# Quantum simulation without ancillae

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1. Motivation: what's the matter with ancillae?

2. Quantum simulation without ancillae
   2.1. Simulating time evolutions
   2.2. Simulating ground-state subspaces

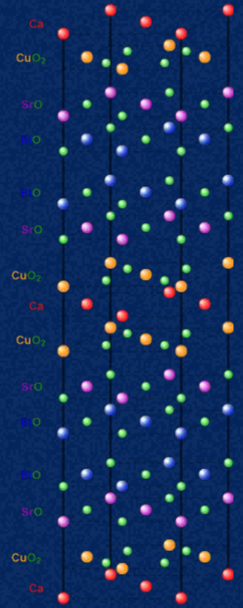
3. Conclusions and open questions

# Motivation

Many-body systems described by *k*-local Hamiltonians:

$$H = \sum_i h_i \in \mathcal{B}((\mathbb{C}^d)^{\otimes n})$$

each term $h_i$ acts nontrivially at most on k of these hilbert spaces



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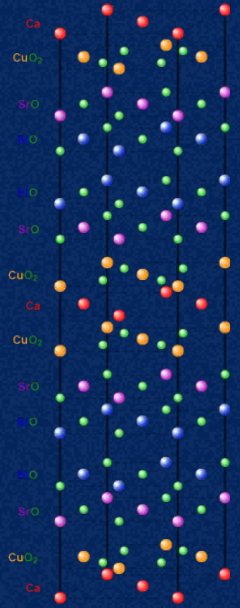
$$\begin{aligned} H^{(3)}_{RSC} = & -X_1X_2X_4X_5 - X_5X_6X_8X_9 \\ & - X_3X_6 - X_4X_7 \\ & - Z_2Z_3Z_5Z_6 - Z_4Z_5Z_7Z_8 \\ & - Z_1Z_2 - Z_8Z_9. \end{aligned}$$



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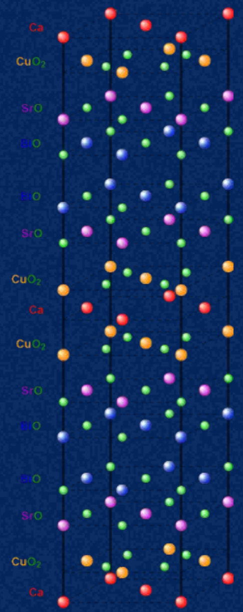
The larger the $k$, the more difficult it is to realize the Hamiltonian experimentally.



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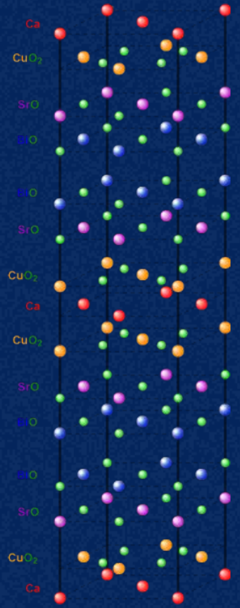
So if you want to measure the properties of such a complex system in the lab, you first need to find a Hamiltonian with a smaller $k$ that describes the same physics, i.e., you need to find the Hamiltonian of your quantum simulator.



PNAS

# Universal quantum Hamiltonians

Toby S. Cubitt, Ashley Montanaro, and Stephen Piddock Authors Info & Affiliations

Edited by Peter W. Shor, Massachusetts Institute of Technology, Cambridge, MA, and approved August 3, 2018 (received for review March 23, 2018)

**August 30, 2018** | 115 (38) 9497-9502 | https://doi.org/10.1073/pnas.1804949115

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- $H_{QS}$ reproduces the whole physics of $H_T$.
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→ Time evolution

→ Ground-state subspace

= Can we do quantum simulation without ancillae?

# Simulating time evolutions

PHYSICAL REVIEW RESEARCH

# Simplifying the simulation of local Hamiltonian dynamics

Ayaka Usui, Anna Sanpera, and María García Díaz

Phys. Rev. Research **6**, 023243 – Published 4 June 2024

Ayaka Usui,
UAB

Anna Sanpera,
UAB, ICREA

*Definition 1.* A Hamiltonian $H_{QS}$ $\epsilon$-simulates a target Hamiltonian $H_T$ at state $|\psi\rangle$ and time $t$ if

$$|\langle\psi|e^{itH_{QS}}e^{-itH_T}|\psi\rangle| \geqslant 1-\epsilon, \tag{1}$$

with $\epsilon \in (0, 1]$.

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$k = 3, k' = 2$

$$H_T = \sum_{j=1}^{4} \left(J_3 \sigma_z^j \sigma_z^{j+1} \sigma_z^{j+2} + h_x \sigma_x^j\right)$$

$$H_{QS} = \sum_{j=1}^{4} \left(J_x \sigma_x^j \sigma_x^{j+1} + J_y \sigma_y^j \sigma_y^{j+1} + J_z \sigma_z^j \sigma_z^{j+1}\right)$$

$(J_x = J_y = J_z)$

Heisenberg model

# b) Approximate simulation $(\epsilon \neq 0)$

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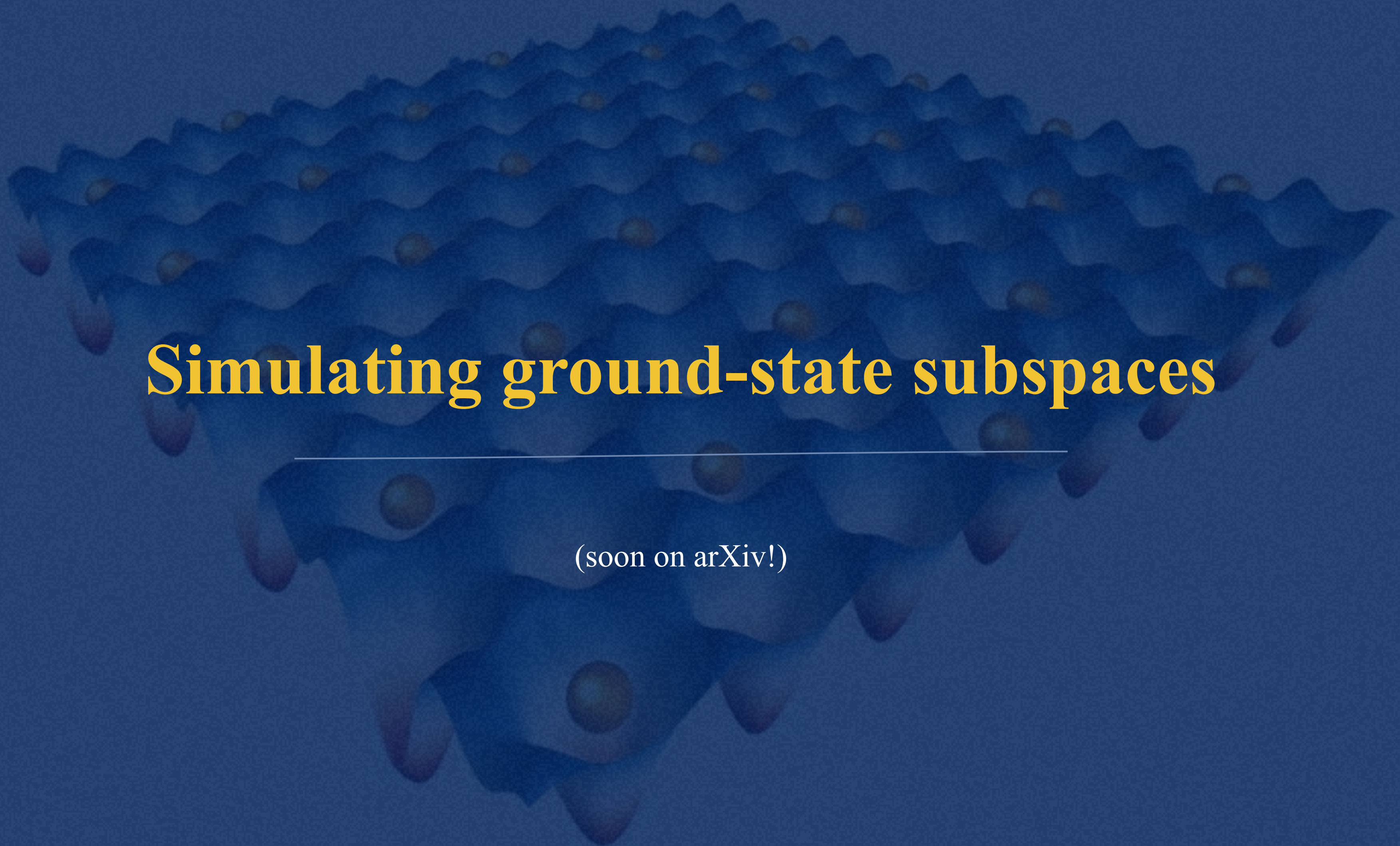
spectral diameter of $h$
(expressible by an SDP)



# Simulating ground-state subspaces

(soon on arXiv!)

# Simulating ground-state subspaces

Given $H_T$:

- $k$-local
- $dim(H_T) = d^n$
- ground-state subspace $\mathcal{S} = \text{span}\{|\psi_i\rangle\}_{i=1}^{M}$

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Does it exist?

If so, how is it built?

# Simulating ground-state subspaces

a) Semidefinite program:

$$\begin{aligned}
\underset{\delta\in\mathbb{R},\ \boldsymbol{\alpha}\in\mathbb{R}^{L+1}}{\text{maximize}} \quad & \delta \\
\text{subject to} \quad & PH_{QS}P = 0 \\
& PH_{QS}Q = 0 \\
& QH_{QS}Q \succeq \delta Q \\
& H_{QS} = \alpha_0 \mathbf{1}_{d^n} + \sum_{j=1}^{L} \alpha_j \Lambda_j^{(k')} \\
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$P = \sum_{i=1}^{M} |\psi_i\rangle\langle\psi_i|$

$Q = 1_{d^n} - P$

generators of $\ell$-body interactions, $\ell = 1, 2, \ldots, k'$

allowed energy range

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$$

$\rightarrow \delta^* = 0 : \nexists\ k'$-local parent to $\mathcal{S}$

$\rightarrow \delta^* \neq 0 : H_{QS}^*\ k'$-local parent to $\mathcal{S}$ with largest spectral gap

# Simulating ground-state subspaces

b) Simple algebraic method (built upon **Method to identify parent Hamiltonians for trial states** Martin Greiter, Vera Schnells, and Ronny Thomale **Physical Review B**)

# Simulating ground-state subspaces

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1) Find $\boldsymbol{\alpha}$ such that

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$$C_{\ell j}^{(i)} := \langle \psi_i | \Lambda_\ell^{(k')} \Lambda_j^{(k')} | \psi_i \rangle$$

$$\Lambda_0^{(k')} := \mathbf{1}_{d^n}$$

# Simulating ground-state subspaces

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2)Pick some $\boldsymbol{\alpha} \in \mathcal{K}$ and build $H_{QS}$ :

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2)Pick some $\boldsymbol{\alpha} \in \mathcal{K}$ and build $H_{QS}$ :

$$H_{QS} = \sum_{j=1}^{L} \alpha_j \Lambda_j^{(k')}$$

3) Diagonalize $H_{QS}$. It will be a suitable parent Hamiltonian if

$$\boxed{-\alpha_0 \text{ is the ground energy}} \text{ and } \boxed{\mathcal{S}' = \mathcal{S}}$$

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-Cons: Diagonalization needed!

-Pros: Allows to check whether the target subspace appears as an excited eigenspace of some $H_{QS}$.

# Example: the Rotated Surface Code Hamiltonian

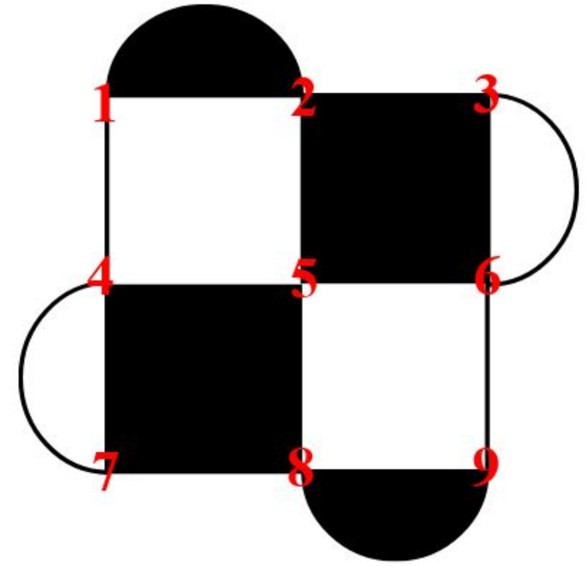
$$H_{RSC}^{(L)} = -\sum_{f=1}^{F_w} A_f - \sum_{f=1}^{F_b} B_f$$

where

$$A_f = \prod_{j\in\partial f} X_j$$

$$B_f = \prod_{j\in\partial f} Z_j$$

- Well-known 4-local Hamiltonian.

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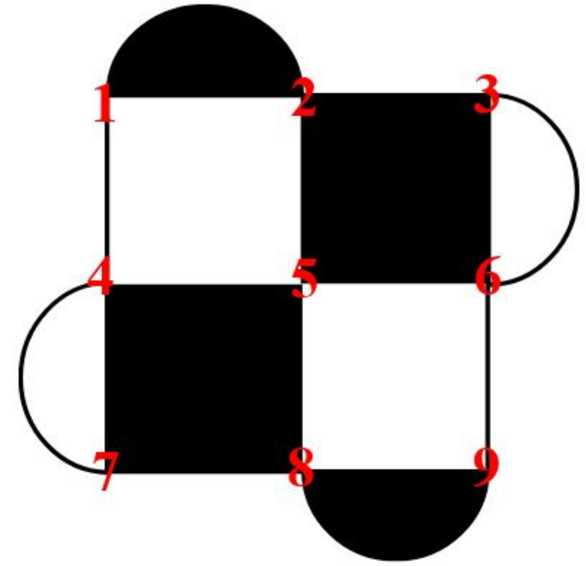


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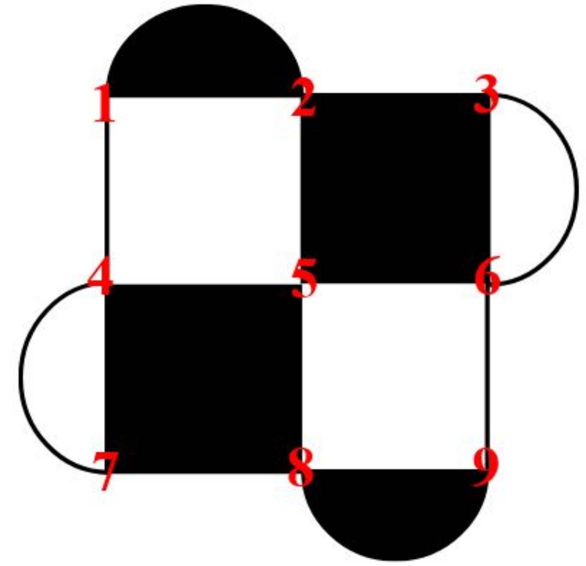


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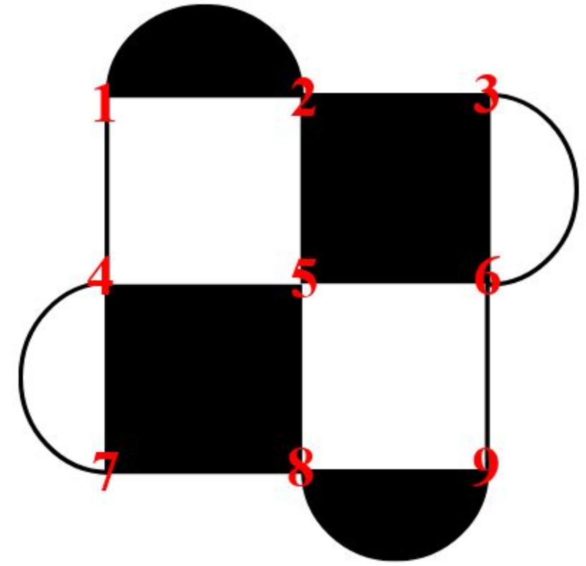


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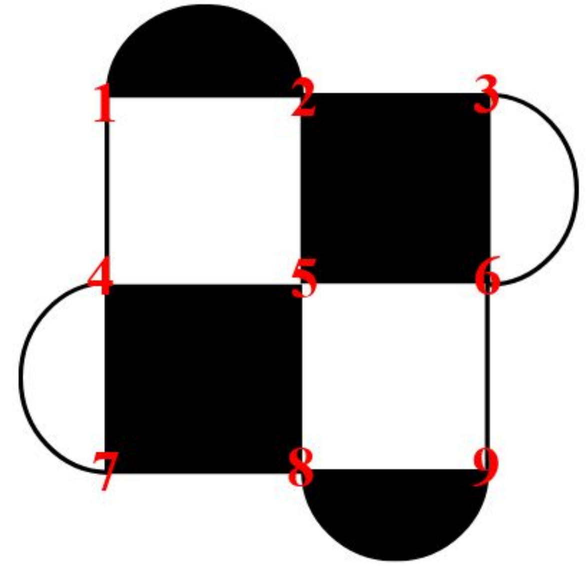


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- Well-known 4-local Hamiltonian.
- 2 degenerate ground states.
- Ground space exhibits topological order (gap, ground states locally indistinguishable).
- 1 protected qubit can be encoded in ground space.
- Ground states exhibit long-range entanglement.

# Example: the Rotated Surface Code Hamiltonian

## Results:

- Sanity check: no 2,3-local $n$-qubit parent Hamiltonian for the ground space of $H_{RSC}$ (proven analytically in **Graph states as ground states of many-body spin-$1/2$ Hamiltonians**
M. Van den Nest[1], K. Luttmer[1], W. Dür[1,2], and H. J. Briegel[1,2]
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- For 2x2 and 3x3 lattices, the ground space of $H_{RSC}$ appears as an excited eigenspace of 2,3-local $n$-qubit Hamiltonians, up to errors of order machine precision.

# Conclusions and open questions

- Instances can be found of $k'$-local Hamiltonians leading to the same dynamics as given $k$-local Hamiltonians of the same dimension, with $k'<k$, for a particular subspace of initial states. Q: given any $H_T$, is there always a suitable $H_{QS}$? If not, what is the relative volume of the manifold of simulatable $H_T$? What is the largest dimension that the subspace of initial states can have for each case?

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- The ground space of $H_{RSC}$ may appear as an excited subspace of 2,3-local Hamiltonians, up to errors of order machine precision. Q: is this true for other stabilizer states, e.g. GHZ?

# Thanks for your attention

**lecturer positions opening soon!**