Time-Like Correlations in Quantum Theory

Francesco Buscemi, Nagoya University

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References

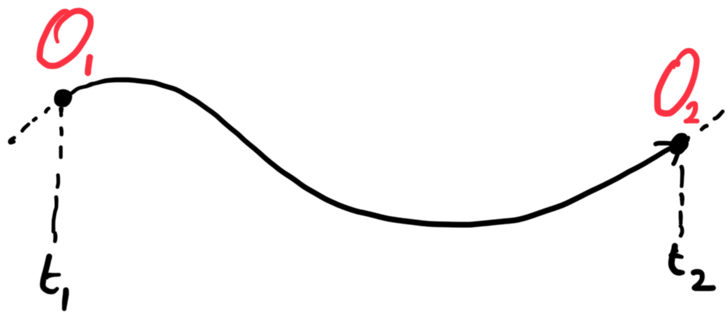
Presentation based on:

- *Direct observation of any two-point quantum correlation function*. Preprint arXiv:1312.4240
- *Universal optimal quantum correlator*. International Journal of Quantum Information 12, 1560002 (2014)
- *Virtual quantum broadcasting*. Physical Review Letters 132, 110203 (2024)

Collaborators on this (long) journey: G. Chiribella, M. Dall'Arno, J. Fullwood, and A. J. Parzygnat

The problem with correlations in time

Consider a single system stochastically evolving from state x at time t_1 to state y at time $t_2 > t_1$, and two observables, $O_1 = O_1(x)$ and $O_2 = O_2(y)$, one for each time.



classical case: $\langle O_1 O_2 \rangle_{t_1 \to t_2} = \sum_{x,y} O_1(x) O_2(y) \, \mathrm{Pr}\{x, t_1; y, t_2\}$

quantum case: observation is intervention and time-correlations depend on how the first observation is performed

Is there a canonical form for quantum time-correlations?

How you do it classically

- fix orthonormal bases $\{|x\rangle\}_x$ and $\{|y\rangle\}_y$
- choose an initial state $\mathbf{p} = \sum_x p(x)|x\rangle\langle x|$
- apply the ideal *classical* correlator $\mathcal{C}(\mathbf{p}) = \sum_x p(x)|x\rangle\langle x|_1 \otimes |x\rangle\langle x|_2$
- let system 2 evolve:

$$\sum_{x,y} \underbrace{p(x)N(y|x)}_{\Pr\{x,t_1;y,t_2\}} |x\rangle\langle x|_1 \otimes |y\rangle\langle y|_2$$

- measure (compute) O_1 on system 1 and O_2 on system 2:

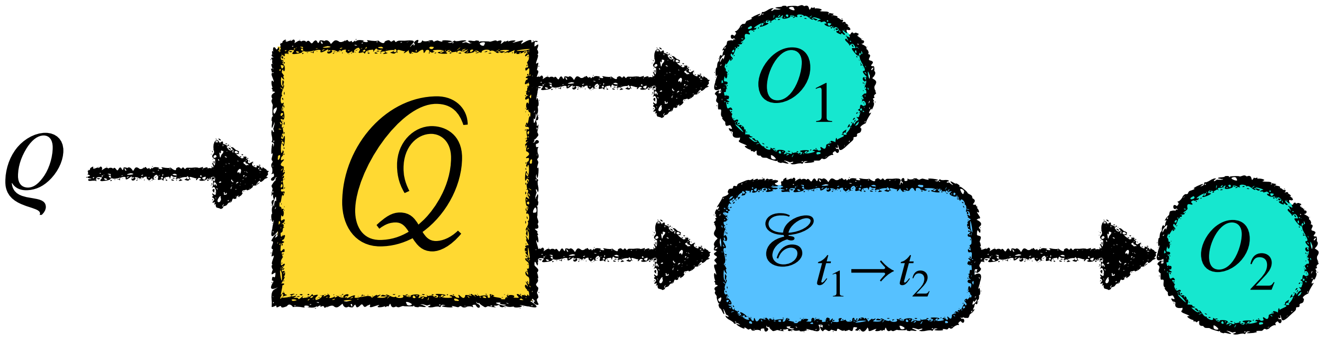
$$\begin{aligned}\langle O_1 O_2\rangle_{t_1\to t_2} &= \sum_{x,y} p(x)N(y|x)O_1(x)O_2(y)\\ &= \sum_x p(x)O_1(x) \underbrace{\sum_y N(y|x)O_2(y)}_{\tilde{O}_2(x)}\end{aligned}$$

The crucial ingredient is the ideal correlator.

What is its quantum analog?

Operational picture

If we had an "ideal quantum correlator" $\mathcal{Q}$, we could do the same:



Properties of $\mathcal{Q}$

Requirements for the ideal *quantum* correlator:

- linearity: $\mathcal{Q}(c_1X_1 + c_2X_2) = c_1\mathcal{Q}(X_1) + c_2\mathcal{Q}(X_2)$
- consistency with the classical correlator: $(\Delta \otimes \Delta)\ \mathcal{Q}(\Delta(\cdot)) = \mathcal{C}(\Delta(\cdot))$, where $\Delta(\cdot) = \sum_x |x\rangle\langle x| \cdot |x\rangle\langle x|$
- symmetry: $\mathtt{SWAP}\ \mathcal{Q}(\cdot)\ \mathtt{SWAP} = \mathcal{Q}(\cdot)$
- universal covariance: for any unitary U, $\mathcal{Q}(U \cdot U^\dagger) = (U \otimes U)\mathcal{Q}(\cdot)(U^\dagger \otimes U^\dagger)$

Remark. Universal covariance guarantees that classical consistency is basis-independent.

Uniqueness result

The ideal quantum correlator (PRL, 2024)

The four requirements of linearity, classical consistency, symmetry, and universal covariance single out a unique map given by

$$\mathcal{Q}(\varrho) := \frac{\mathtt{SWAP}(\varrho \otimes \mathbb{1}) + (\varrho \otimes \mathbb{1})\mathtt{SWAP}}{2} ,$$

where $\mathtt{SWAP}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle$ for all vectors $|\psi\rangle$ and $|\phi\rangle$.

For example, if $\varrho = |\psi\rangle\langle\psi|$, then

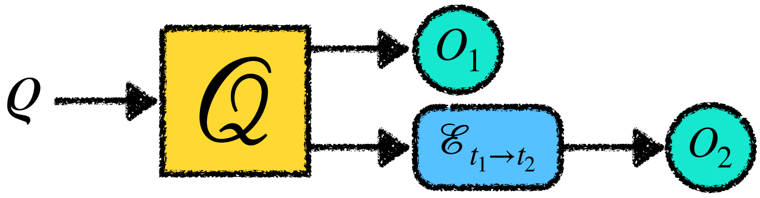
$$\mathcal{Q}(\varrho) = \frac{1}{2}\sum_x \Big(|x\rangle\langle\psi| \otimes |\psi\rangle\langle x| + \text{cc} \Big) .$$

Consequences

As a *result*, $\mathcal{Q}$ satisfies:

- hermite-preservation (HP): $X = X^\dagger \implies \mathcal{Q}(X) = [\mathcal{Q}(X)]^\dagger$
- trace-preservation (TP): $\mathrm{Tr}[\mathcal{Q}(X)] = \mathrm{Tr}[X]$
- ideal broadcasting: $\mathrm{Tr}_1[\mathcal{Q}(\varrho)] = \mathrm{Tr}_2[\mathcal{Q}(\varrho)] = \varrho$, for all ϱ

Use of $\mathcal{Q}$ to generate quantum time-correlations

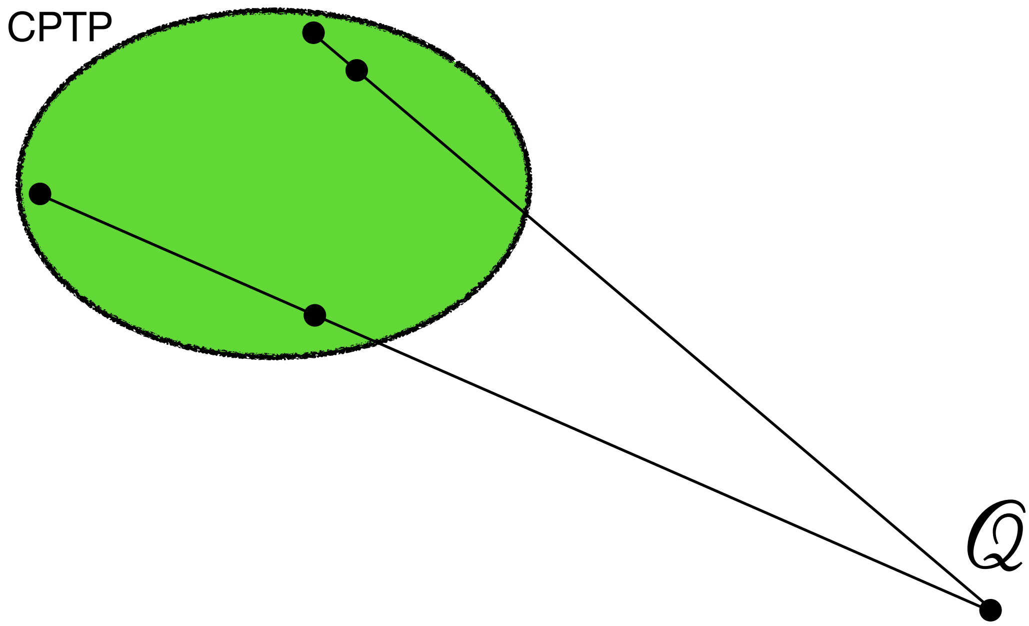


$$\langle O_1 O_2 \rangle^{Q}_{t_1 \to t_2} := \mathrm{Tr}[(\mathsf{id} \otimes \mathcal{E}_{t_1 \to t_2})\mathcal{Q}(\varrho)\ (O_1 \otimes O_2)] = \mathrm{Tr}\left[\varrho\ \frac{O_1 \overbrace{\mathcal{E}^\dagger(O_2)}^{\tilde{O}_2} + \mathcal{E}^\dagger(O_2) O_1}{2}\right]$$

The above corresponds to the symmetric part of the Kirkwood–Dirac quasi-probability distribution, which thus gets a **new operational interpretation** as the unique "canonical" analog of classical broadcasting.

However, $\mathcal{Q}$ is not a valid quantum channel (it is not positive)

Optimal physical approximation



Optimal physical approximation

arXiv:1312.4240; PRL (2024)

The ideal quantum correlator satisfies

$$\arg \min_{\mathcal{E}:\text{CPTP}} \|\mathcal{Q} - \mathcal{E}\|_\diamond = \{\mathcal{S}\}$$

where $\mathcal{S}$ is the Bužek–Hillery–Werner symmetric optimal one-to-two universal quantum cloning map given by

$$\mathcal{S}(\cdot) := \frac{2}{d+1}\Pi_{\text{symm}}(\varrho \otimes \mathbb{1})\Pi_{\text{symm}} \ ,$$

where $\Pi_{\text{symm}} := \frac{1}{2}(\mathbb{1} + \mathtt{SWAP})$ is the projector on the symmetric subspace of $\mathbb{C}^d \otimes \mathbb{C}^d$.

Remark. A connection between Kirkwood–Dirac distribution and cloning is also noted in [Hofmann, PRL (2012)], but without any discussion about optimality or uniqueness.

Optimal simulation protocol

arXiv:1312.4240

The unique optimal* statistical decomposition for the ideal quantum correlator is given by

$$\mathcal{Q}(\cdot) = \frac{d+1}{2}\mathcal{S}(\cdot) - \frac{d-1}{2}\mathcal{A}(\cdot) \ ,$$

where

$$\mathcal{S}(\cdot) := \frac{2}{d+1}\Pi_{\mathsf{symm}}(\varrho \otimes \mathbb{1})\Pi_{\mathsf{symm}} \qquad \mathcal{A}(\cdot) := \frac{2}{d-1}\Pi_{\mathsf{anti}}(\varrho \otimes \mathbb{1})\Pi_{\mathsf{anti}} \ ,$$

where $\Pi_{\mathsf{symm}} := \frac{1}{2}(\mathbb{1} + \mathtt{SWAP})$ and $\Pi_{\mathsf{anti}} := \frac{1}{2}(\mathbb{1} - \mathtt{SWAP})$ are, respectively, the projector on the symmetric and on the antisymmetric subspaces of $\mathbb{C}^d \otimes \mathbb{C}^d$.

* optimal because $\|\mathcal{Q}\|_\diamond = \left\|\frac{d+1}{2}\mathcal{S} - \frac{d-1}{2}\mathcal{A}\right\|_\diamond = \frac{d+1}{2}\|\mathcal{S}\|_\diamond + \frac{d-1}{2}\|\mathcal{A}\|_\diamond = d.$

Interferometric scheme (arXiv:1312.4240)

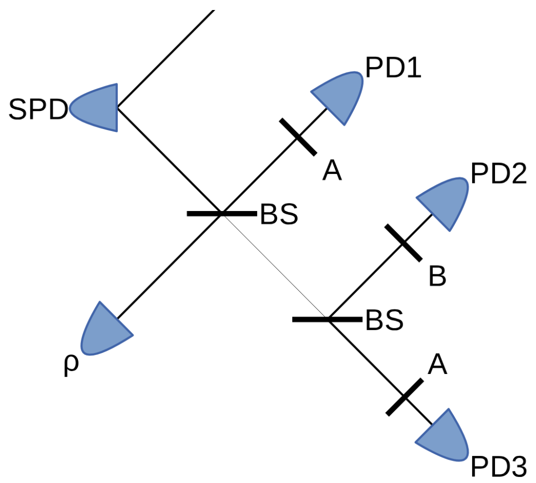


FIG. 4. Experimental proposal for the probabilistic implementation of the real part of the universal optimal two-point correlator for qubits. Thin lines represent optical qubits encoded in the polarization of photons, BS is a 50/50 beamsplitter, SPD is a source of maximally entangled photons through spontaneous parametric downconversion, ρ is the input state fed into the universal optimal two-point correlator, A and B are the phase shifters implementing the corresponding observables, PD1, PD2, and PD3 are photodetectors. A coincidence occurs between PD2 and PD3 (resp., PD1 and PD2) with probability 3/16 (resp., 1/16), in this case optimal universal symmetric (resp., antisymmetric) cloning has been per-

Conclusion

Take-home messages

- there is a unique, canonical, ideal quantum correlator
- no *ad hockeries* or arguments from authority, but only four natural requirements (linearity, classical consistency, symmetry, unitary covariance)
- the ideal quantum correlator generates a canonical representation of time correlations (a canonical state-over-time)
- the ideal quantum correlator is not positive: another manifestation of the negative signature of time?
- the optimal universal quantum cloning is the unique optimal physical approximation

Time to venture into new territories of quantum theory!