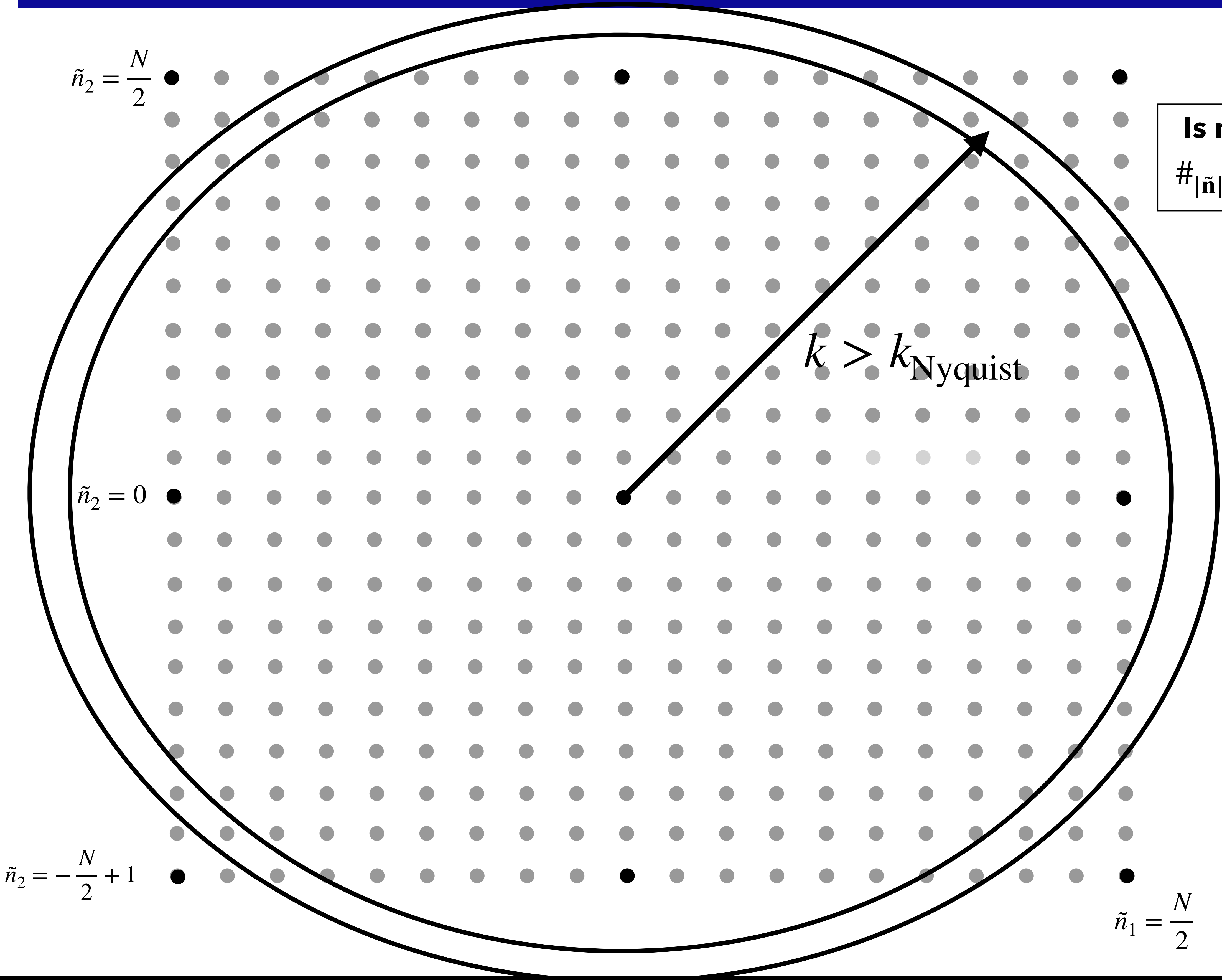
5. Discrete power spectrum



Is multiplicity $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi |\tilde{\mathbf{n}}|^2$?

Excellent!

5. Discrete power spectrum



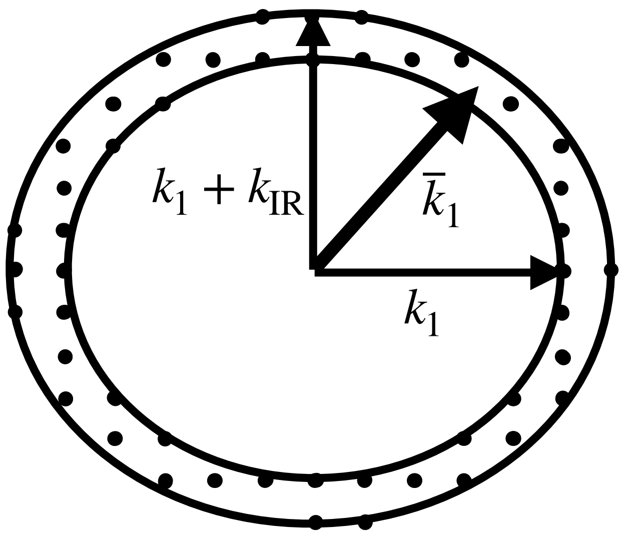
Is multiplicity $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi |\tilde{\mathbf{n}}|^2$?

~ works
Badly

5. Discrete power spectrum

Assuming multiplicity $4\pi|\tilde{\mathbf{n}}|^2$ we found:

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathtt{f}(\tilde{\mathbf{n}})\right|^2\right\rangle_{R(\tilde{\mathbf{n}})} \quad \left(\begin{array}{c}\text{Multiplicity}\\ \simeq 4\pi|\tilde{\mathbf{n}}|^2\end{array}\right)$$



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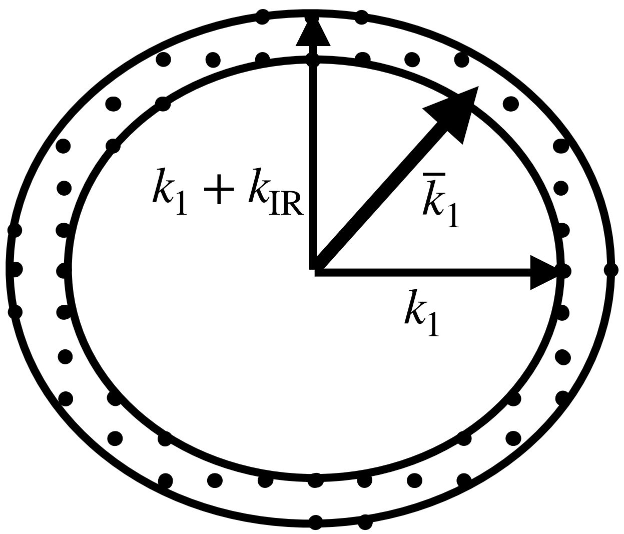
Using real multiplicity we find (similar computation as before):

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2\right\rangle_{R(\tilde{\mathbf{n}})} \quad \left(\begin{array}{c}\#_{|\tilde{\mathbf{n}}|}\ \text{Real}\\ \text{Multiplicity}\end{array}\right)$$



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$1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$

(sub-Nyquist freq.'s)

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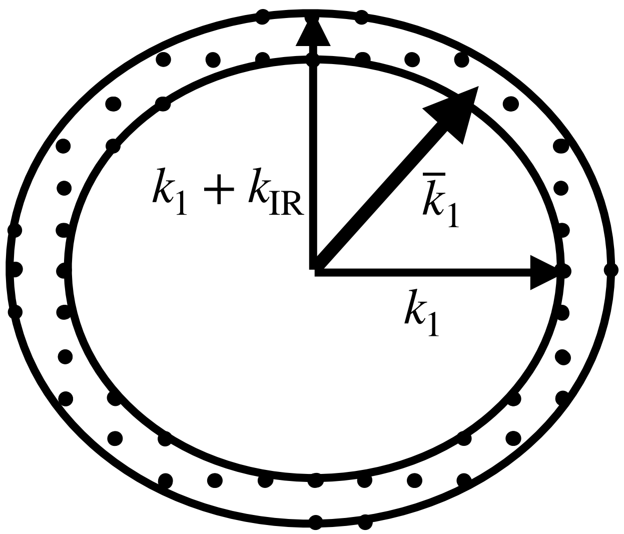

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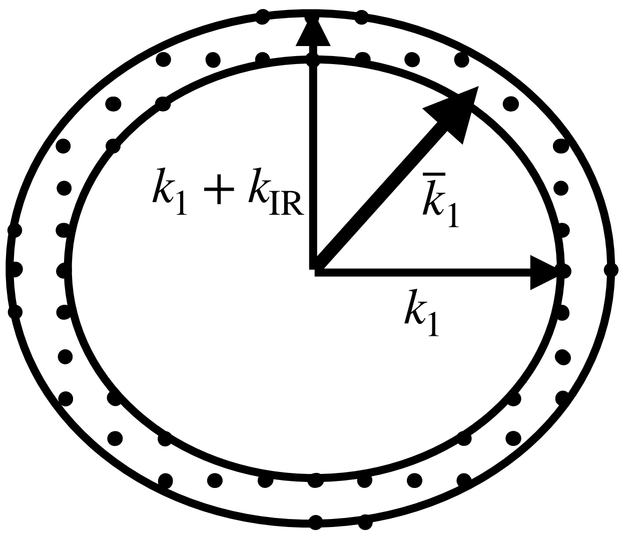

See Technical Note I, on the definition of Power Spectrum
https://cosmolattice.net/technicalnotes/

5. Discrete power spectrum

CosmoLattice → Choose: ***Type*** **+** ***Version*** **+** ***Verbosity***

5. Discrete power spectrum

CosmoLattice → Choose: ***Type*** **+** ***Version*** **+** ***Verbosity***

Type-I Lattice PS:

(Correct for all lattice sites)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2\right\rangle_{R(\tilde{\mathbf{n}})}$$

($\#_{|\tilde{\mathbf{n}}|}$ Real Multiplicity)

5. Discrete power spectrum

CosmoLattice → Choose: ***Type*** + ***Version*** + ***Verbosity***

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(Correct for all lattice sites)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

($\#_{|\tilde{\mathbf{n}}|}$ Real Multiplicity)

Type-II Lattice PS:

(Correct for Sub-Nyquist modes only: $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left| \mathtt{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

(Multiplicity: $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi|\tilde{\mathbf{n}}|^2$)

5. Discrete power spectrum

CosmoLattice → Choose: ***Type*** + ***Version*** + ***Verbosity***

Type-I Lattice PS:

(Correct for all lattice sites)

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($\#_{|\tilde{\mathbf{n}}|}$ Real Multiplicity)

Type-II Lattice PS:

(Correct for Sub-Nyquist modes only: $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathfrak{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

(Multiplicity: $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi|\tilde{\mathbf{n}}|^2$)

Version (for e.g. Type I)

CosmoLattice → **Choose: *Type* + *Version* + *Verbosity***

Type-I Lattice PS:

(Correct for all lattice sites)

$$\Delta_f^{(L)}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|dx}{2\pi N^5} \#_{|\tilde{\mathbf{n}}|} \left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

($\#_{|\tilde{\mathbf{n}}|}$ Real Multiplicity)

Type-II Lattice PS:

(Correct for Sub-Nyquist modes only: $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

$$\Delta_f^{(L)}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2} \frac{L^3}{N^6} \left\langle \left| \mathfrak{f}(\tilde{\mathbf{n}}) \right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

(Multiplicity: $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi |\tilde{\mathbf{n}}|^2$)

Version (for e.g. Type I)

1: $$\Delta_f^{(\mathrm{I})}(l) = \frac{k(l)\delta x}{2\pi N^5} \#_l \left\langle \left| f(\tilde{\mathbf{n}}') \right|^2 \right\rangle_{R(l)}$$

$$k(l) = \frac{1}{2}(k_{\max}^{(l)} + k_{\min}^{(l)}(l))$$

(Mean bin momentum)

CosmoLattice → Choose: *Type* + *Version* + *Verbosity*

Type-I Lattice PS:

(Correct for all lattice sites)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left| f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

($\#_{|\tilde{\mathbf{n}}|}$ Real Multiplicity)

Type-II Lattice PS:

(Correct for Sub-Nyquist modes only: $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left| \mathtt{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

(Multiplicity: $\#_{|\tilde{\mathbf{n}}|} \simeq 4\pi|\tilde{\mathbf{n}}|^2$)

Version (for e.g. Type I)

1: $$\Delta_f^{(\mathrm{I})}(l) = \frac{k(l)\delta x}{2\pi N^5}\#_l\left\langle \left| f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$$

$$k(l) = \frac{1}{2}(k_{\max}^{(l)} + k_{\min}^{(l)}(l))$$

(Mean bin momentum)

2: $$\Delta_f^{(\mathrm{I})}(l) = \frac{\langle k(\tilde{\mathbf{n}}')\rangle_l \delta x}{2\pi N^5}\#_l\left\langle \left| f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$$

$$\langle k(\tilde{\mathbf{n}}')\rangle_l \equiv \frac{k_{\mathrm{IR}}}{\#_l}\sum_{\tilde{\mathbf{n}}'\in R(l)} |\tilde{\mathbf{n}}'|$$

(Weighted bin momentum)

CosmoLattice → Choose: *Type* + *Version* + *Verbosity*

Type-I Lattice PS:
(Correct for all lattice sites)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \equiv \frac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

($\#_{|\tilde{\mathbf{n}}|}$ Real Multiplicity)

Type-II Lattice PS:
(Correct for Sub-Nyquist modes only: $1 \lesssim |\tilde{\mathbf{n}}| \leq N/2$)

$$\Delta_f^{(\mathrm{L})}(|\tilde{\mathbf{n}}|) \simeq \frac{k^3(\tilde{\mathbf{n}})}{2\pi^2}\frac{L^3}{N^6}\left\langle \left|\mathfrak{f}(\tilde{\mathbf{n}})\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$$

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3: $$\Delta_f^{(\mathrm{I})}(l) = \frac{\delta x\#_l}{2\pi N^5}\left\langle k(\tilde{\mathbf{n}}')\left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(l)}$$

CosmoLattice → Choose: *Type* + *Version* + *Verbosity*

Type-I Lattice PS: $\Delta_f^{(L)}(|\tilde{\mathbf{n}}|) \equiv \dfrac{|\mathbf{k}(\tilde{\mathbf{n}})|\,dx}{2\pi N^5}\,\#_{|\tilde{\mathbf{n}}|}\left\langle \left|f(\tilde{\mathbf{n}}')\right|^2 \right\rangle_{R(\tilde{\mathbf{n}})}$ ($\#_{|\tilde{\mathbf{n}}|}$ Real Multiplicity)

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Verbosity (output)

0: $\{k(l),\ \Delta_f^{(X)}(l)\}$

1: $\{\langle k(\tilde{\mathbf{n}}')\rangle_l,\ \Delta_f^{(X)}(l)\}$

2: $\{k(l),\ \langle k(\tilde{\mathbf{n}}')\rangle_l,\ rms(k(\tilde{\mathbf{n}}')),\ k_{\rm min}^{(l)},\ k_{\rm max}^{(l)}$
$\Delta_f^{(X)}, \langle\Delta_f^{(X)}\rangle, rms(\Delta_f^{(X)}), \Delta_{\rm min}^{(X)}, \Delta_{\rm max}^{(X)}\}$

Summary

* Lattice Definition
* Fourier Transform
* Fourier Lattice (Reciprocal)
* Lattice Derivatives
* Lattice Momenta
* Power Spectrum

THANK YOU!

How to choose the time step?

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Set by
physics problem