Some quantum algorithms for black-box algebraic structures

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Introduction and preliminaries

- Some algorithms will be provided for black-box structures: for a structure M, its elements are represented as strings of n qubits; we assume efficient access to an oracle that implements the operations in M. This implies that $|M| \leq 2^n$. We focus on QMA^B and BQP^B.

  Definition 1 (BQP^B). *A language $L \subset \{0,1\}^*$ is in BQP^B if there exists a uniform family of quantum poly-generated circuits $\{Q_n\}$ with poly-width and efficient access to an oracle B, such that, $\forall x \in \{0,1\}^n$: if $x \in L$, Q_n accepts x in poly-time with probability $\geq 2/3$; if $x \notin L$, Q_n accepts x in poly-time with probability $< 1/3$.*

  Definition 2 (QMA^B). *A language $L \subset \{0,1\}^*$ is in QMA^B if there exists a uniform family of quantum poly-generated circuits $\{Q_n\}$ with poly-width and efficient access to an oracle B, such that, $\forall x \in \{0,1\}^n$: if $x \in L$, there exists $|\psi\rangle \in (\mathbb{C}^2)^{\otimes q(n)}$ so that Q_n accepts $(x, |\psi\rangle)$ in poly-time with probability $\geq 2/3$; if $x \notin L$, then for all $|\psi\rangle \in (\mathbb{C}^2)^{\otimes q(n)}$, Q_n accepts $(x, |\psi\rangle)$ in poly-time with probability $< 1/3$.*

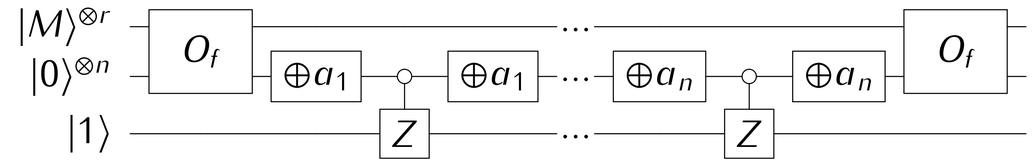
- Initial result: for any magma M, the state $|M\rangle := 1/\sqrt{|M|}\sum_{m\in M}|m\rangle$ can be created:
  1. *Prepare the state $|+\rangle^n$ in register **R** and $|0\rangle$ in register **S**.*
  2. *Check if register **R** belongs to M, storing the output in register **S**.*
  3. *Measure the register **S**.*

Ring problems in QMA^B or BQP^B

- Finding a generating set of a group G: generate from $|G\rangle$ a total of n^2 random elements.
- Finding additive generators for a subring $S = \langle s_0, \ldots, s_k\rangle$: let H_0 be the group generated by $\tilde{H}_0 := \{s_0, \ldots, s_k\}$ and set $l = 0$.
  1. *For all $s_i \in \tilde{H}_0$ and $h_j \in \tilde{H}_l$, check, one by one, if $s_i h_j \in H_l$ [1].*
  2. *If $s_i h_j \notin H_l$, define $\tilde{H}_{l+1} = \tilde{H}_l \cup \{s_i h_j\}$ and start again for $l = l + 1$.*
  3. *Continue until $s_i h_j \in H_l, \forall i, j$.*
- Hence, any result applicable to abelian subgroups is also applicable to subrings, such as creating the state $|S\rangle$ for any subring S [1].
- Combining the previous three results, we can deduce:
  - → Membership to a subring: given an element $r \in R$, we can efficiently determine if $r \in S$ [1].
  - → Equality of subrings: construct $|S\rangle$, $|T\rangle$ and perform a swap test.
  - → Elimination of redundant generators.
  - → Generators for the intersection of two subrings: the function $f(s) = |s + T\rangle$ hides the subgroup $(S \cap T)^+$ inside S^+. The hidden subgroup problem can be solved for abelian groups [2].
- Left-ideality of a subring I:
  1. Prepare $|R\rangle$, $|I\rangle$ and $|0\rangle$ in registers **R**, **I** and **M**, respectively.
  2. Multiply $|R\rangle$ and $|I\rangle$ on register **M** and measure **M**.
  3. Declare that I is an ideal only if the result belongs to I.
- 'Non-division ring' in QMA^B: let **S** be the certificate register.
  1. Measure **S** and reject if the result is not in R or is equal to 0.
  2. In **S**, we are left with an $r \in R$. Accept if and only if $rR \neq R$.

Magmas: existence problems

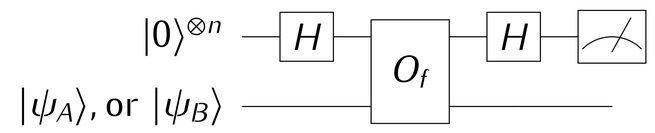
- Given $f : M^r \longrightarrow \{0,1\}^n$, we want to decide if there exists an element in a fixed $A \subseteq \{0,1\}^n$ such that $A \cap f(M^r) \neq \emptyset$.
- This includes the problems of deciding if an element is invertible, if its annihilator is different from 0, the existence of divisors of 0 or idempotent elements, if an element behaves like a 0 for the product, or if M is commutative, associative, distributive, etc.
- Approach based on Quantum Abstract Detecting Systems [3]:



- Exact if $A \cap f(M^r) \neq \emptyset$ or if $|\{\vec{x} \in M^r \text{ such that } f(\vec{x}) \in A\}| = |f(M^r)|/2$. Overall behavior equivalent to the classical random pooling method: if we repeat it R times, the probability of error is $\frac{1}{2R+1}$.
- However, if we repeat our quantum method $R-1$ times and then apply the classical method just once, the probability of error is $\frac{R}{(2R-1)(2R+1)}$.

Deutsch-Jozsa generalization

- We are given $f : \{0,1\}^n \longrightarrow \{0,1\}^m$ and two disjoint subsets of $\{0,1\}^m$, $A = \{a_1, a_2, \ldots, a_{d_A}\}$ and $B = \{b_1, b_2, \ldots, b_{d_B}\}$. We assume that $f(\{0,1\}^n)$ is either contained in one of both sets, or balanced between both, and must decide which is the case.



- For $A = \{a_1\}, B = \{b_1, b_2\}$, $|\psi_A\rangle$ and $|\psi_B\rangle$ are, respectively:

$$\frac{1}{\sqrt{2^{1+2}}}(|0\rangle - |a_1\rangle + |b_1\rangle + |b_2\rangle - |a_1 \oplus b_1\rangle - |a_1 \oplus b_2\rangle + |b_1 \oplus b_2\rangle - |a_1 \oplus b_1 \oplus b_2\rangle);$$

$$\frac{1}{\sqrt{2^{1+2}}}(|0\rangle + |a_1\rangle - |b_1\rangle - |b_2\rangle - |a_1 \oplus b_1\rangle - |a_1 \oplus b_2\rangle + |b_1 \oplus b_2\rangle + |a_1 \oplus b_1 \oplus b_2\rangle)$$

Definition 3. *We say A and B are $\oplus$-incoherent if an equation of the form $\bigoplus_{p=1}^{k} a_{i_p} = \bigoplus_{q=1}^{l} b_{j_q}$ holds, being both k and l odd.*

Theorem 1. *The $\oplus$-coherence of A and B can be deterministically decided in $O(m(d_A + d_B)^2)$ operations. At least one of $|\psi_A\rangle$ or $|\psi_B\rangle$, is deterministically constructible $\Leftrightarrow$ A and B are $\oplus$-coherent.*

Conclusions and future works

- Every quantum procedure that works for abelian subgroups, works for any subring as well.
- Connection between magmas and the QADS paradigm.
- New generalization of the Deutsch-Jozsa algorithm, where the function is either balanced between two sets or constant on one.
- Other approaches to the construction of $|\psi_A\rangle$ or $|\psi_B\rangle$ might gain efficiency. Starting an algorithm with a random distribution of amplitude signs in $|M\rangle$ could mitigate the arbitrariness of a magma operation.

References

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