Bayesian Hierarchical Model for cross calibration of hybrid air shower detectors

Anatoli Fedynitch, Anton Prosekin, Kozo Fujisue
Academia Sinica, Taipei, Taiwan

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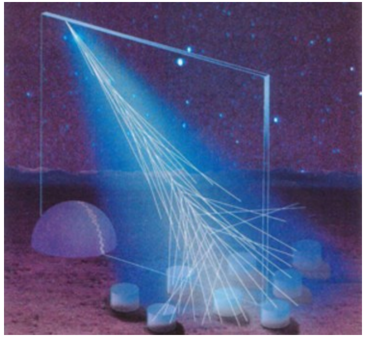
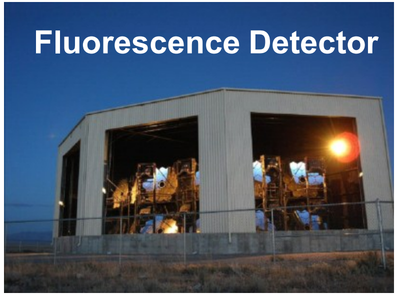
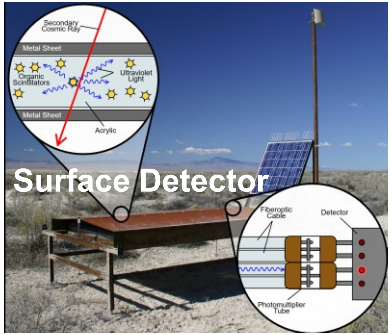
Hybrid air shower detectors







Surface Detectors (SD):

- Operate continuously (~100% duty cycle)
- Large event statistics but higher uncertainty in energy estimation

Fluorescence Detectors (FD):

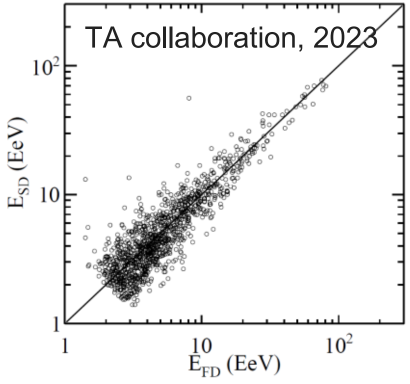
- Operate only on clear, moonless nights (~10–15% duty cycle)
- Provide nearly calorimetric energy measurement with smaller systematic uncertainty

Hybrid Detection:

- Events observed simultaneously by SD and FD
- Combine precise FD energy with SD’s high-statistics sampling
- Used to calibrate the SD energy scale

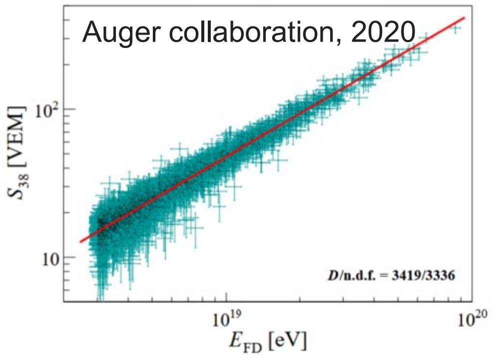
SD–FD cross-calibration approaches

Least square fit



$$E_{FD} = E_{SD}/1.27$$

Loglikelihood minimization

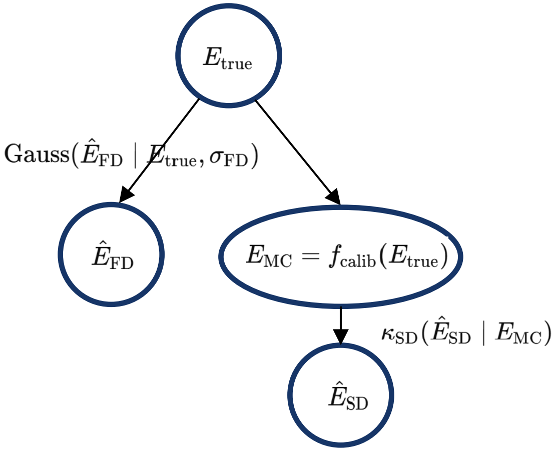


$$E_{FD} = A\ {S_{38}}^{B}$$

$$A = (1.86 \pm 0.03) \times 10^{17}\ \text{eV}$$

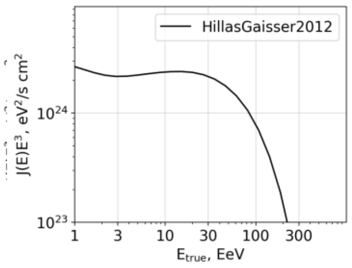
$$B = 1.031 \pm 0.004$$

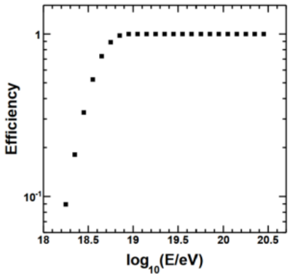
Hierarchical MCMC

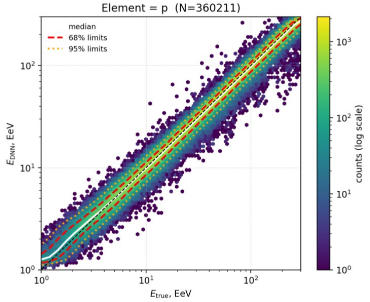


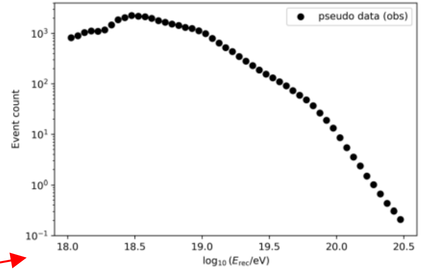
Motivation: Forward folding

CR spectrum × Efficiency × Response matrix/function = Event counts distribution







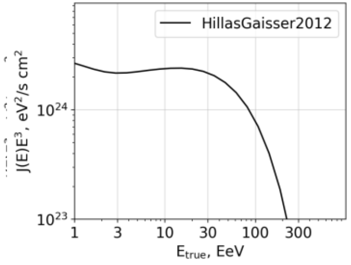


$$J^{\mathrm{raw}}(E_{\mathrm{SD}};\mathbf{s}) = \frac{\int \mathrm{d}\Omega \cos\theta \int \mathrm{d}E \epsilon(E,\theta) J(E;\mathbf{s}) \kappa(E_{\mathrm{SD}}|E;\theta)}{\int \mathrm{d}\Omega \cos\theta}$$

- Forward folding maps model energy spectrum through detector response to compare with observed measurements

Forward folding: log-likelihood minimization

CR spectrum × Efficiency × Response matrix/function = Event counts distribution



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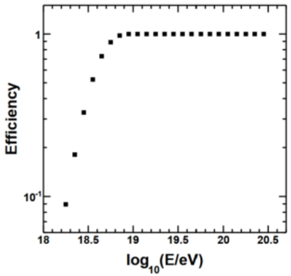

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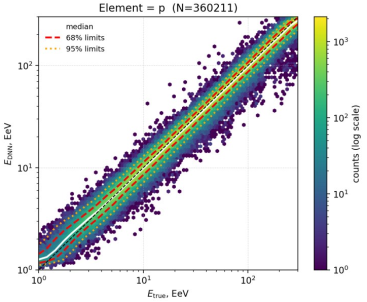

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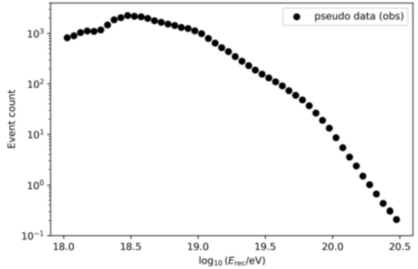


$$J^{\rm raw}(E_{\rm SD};\mathbf{s}) = \frac{\int d\Omega \cos\theta \int dE\, \epsilon(E,\theta) J(E;\mathbf{s}) \kappa(E_{\rm SD}|E;\theta)}{\int d\Omega \cos\theta}$$

binning ⇒

$$R_{ij} = \frac{\int_{\Delta E_i} dE_{\rm SD} \int_{\Delta E_j} dE \int d\Omega \cos\theta\, \kappa(E_{\rm SD}|E,\theta)\epsilon(E,\theta)J(E;\mathbf{s})}{\int_{\Delta E_j} dE \int d\Omega \cos\theta\ J(E;\mathbf{s})}$$

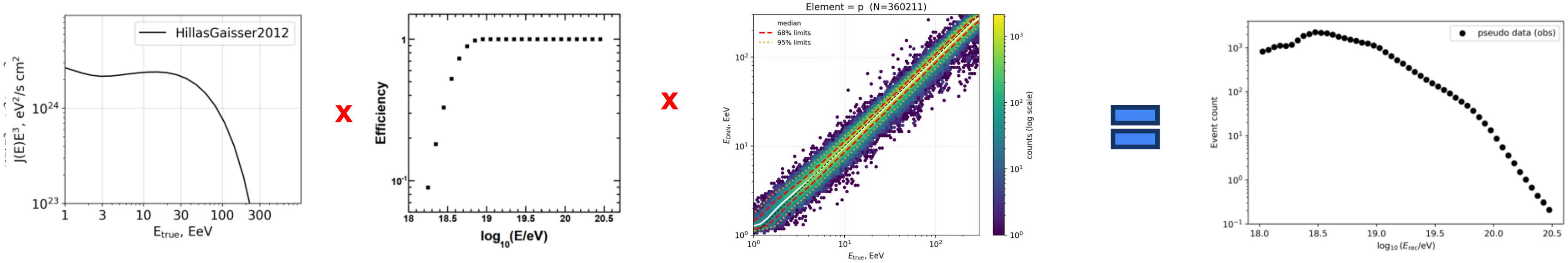
$$\nu_i = \sum_j R_{ij}\mu_j$$

Log Likelihood minimization:

$$-\ln \mathcal{L}(\mathbf{s}) = \sum_i \left(\nu_i(\mathbf{s}) - N_i \ln \nu_i(\mathbf{s})\right)$$

Forward folding: MCMC

CR spectrum x Efficiency x Response matrix/function = Event counts distribution



$$J^{\rm raw}(E_{\rm SD};\mathbf{s}) = \frac{\int {\rm d}\Omega \cos\theta \int {\rm d}E \epsilon(E,\theta) J(E;\mathbf{s}) \kappa(E_{\rm SD}|E;\theta)}{\int {\rm d}\Omega \cos\theta}$$

binning ⇒

$$R_{ij} = \frac{\int_{\Delta E_i} {\rm d}E_{\rm SD} \int_{\Delta E_j} {\rm d}E \int {\rm d}\Omega \cos\theta \kappa(E_{\rm SD}|E,\theta)\epsilon(E,\theta)J(E;\mathbf{s})}{\int_{\Delta E_j} {\rm d}E \int {\rm d}\Omega \cos\theta \; J(E;\mathbf{s})}$$

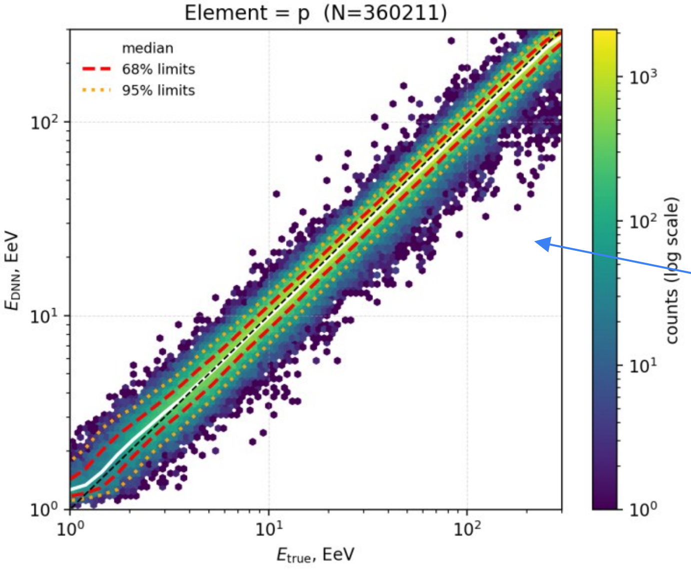
$$\nu_i = \sum_j R_{ij}\mu_j$$

Log posterior sampling using MCMC:

$$p(\theta \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \theta)\, p(\theta)}{p(\mathcal{D})}$$

$$\log p(\theta \mid D) \propto \sum_{i=1}^{N_{E,\rm rec}} \left[N_i \log \nu_i(\theta) - \nu_i(\theta) - \log(N_i!) \right] + \log p(\theta)$$

Response matrix



$$\nu_i = \int dE\, f(E) \int du\, \varepsilon(E,u) \int_{\hat{E}\in \text{bin } i} \left[\int ds\, p_{\text{det}}(\hat{E} \mid s, E, u)\, p_s(s \mid E, u) \right] d\hat{E}$$

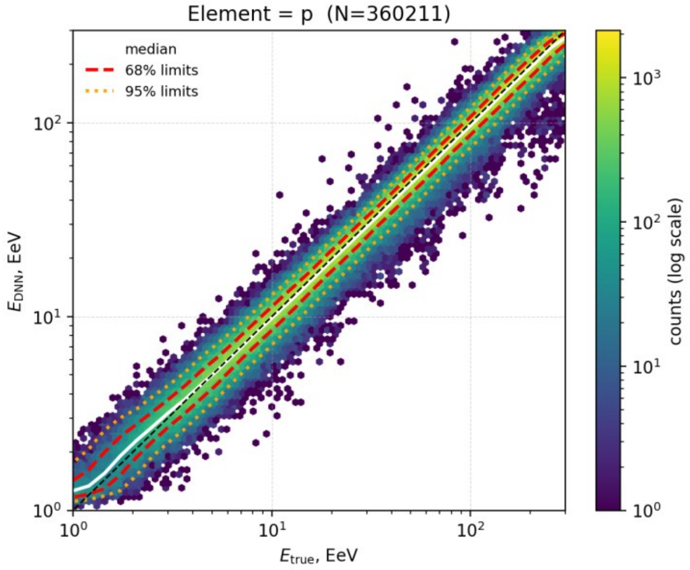
Resolution/smearing kernel:

$$\kappa(\hat{E} \mid E, u) = \int ds\, p_{\text{det}}(\hat{E} \mid s, E, u)\, p_s(s \mid E, u)$$

$$\nu_i = \int dE\, f(E) \int du\, \varepsilon(E,u) \int_{\hat{E}\in \text{bin } i} \kappa(\hat{E} \mid E, u)\, d\hat{E}$$

Response matrix

Simulated resolution kernel

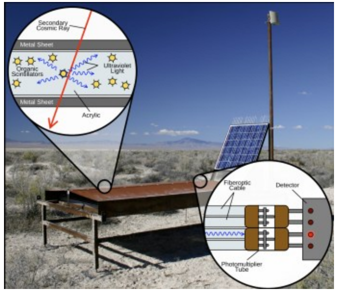


Real resolution/smearing kernel:

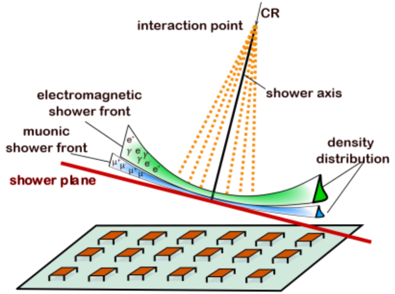
$$\kappa(\hat{E} \mid E, u) = \int ds\, p_{\mathrm{det}}(\hat{E} \mid s, E, u)\, p_s(s \mid E, u)$$

Detector fluctuations/smearing:

- sampling/LDF
- Scintillator/PMT

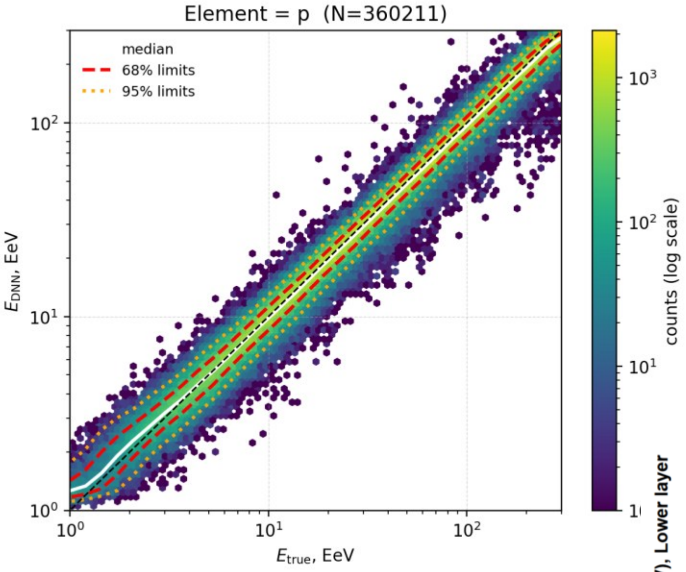


Shower size (s) fluctuations



Response matrix

Simulated resolution kernel



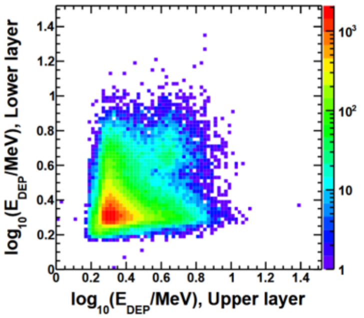
Simulated resolution/smearing kernel:

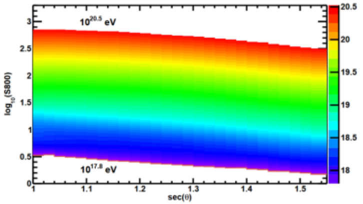
$$\kappa(\hat{E} \mid E, u) = \int ds\, p_{\text{det}}(\hat{E} \mid s, E, u)\, p_s(s \mid E, u)$$

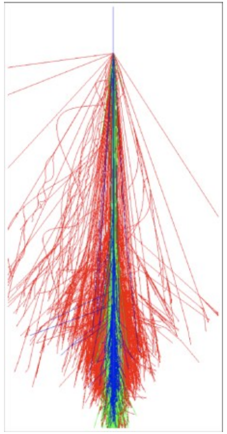
Detector fluctuations/smearing:

- Geant4
- Reconstruction (Std or DNN)

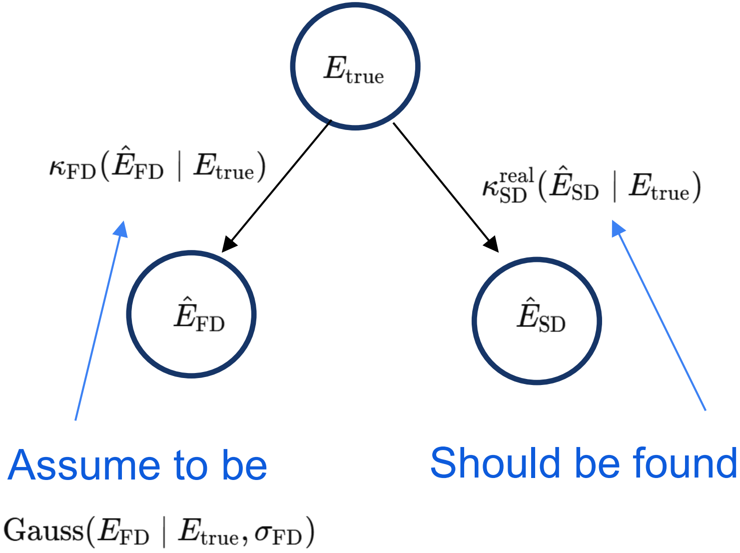
Shower size (s) fluctuations (CORSKA MC)

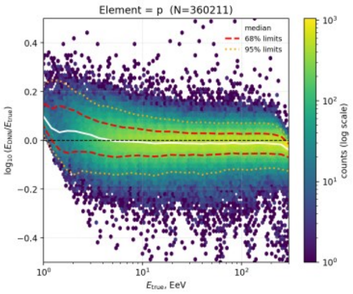




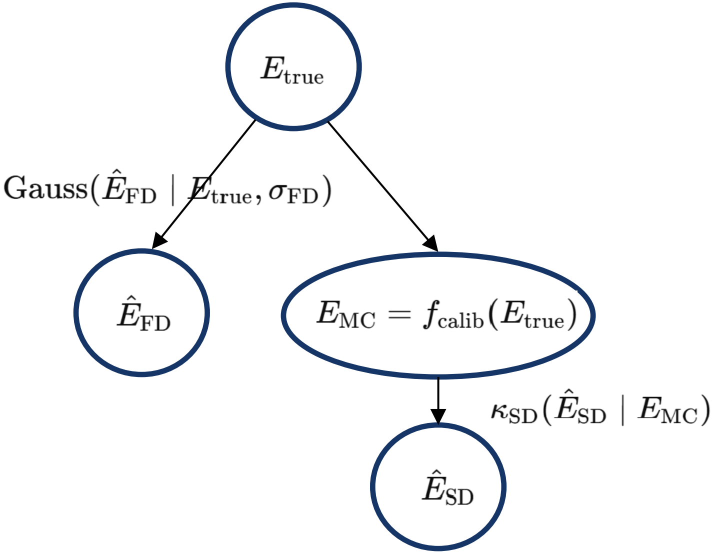


Calibration with Hierarchical MCMC





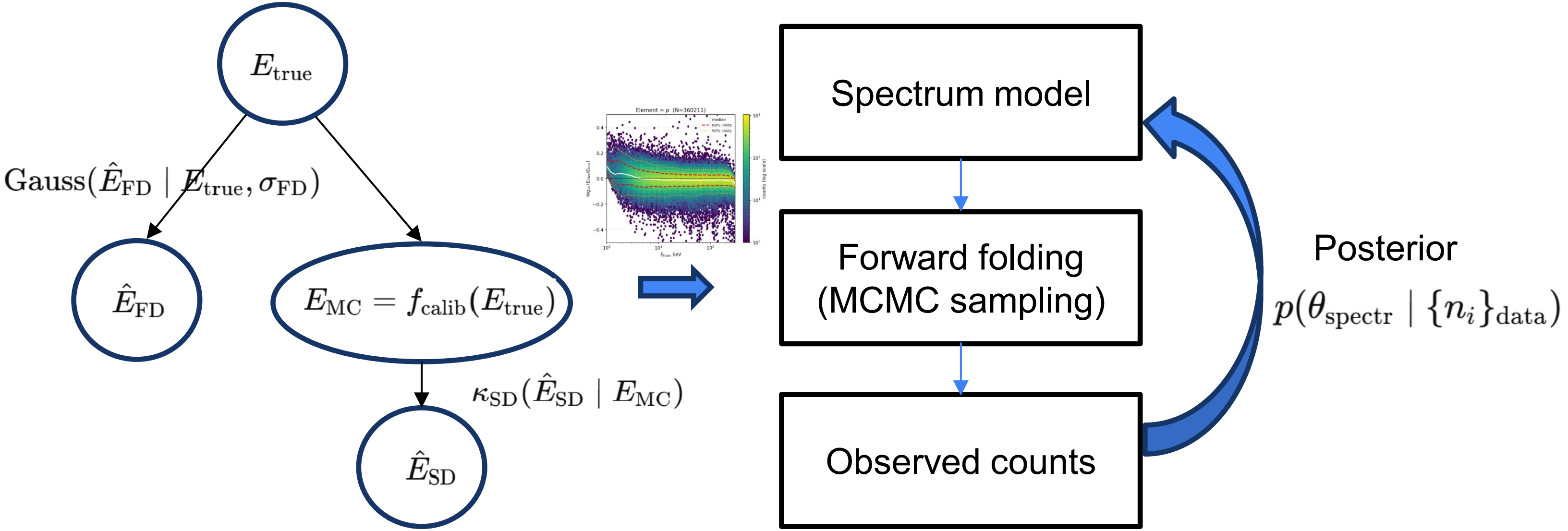
$\kappa_{\text{SD}}(\hat{E}_{\text{SD}} \mid E_{\text{MC}})$



- Response function can, in principle, be derived from hybrid events
- Hybrid statistics are limited vs. full SD sample

- Use MC-based κ_{SD} modeled with Gauss/Student-t (mixture) models
- Calibrate the model using hybrid events

Motivation: Calibration and forward folding



- MCMC calibration yields posterior distributions of calibration parameters
- Enables propagation of these uncertainties into forward folding
- Captures parameter correlations and avoids bias from fixed (point-estimated) values

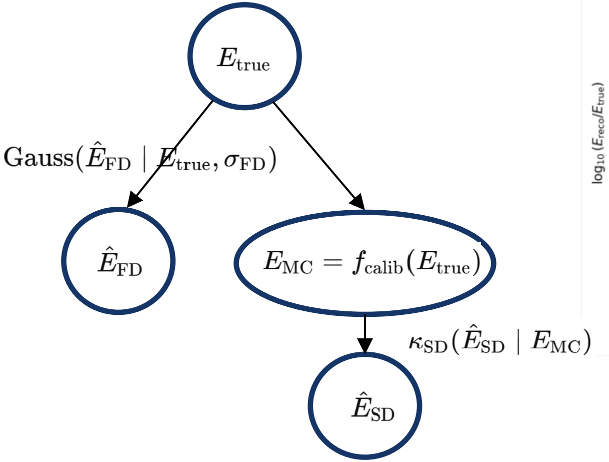
Parametrization

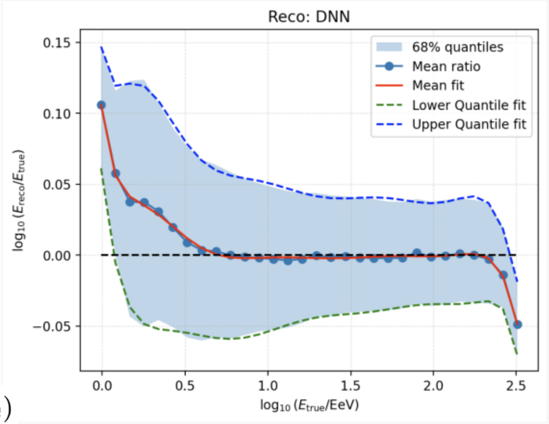
Gauss (or Student-t) mixture model

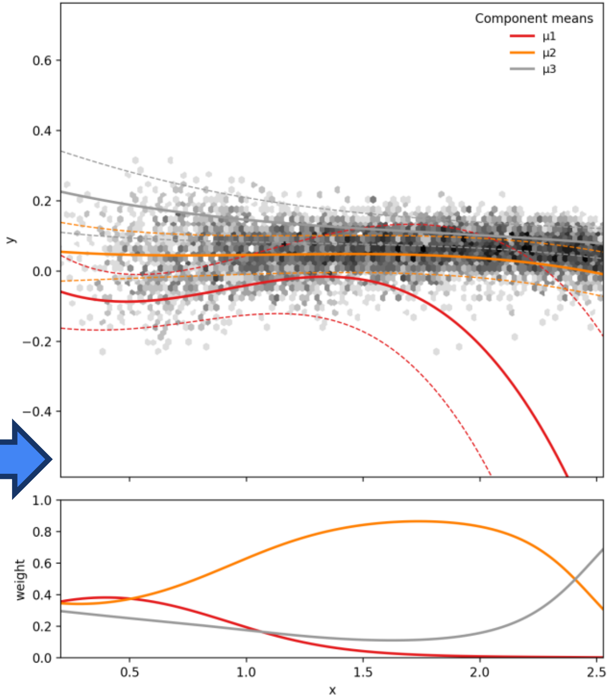
$$\kappa_{\mathrm{SD}}(E_{\mathrm{SD}} \mid E_{\mathrm{true}}) = \sum_{n=1}^{N} w_n(x)\, G(r \mid \mu_n(x), \sigma_n(x))$$

$$r = \log_{10}(E_{\mathrm{SD}}/E_{\mathrm{true}}) \qquad x = \log_{10} E_{\mathrm{true}}$$

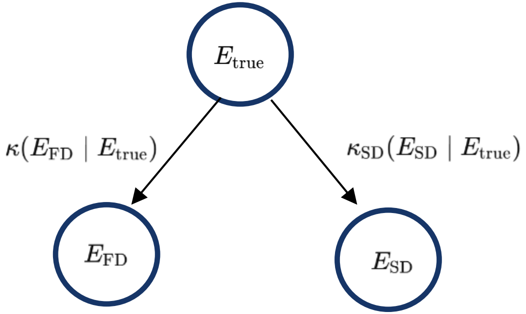
$$f_{\mathrm{calib}} = ax^2 + bx + c + x$$





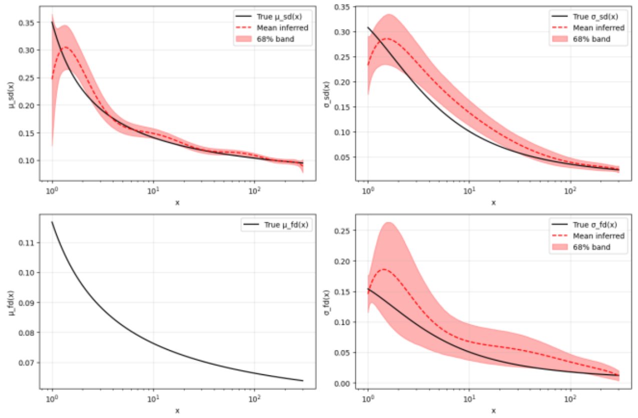


Case: full resolution reconstruction

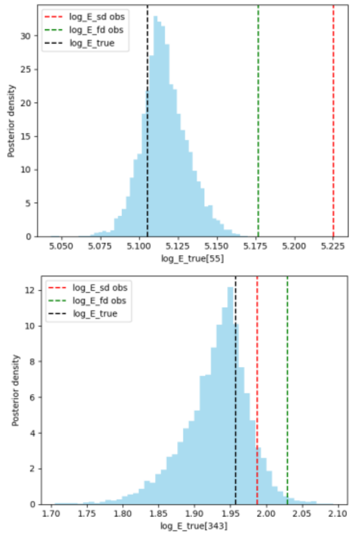


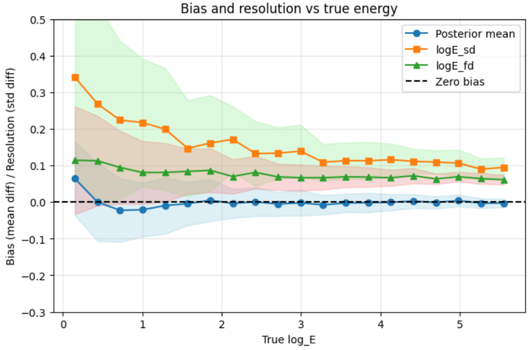
The model is identifiable if we know bias of FD detector

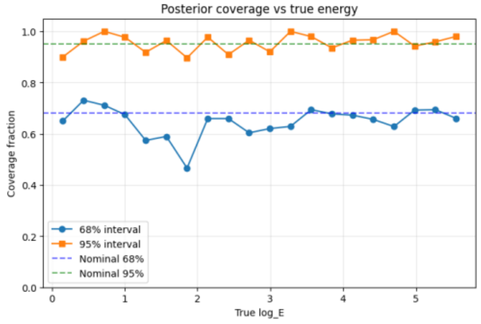
Response functions reconstruction



True energy reconstruction

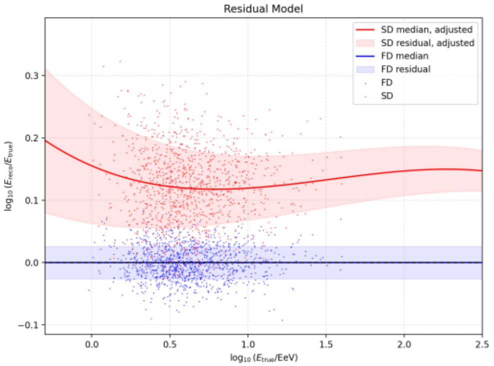






Case: toy model

Response functions residuals vs E_{true}



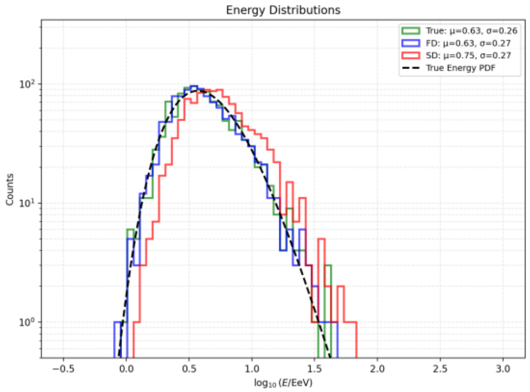
- SD response is normal distribution:

$$\mathrm{Gauss}(\hat{E}_{\mathrm{SD}} \mid \mu_{\mathrm{SD}}(E_{\mathrm{true}}) + E_{\mathrm{true}},\ \sigma_{\mathrm{SD}}(E_{\mathrm{true}}))$$

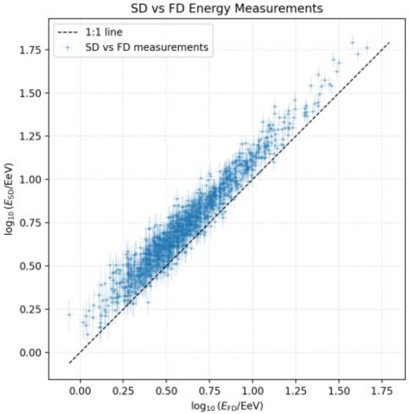
- FD response:

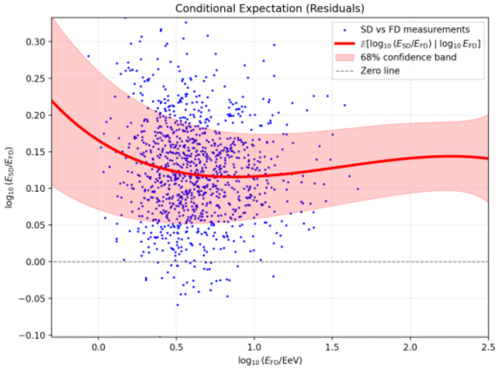
$$\mathrm{Gauss}(\hat{E}_{\mathrm{FD}} \mid E_{\mathrm{true}},\ \sigma_{\mathrm{FD}}(E_{\mathrm{true}}))$$

Energy distributions

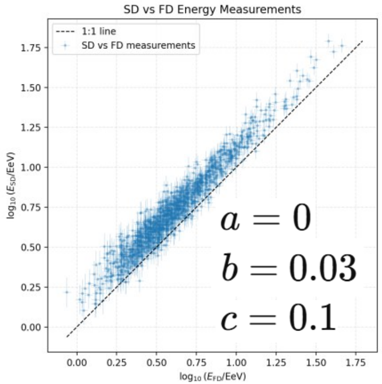


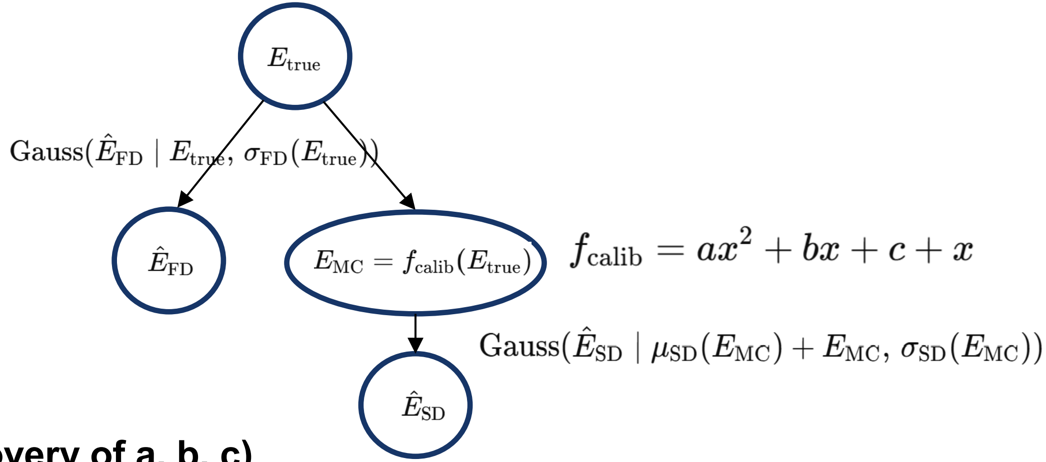
E_{SD} vs E_{FD} plane





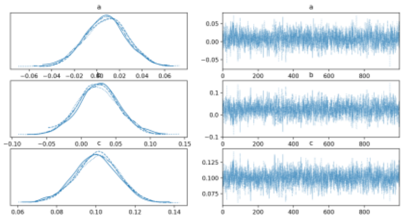
Case: toy model

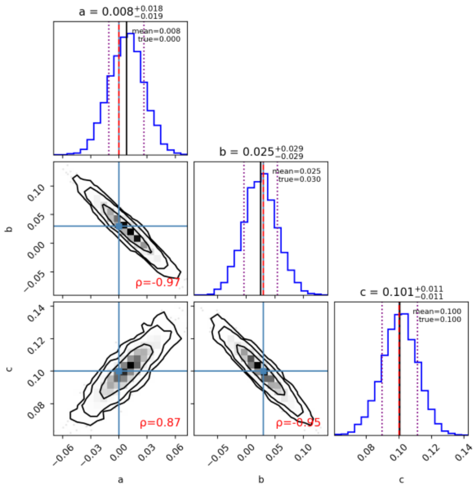




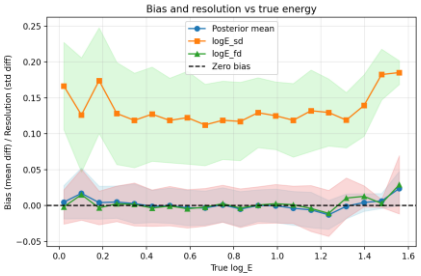
Posterior (recovery of a, b, c)

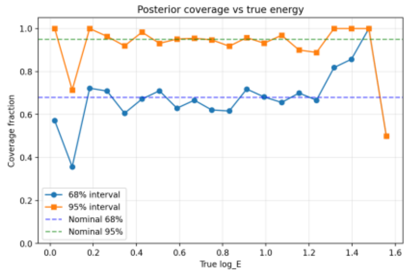
Trace plot





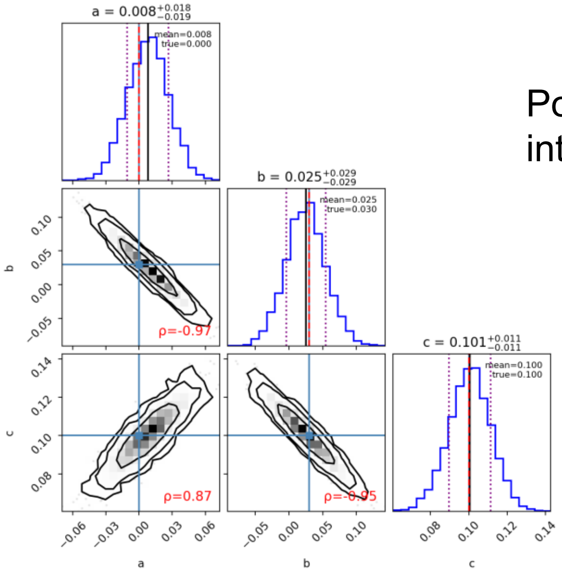
E_{true} recovery



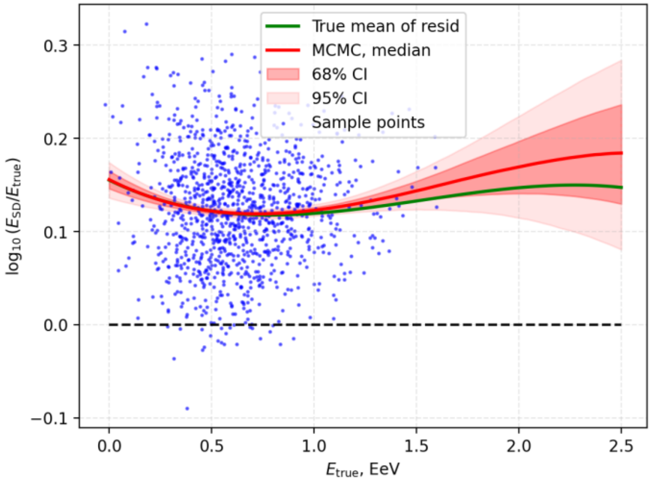


Uncertainties

Posterior (recovery of a, b, c)

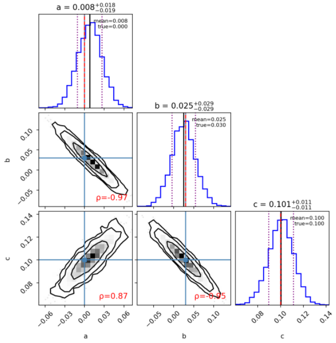


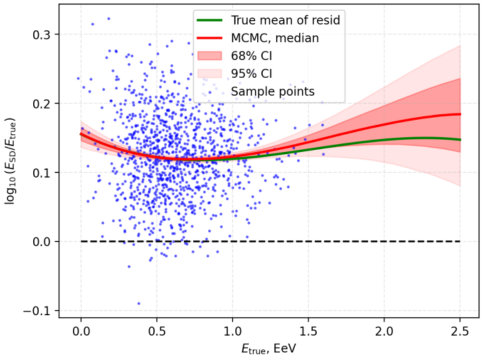
Posterior distribution translates into uncertainty band



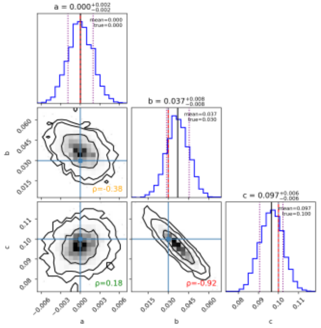
Priors (model selection)

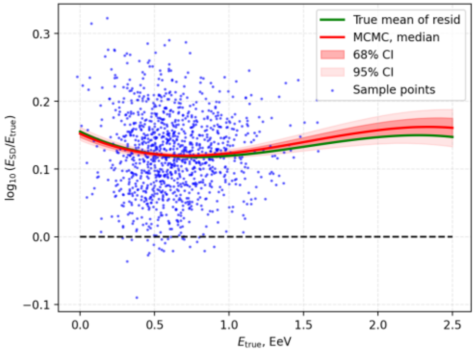
a*x^2 + b*x + c



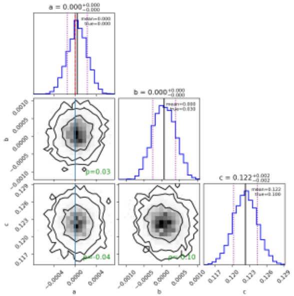


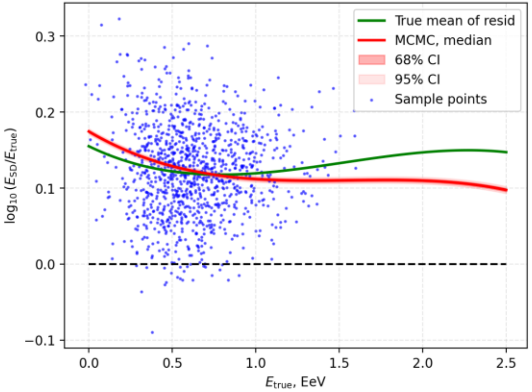
b*x + c (a=0)





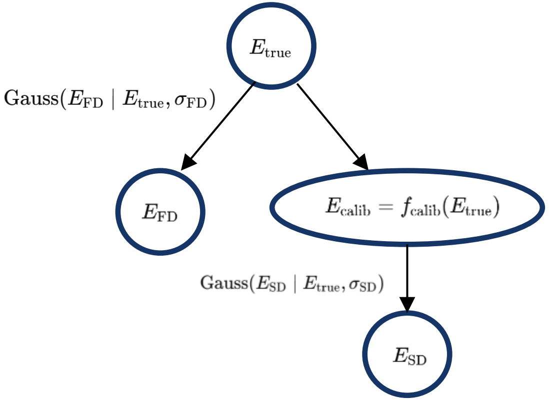
c (a=0, b=0)

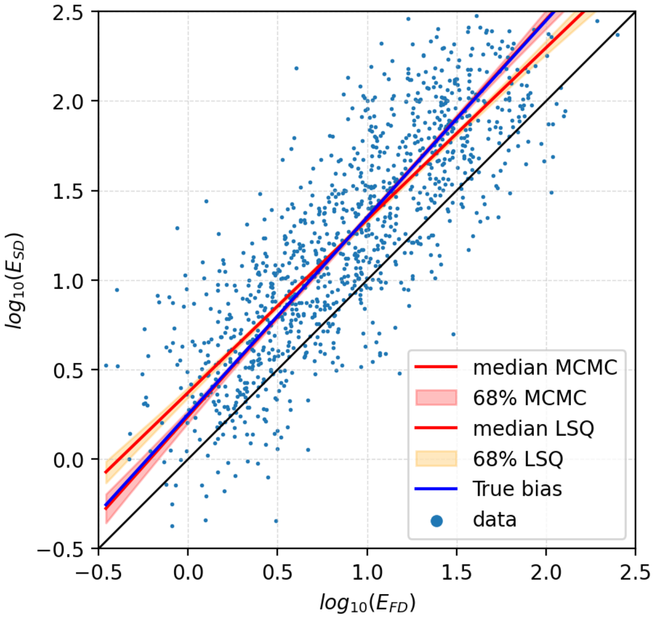




- Flexible models capture the true trend and yield realistic uncertainties
- Restrictive models miss dependencies, causing bias and narrow bands

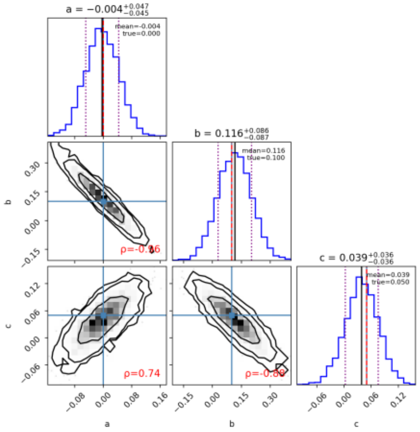
MCMC vs least square minimization (LSQ)






```
Nsample = 1100

a_true = 0
b_true = 0.1
c_true = 0.05

mu_resid = 0.2
sigma_resid = 0.3
sigma_fd_frac = 0.2
```

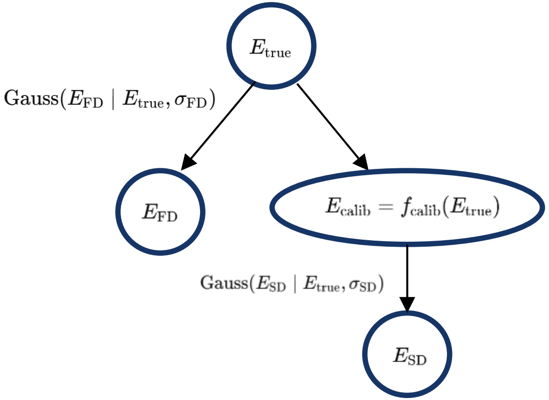
$$f_{\rm calib} = ax^2 + bx + c + x$$




```
MCMC:
a = -0.004 ± 0.045
b = 0.117 ± 0.085
c = 0.038 ± 0.035
```

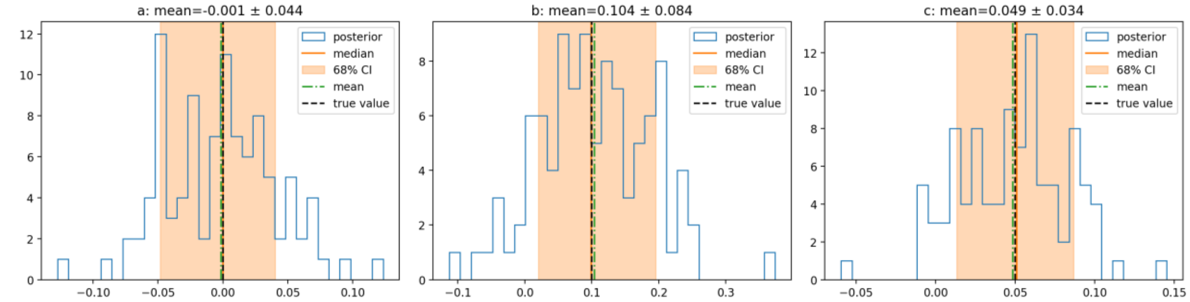
```
LSQ:
a = -0.001 ± 0.033
b = -0.034 ± 0.064
c = 0.372 ± 0.028
```

MCMC vs least square minimization (LSQ)

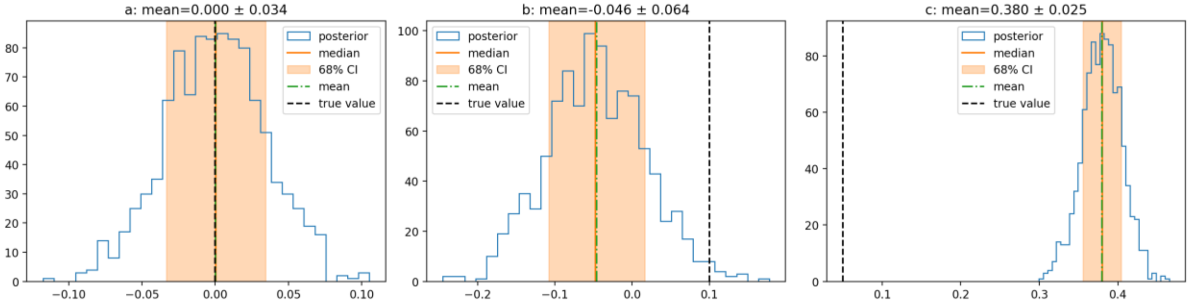



```
Nsample = 1100

a_true = 0
b_true = 0.1
c_true = 0.05

mu_resid = 0.2
sigma_resid = 0.3
sigma_fd_frac = 0.2
```

MCMC



LSQ

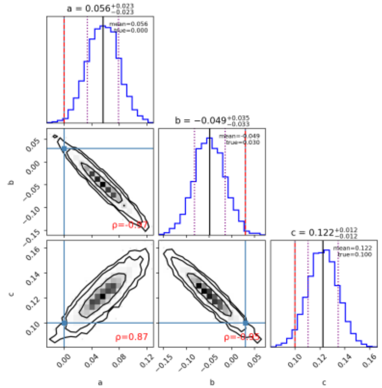


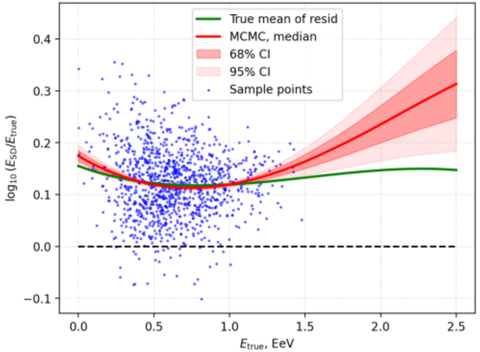
Conclusions

- Calibration of the response function is required to correctly perform forward folding
- Least-squares fits are biased because they neglect uncertainties in the FD (x-axis) variable
- MCMC is a powerful analog of the log-likelihood approach, as it does not require explicit multidimensional integration
- Bayesian MCMC calibration allows a natural extension of the MCMC forward folding method and enables uncertainty propagation

Atypical realizations

- Some realizations are atypical and lead to posterior outliers:





- Their number is small