Thermodynamic Approach to Quantifying Incompatible Instruments

University of Bristol

Chung-Yun Hsieh

Leverhulme Trust

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Chung-Yun Hsieh, Shin-Liang Chen, *Phys. Rev. Lett.* **133**, 170401 (2024)

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Thermodynamically Quantifying Uncertainty Principle's Signature

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University of BRISTOL

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Uncertainty principle's signature

Cannot be measured simultaneously

Uncertainty principle's signature

↓

Cannot be measured simultaneously

mathematically described by

Incompatible Instruments

What is incompatibility

Here!

Details

Position

Cannot be measured simultaneously

Incompatible

Details ⟷ Position

Cannot be measured simultaneously

Incompatible

Details ⟷ Position

(classical)

Cannot be measured simultaneously

(momentum)

Details

of motion

Incompatible

(quantum)

Position

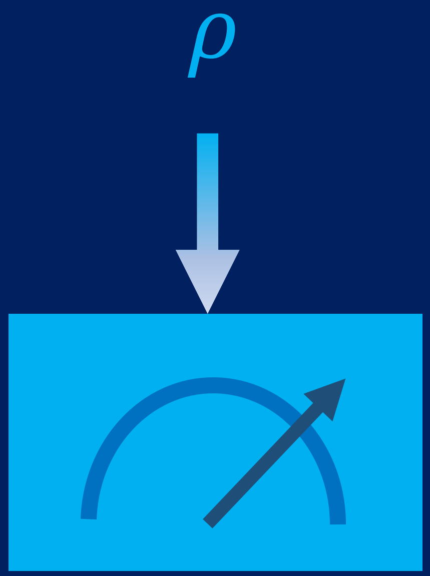
Cannot be measured simultaneously

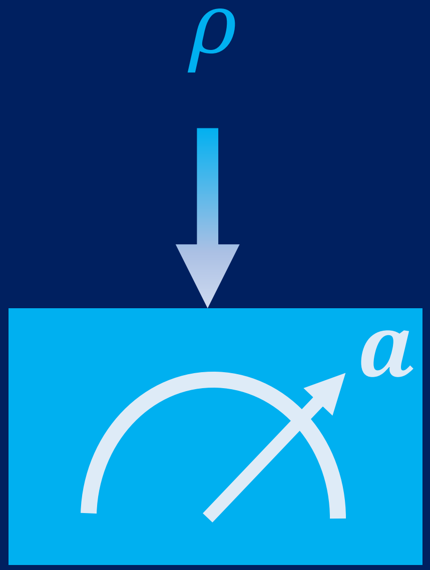
Incompatible

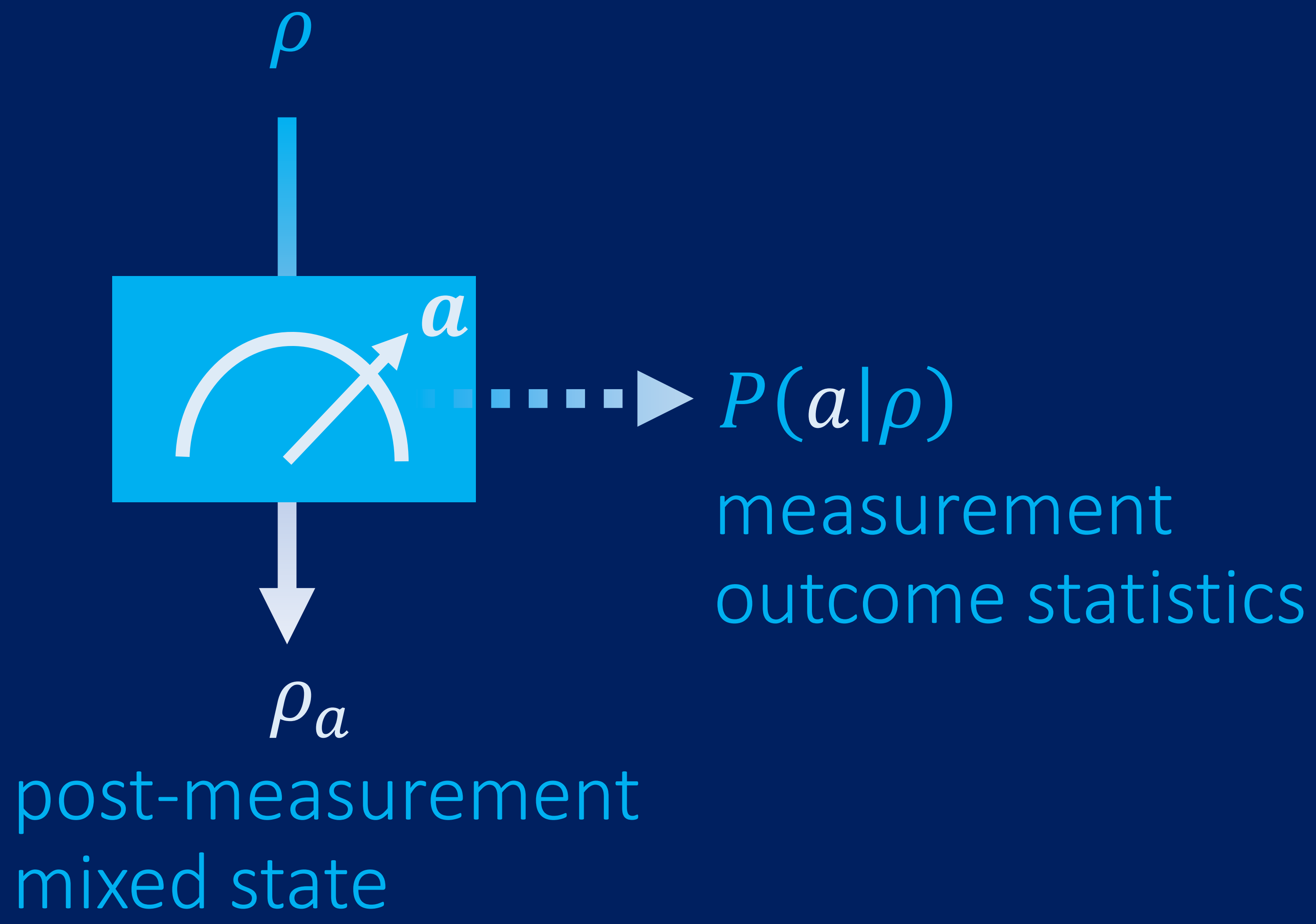
Spin X ⟷ Spin Z

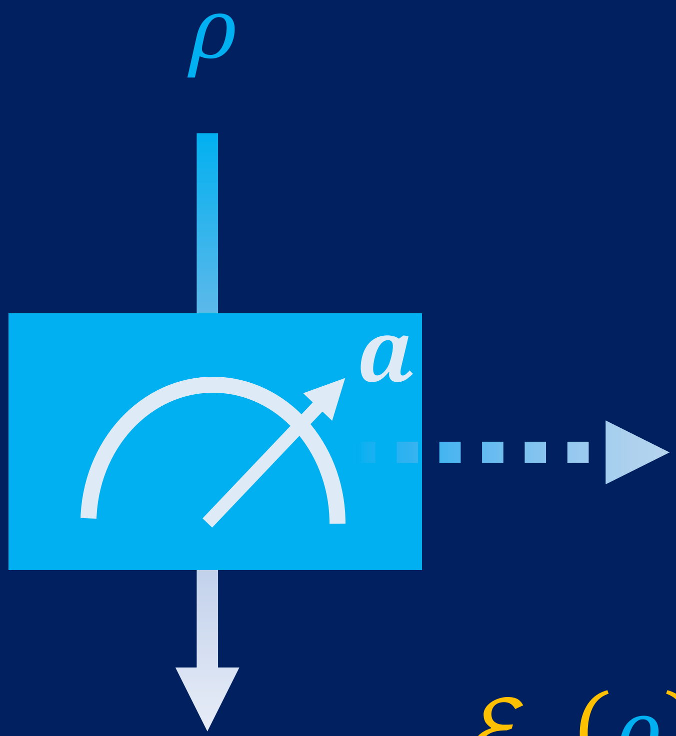
(quantum)

What is instrument


ρ


ρ
a

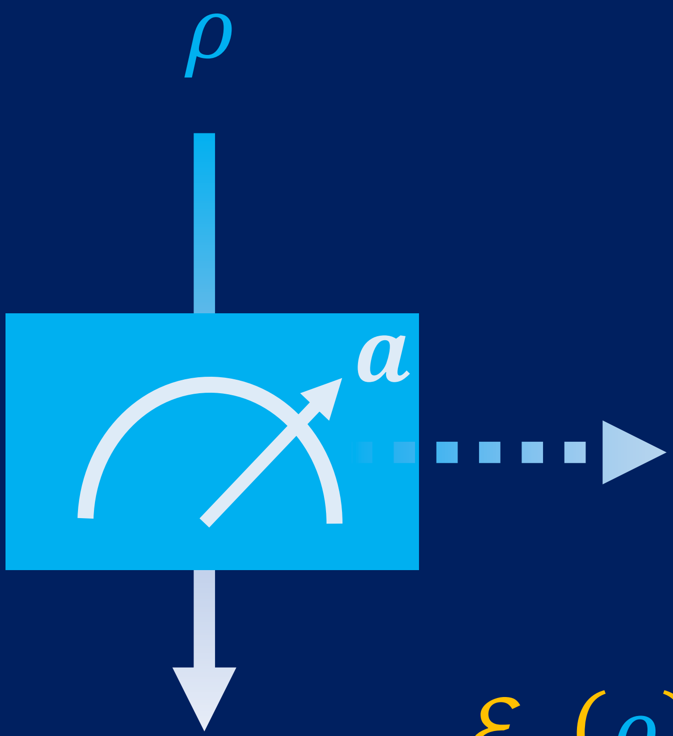

ρ
a
$P(a|\rho)$
measurement
outcome statistics
ρ_a
post-measurement
mixed state



$$P(a|\rho) = \mathrm{tr}[\mathcal{E}_a(\rho)]$$

$$\rho_a = \frac{\mathcal{E}_a(\rho)}{\mathrm{tr}[\mathcal{E}_a(\rho)]}$$

$$\{\mathcal{E}_a\}_a$$



$$P(a|\rho) = \mathrm{tr}[\mathcal{E}_a(\rho)]$$

$$\rho_a = \frac{\mathcal{E}_a(\rho)}{\mathrm{tr}[\mathcal{E}_a(\rho)]}$$

Instrument

$\mathcal{E}_a$ is completely-positive trace-non-increasing

$\sum_a \mathcal{E}_a$ is completely-positive trace-preserving

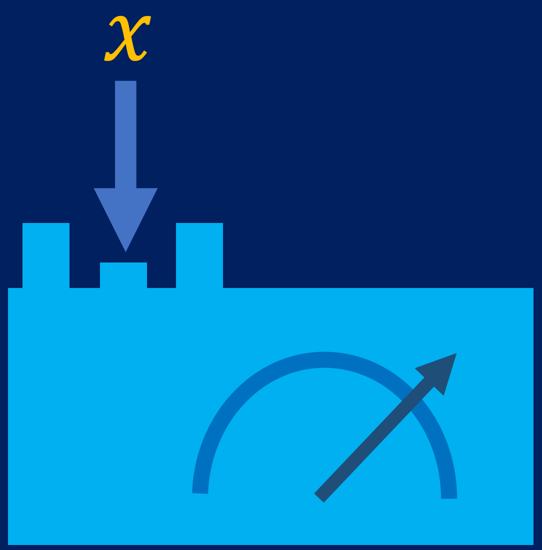
An Instrument describes the process of measuring a single thing

To measure multiple things …

$$\{\mathcal{E}_{a|x}\}_{a,x}$$

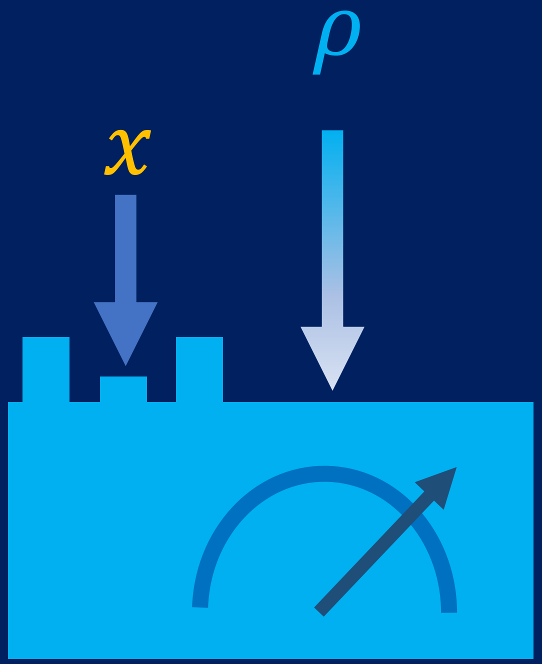
A set of Instruments

$$\{\mathcal{E}_{a|x}\}_{a,x}$$



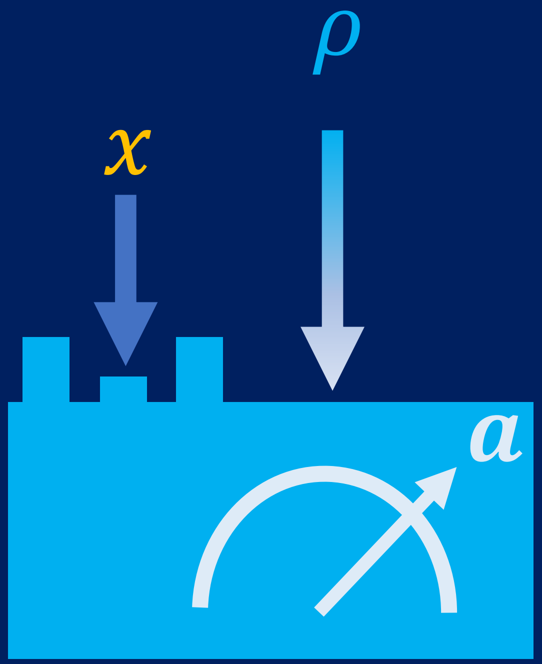
A set of Instruments

$$\{\mathcal{E}_{a|x}\}_{a,x}$$



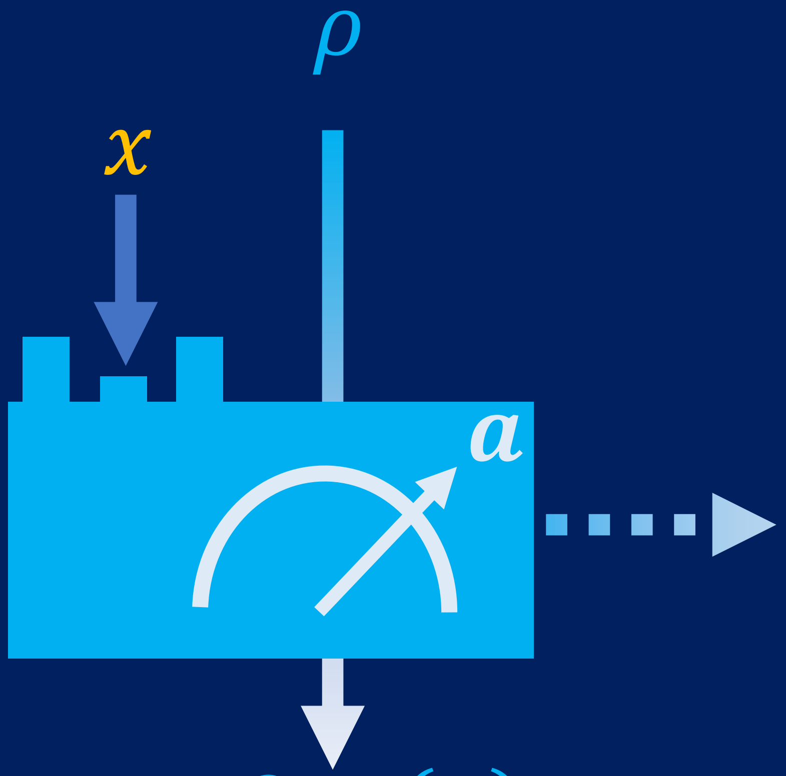
A set of Instruments

$$\{\mathcal{E}_{a|x}\}_{a,x}$$



A set of Instruments

$$\{\mathcal{E}_{a|x}\}_{a,x}$$

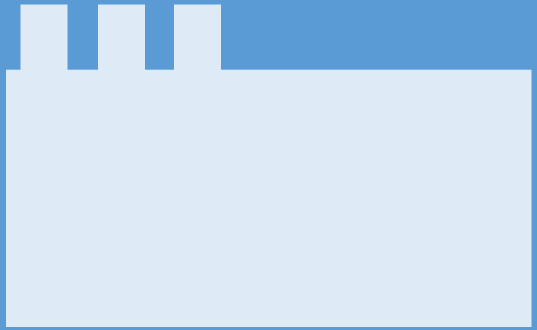


$$\mathrm{tr}[\mathcal{E}_{a|x}(\rho)]$$

$$\frac{\mathcal{E}_{a|x}(\rho)}{\mathrm{tr}[\mathcal{E}_{a|x}(\rho)]}$$

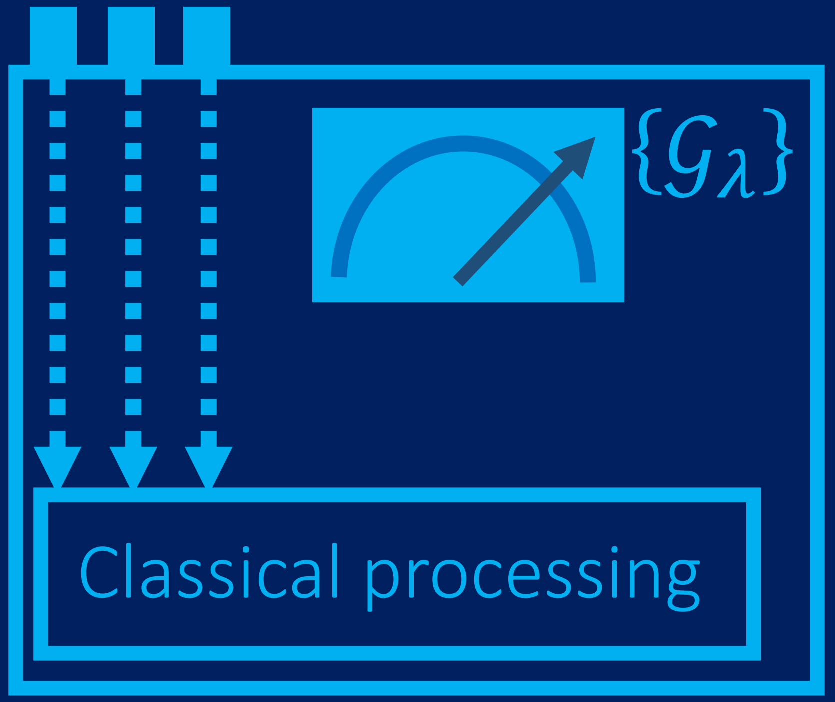
A set of Instruments

What if

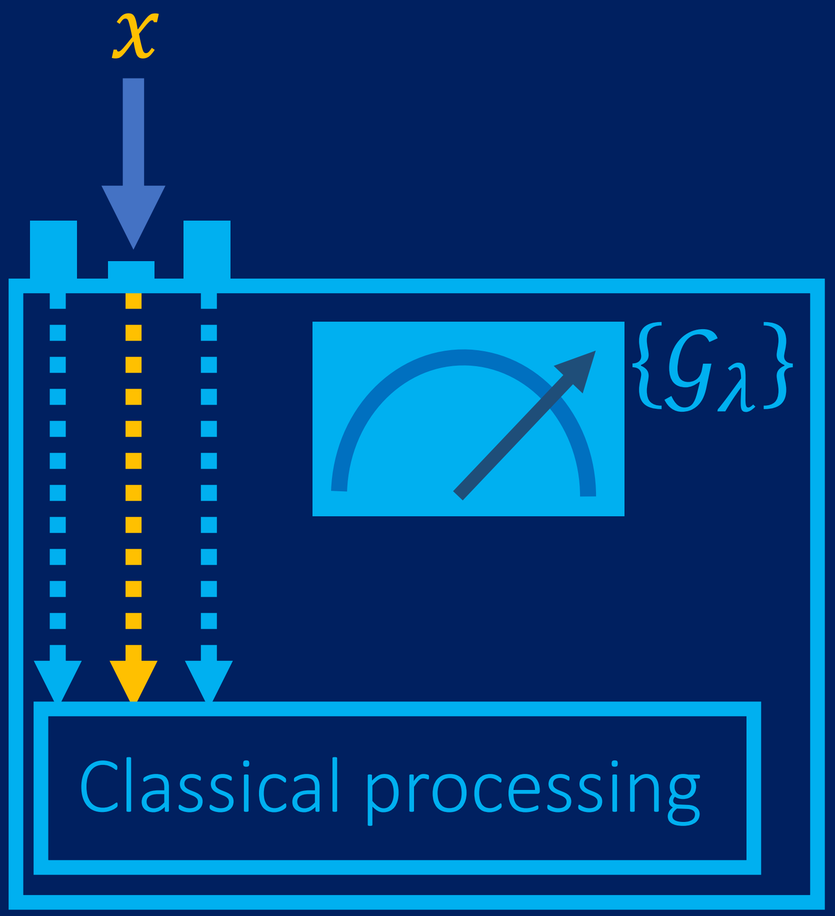


can be simultaneously measured?

$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument



$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument



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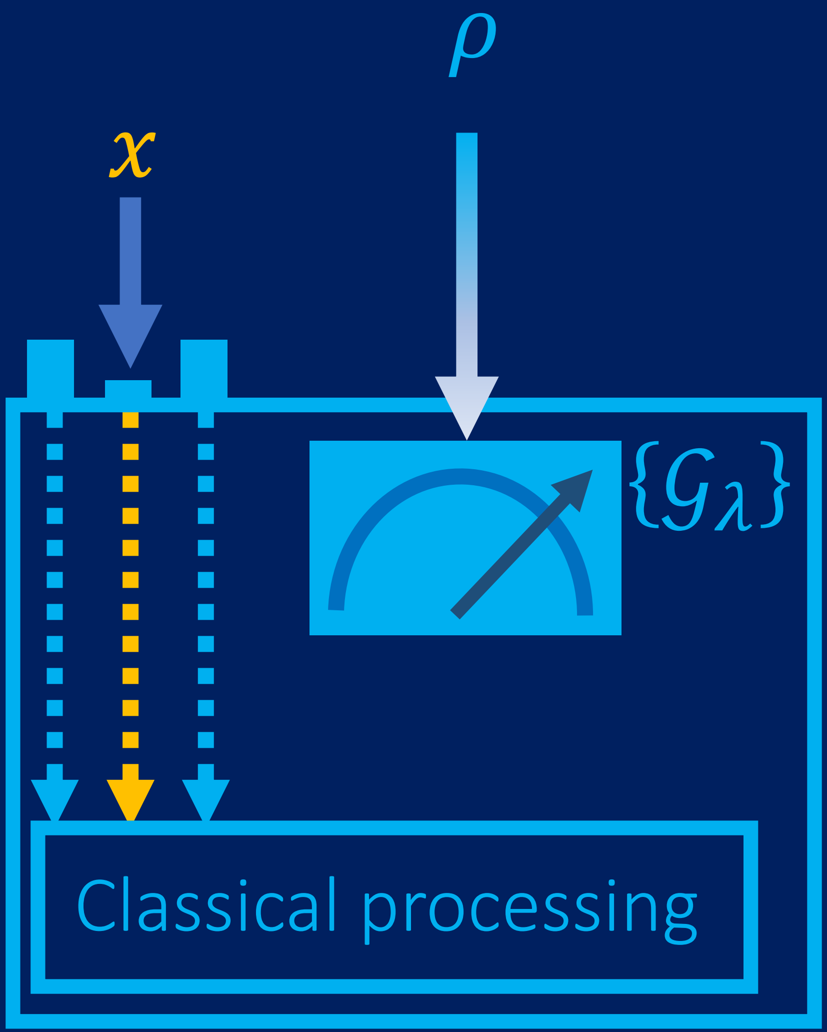
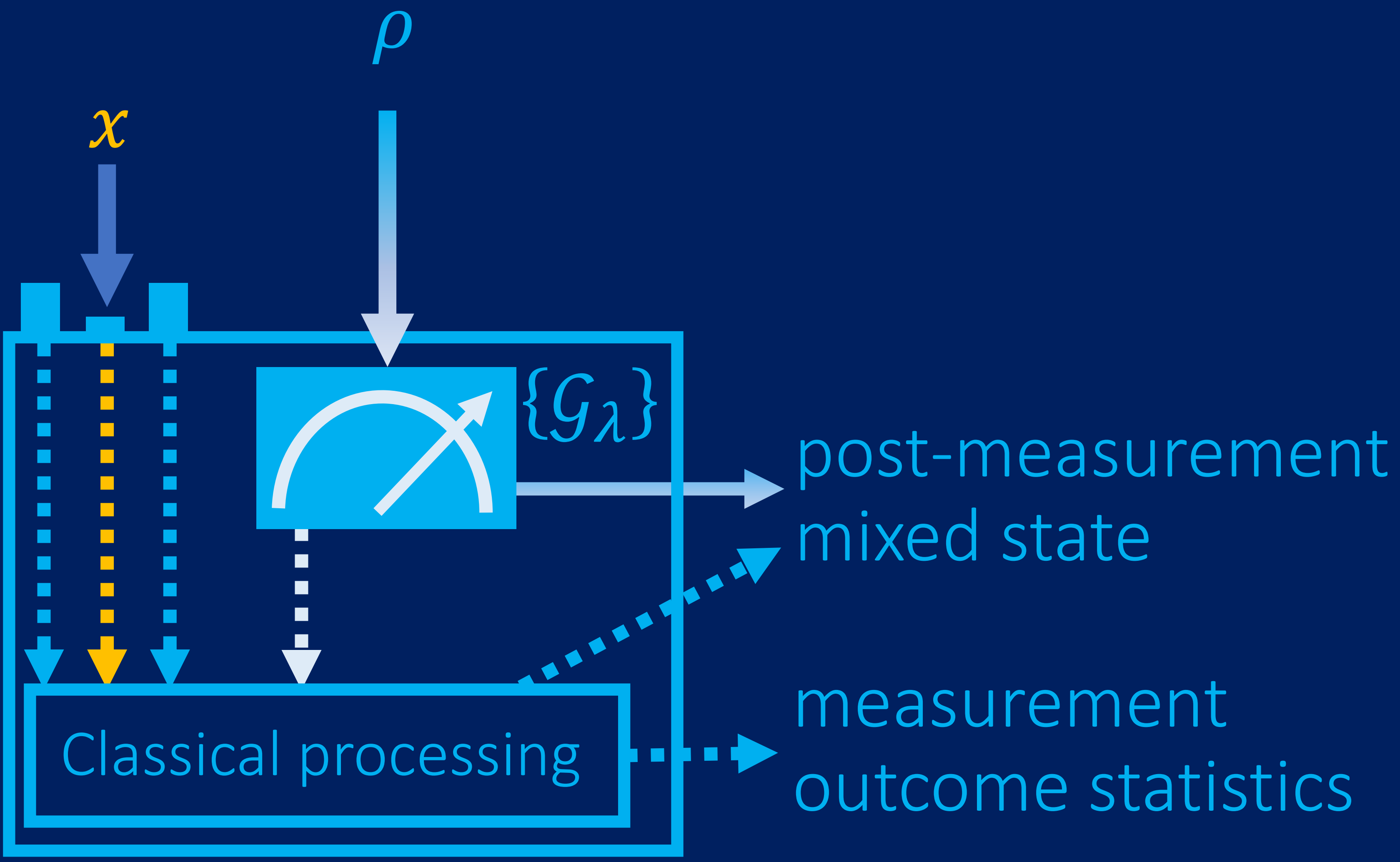
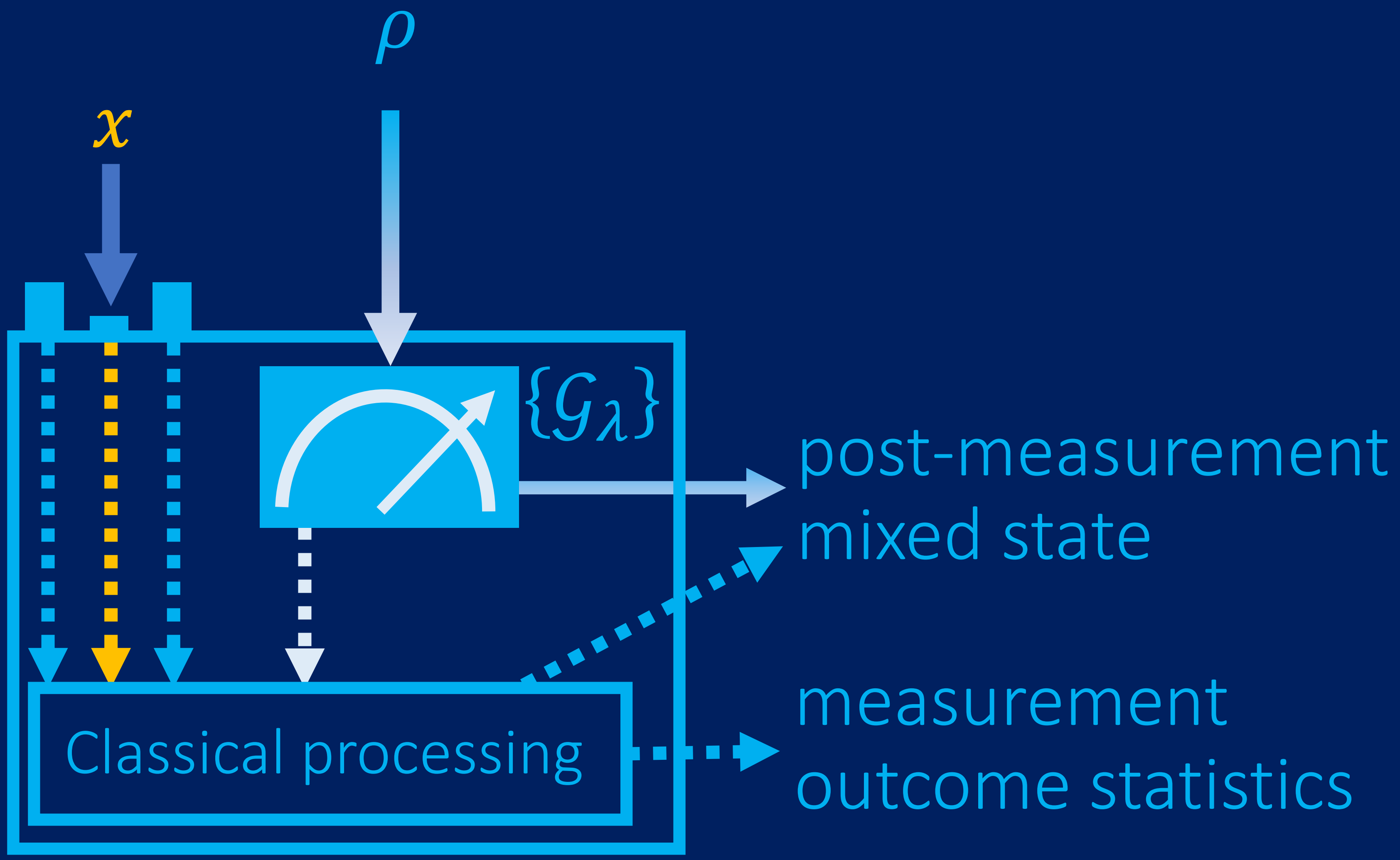


$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument



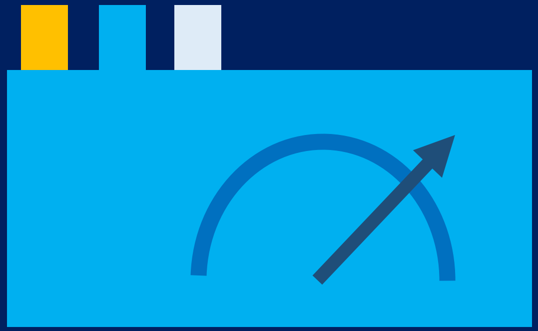
$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument

$$\mathcal{E}_{a|x} = \sum_{\lambda} P(a|x,\lambda)\mathcal{G}_{\lambda}$$

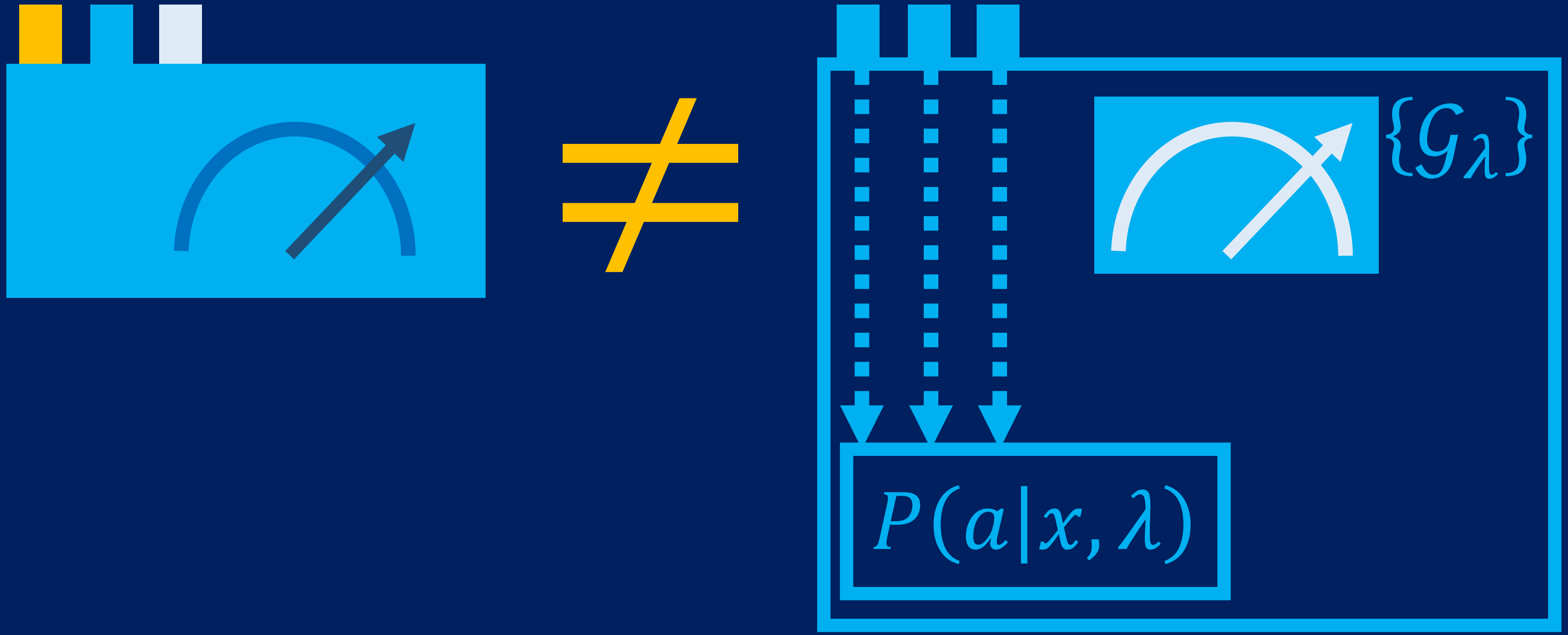


Compatible Instruments

Some instruments cannot be measured simultaneously!



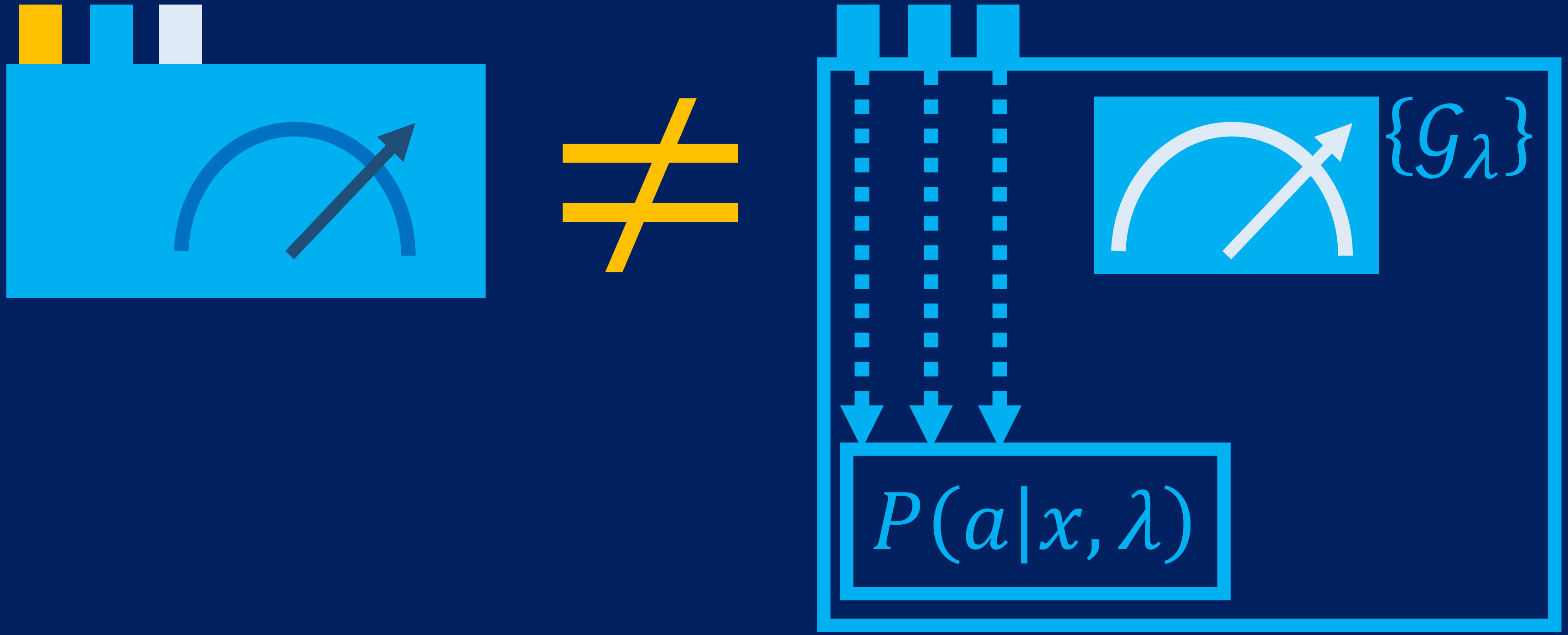
Some instruments cannot be measured simultaneously!



$$\sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda$$

Compatible Instruments

Some instruments cannot be measured simultaneously!



$$\sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda$$

Compatible Instruments

Incompatible Instruments

Thermodynamic Approach to Quantifying Incompatible Instruments

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Thermodynamic Approach to Quantifying Incompatible Instruments

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System


Bath
System

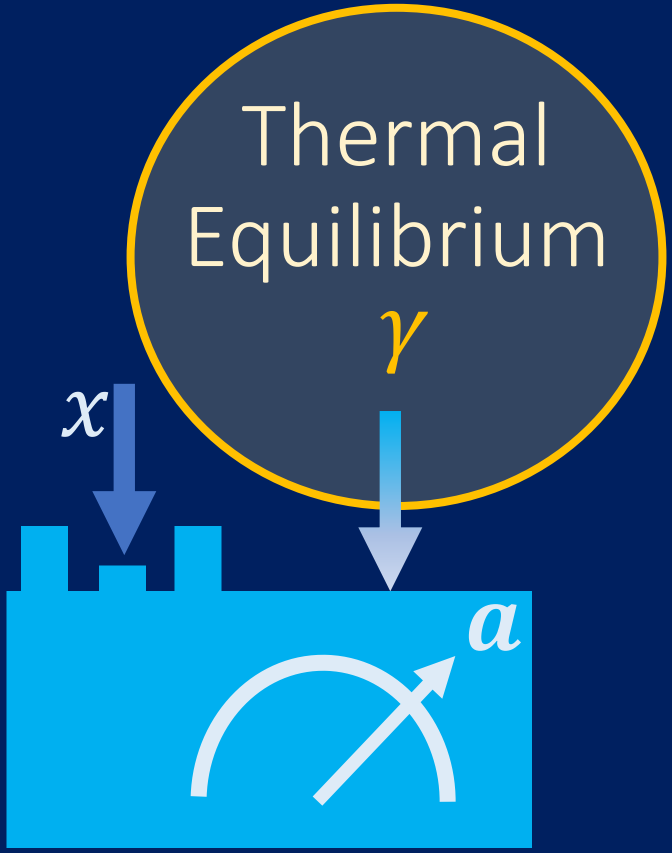
Thermal Equilibrium

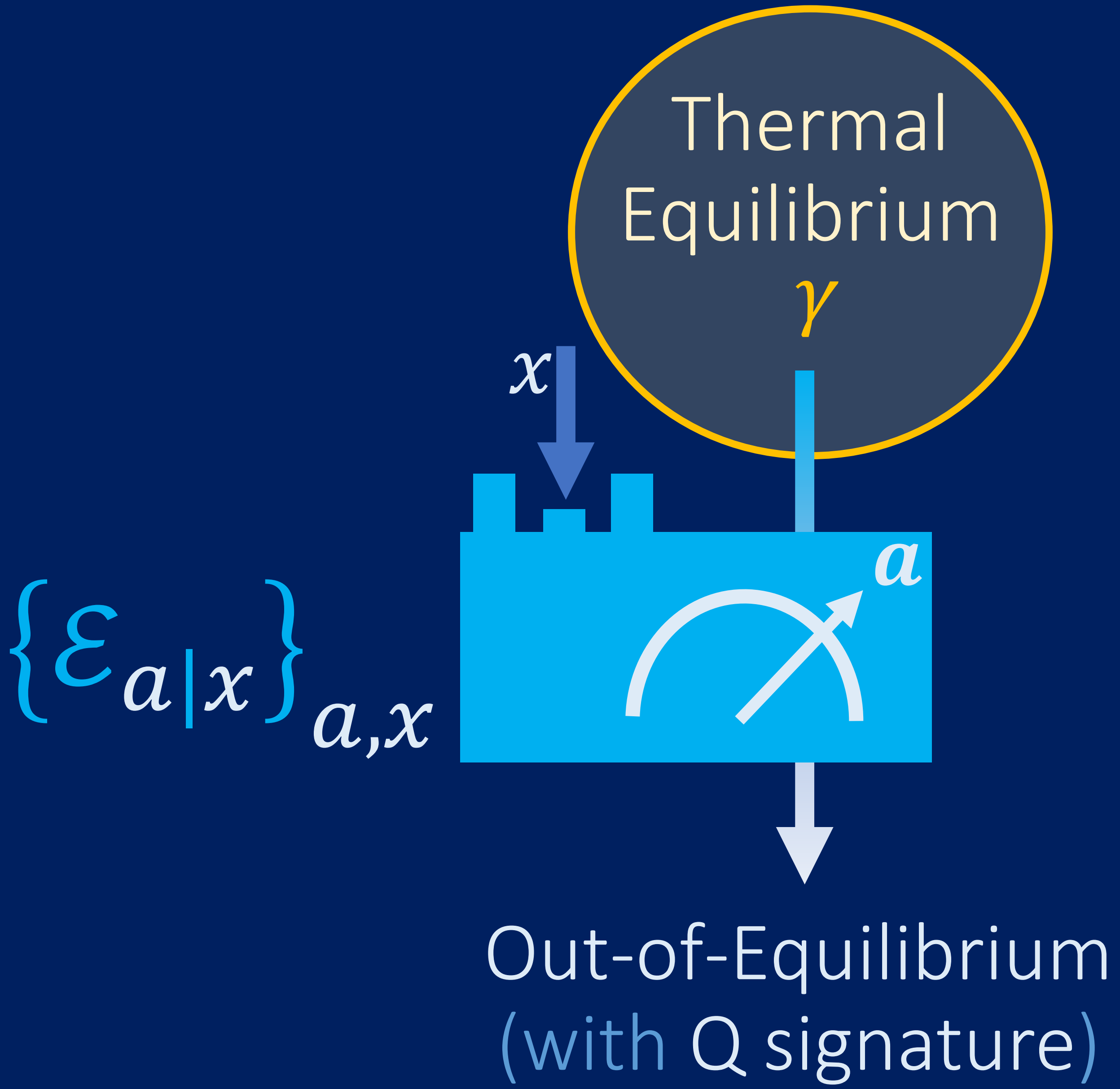
γ

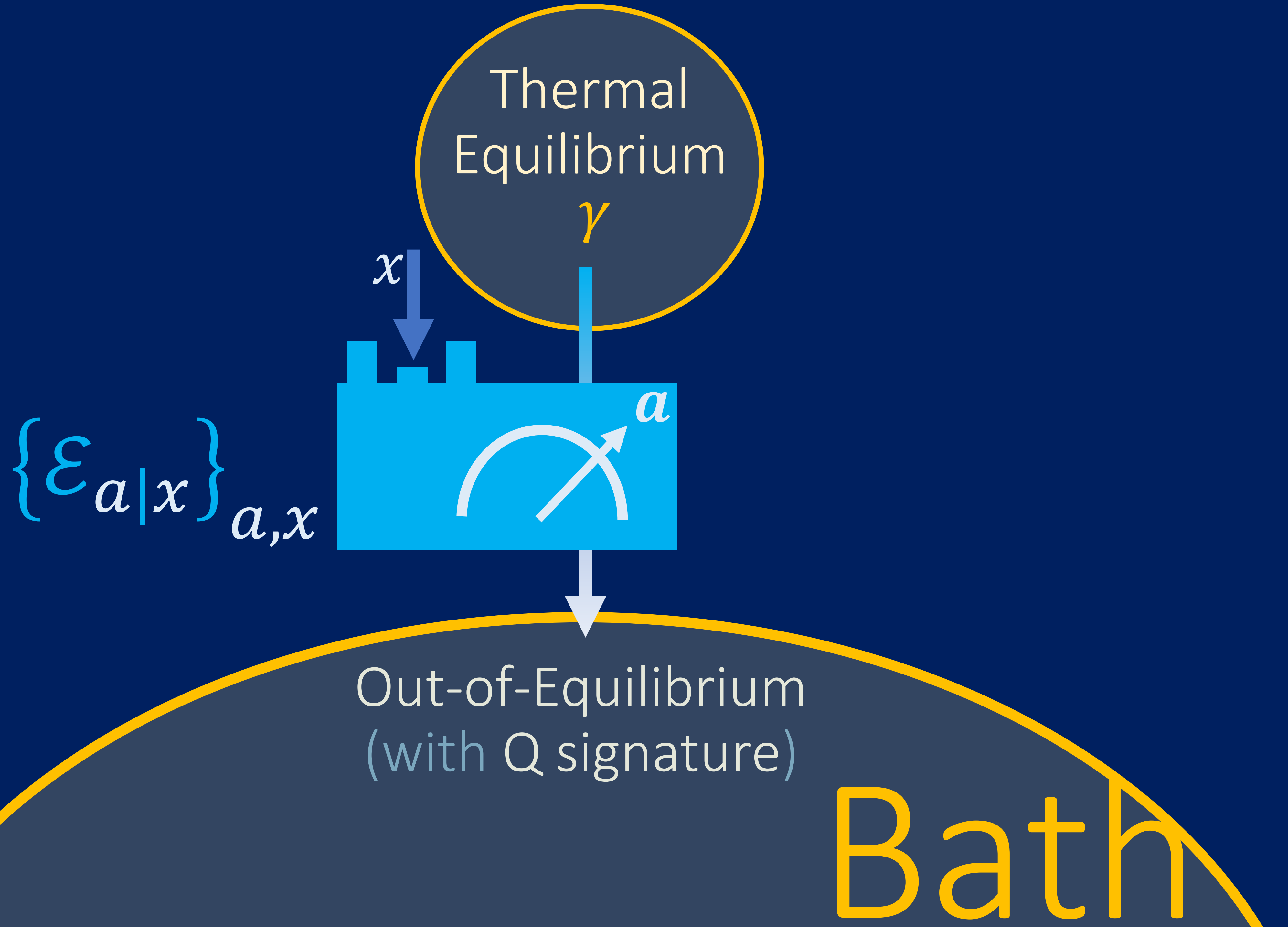
Thermal
Equilibrium
γ

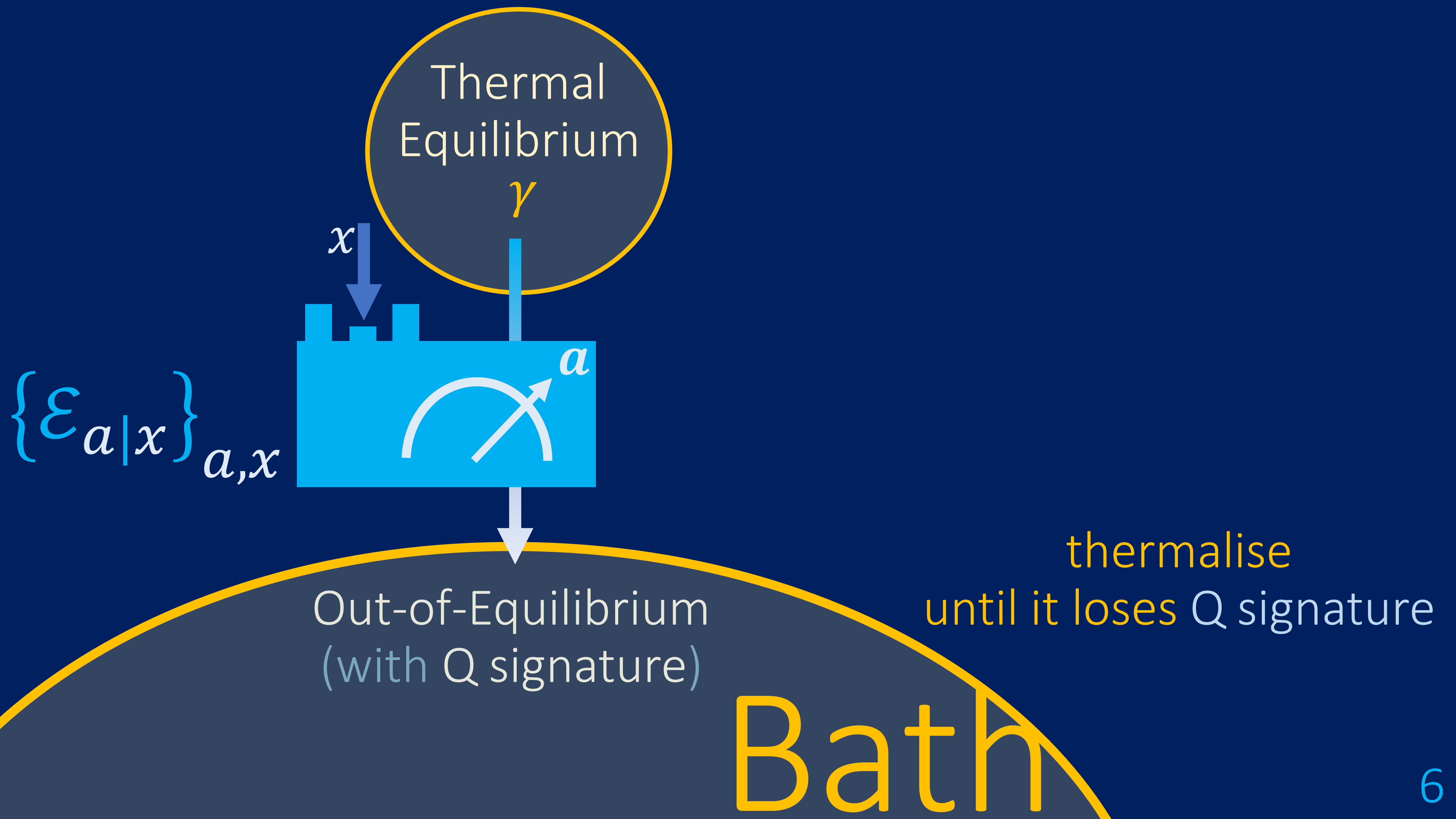
$\{\mathcal{E}_{a|x}\}_{a,x}$ 

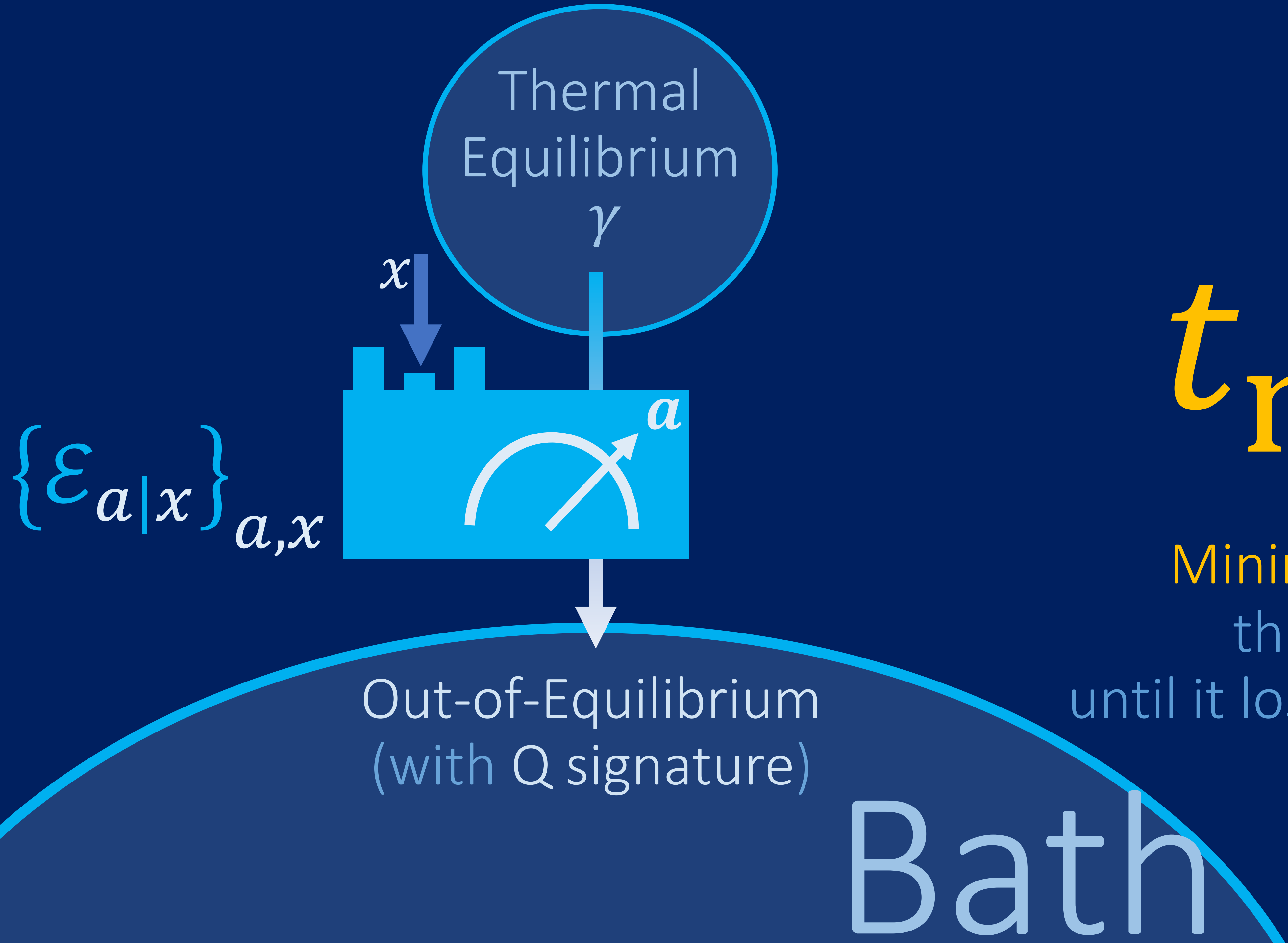
$\{\mathcal{E}_{a|x}\}_{a,x}$




Thermal
Equilibrium
γ
x
a
$\{\mathcal{E}_{a|x}\}_{a,x}$
Out-of-Equilibrium
(with Q signature)


Thermal
Equilibrium
γ
x
a
$\{\mathcal{E}_{a|x}\}_{a,x}$
Out-of-Equilibrium
(with Q signature)
Bath


Thermal
Equilibrium
γ
x
a
$\{\mathcal{E}_{a|x}\}_{a,x}$
Out-of-Equilibrium
(with Q signature)
thermalise
until it loses Q signature
Bath
6



$$t_{\min}$$

Minimal time to
thermalise
until it loses Q signature

How to compute $t_{\min}$

System


System Hamiltonian
H
Background Temperature
T

When system thermalises …

System Hamiltonian H

Background Temperature T

When system thermalises …

$$\rho \longrightarrow \mathcal{D}_t^{(h)}(\rho) = h(t)\rho + [1 - h(t)]\gamma$$

Thermalisation Model

h is strictly decreasing, $h(0) = 1, \lim_{t\to\infty} h(t) = 0$

$\gamma = e^{-H/k_B T}/\mathrm{tr}\left(e^{-H/k_B T}\right)$ is thermal state

$$t_{\min} := \min \; t \quad \text{s.t.} \quad \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)] = \sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda(\gamma)$$

Assumption $\sum_a \mathcal{E}_{a|x}(\gamma) = \gamma$

$$t_{\min} := \min \ t \quad \text{s.t.} \quad \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)] = \underbrace{\sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda(\gamma)}_{\text{LHS}}$$

$$\boxed{\text{Assumption} \quad \sum_a \mathcal{E}_{a|x}(\gamma) = \gamma}$$

$$t_{\min} := \min \ t \quad \text{s.t.} \quad \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)] = \underbrace{\textstyle\sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda(\gamma)}_{\text{LHS}}$$

$$\boxed{\text{Assumption} \quad \textstyle\sum_a \mathcal{E}_{a|x}(\gamma) = \gamma}$$

can be found by solving...

$$\boxed{\begin{aligned} &\min \quad \mu \geq 1 \\ &\text{s.t.} \quad \frac{\mathcal{E}_{a|x}(\gamma)+(\mu-1)\mathrm{tr}[\mathcal{E}_{a|x}(\gamma)]\gamma}{\mu} \in \text{LHS} \end{aligned}}$$

R. Uola , A. C.S. Costa , H. C. Nguyen , O. Gühne, *Rev. Mod. Phys.* **92**, 15001 (2020).

$$t_{\min} := \min\ t \quad \text{s.t.}\ \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)] = \underbrace{\textstyle\sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda(\gamma)}_{\text{LHS}}$$

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can be found by solving...

$$\boxed{\min\ \mu \geq 1 \quad \text{s.t.}\ \frac{\mathcal{E}_{a|x}(\gamma)+(\mu-1)\mathrm{tr}[\mathcal{E}_{a|x}(\gamma)]\gamma}{\mu} \in \text{LHS}}$$

A robustness measure

$$t_{\min} := \min \; t \quad \text{s.t.} \; \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)] = \underbrace{\textstyle\sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda(\gamma)}_{\text{LHS}}$$

$$\boxed{\text{Assumption} \quad \textstyle\sum_a \mathcal{E}_{a|x}(\gamma) = \gamma}$$

can be found by solving...

$$\boxed{\begin{aligned} &\min \quad \mu \geq 1 \\ &\text{s.t.} \quad \frac{\mathcal{E}_{a|x}(\gamma)+(\mu-1)\mathrm{tr}[\mathcal{E}_{a|x}(\gamma)]\gamma}{\mu} \in \text{LHS} \end{aligned}}$$

A robustness measure

A semi-definite program (SDP)

How to measure t_{min}

$$t_{\min}$$

$$t_{\min} \sim W$$

Score of a
work extraction game

$$t_{\min} \sim W$$

Score of a
work extraction game

We can use energy extraction experiments
to obtain time value $t_{\min}$

Summary

Thermodynamically Quantifying Uncertainty Principle's Signature

Thermodynamic Approach to Quantifying Incompatible Instruments

Thermalisation time needed to remove Q signature

$$t_{\mathrm{min}}$$

Thermalisation time needed to remove Q signature

$$t_{\min}$$

A robustness & semi-definite program (SDP)

Thermalisation time needed to remove Q signature

$$t_{\min}$$

A robustness & semi-definite program (SDP)

Can be obtained in energy extraction experiments

Thermalisation time needed to remove Q signature

$$t_{\mathrm{min}}$$

A robustness & semi-definite program (SDP)

Can be obtained in energy extraction experiments

Induce a resource theory of incompatible instruments

Thermalisation time needed to remove Q signature

$$t_{\mathrm{min}}$$

A robustness & semi-definite program (SDP)

Can be obtained in energy extraction experiments

Induce a resource theory of incompatible instruments

C.-Y. Hsieh, S.-L. Chen, *Phys. Rev. Lett.* **133**, 170401 (2024)

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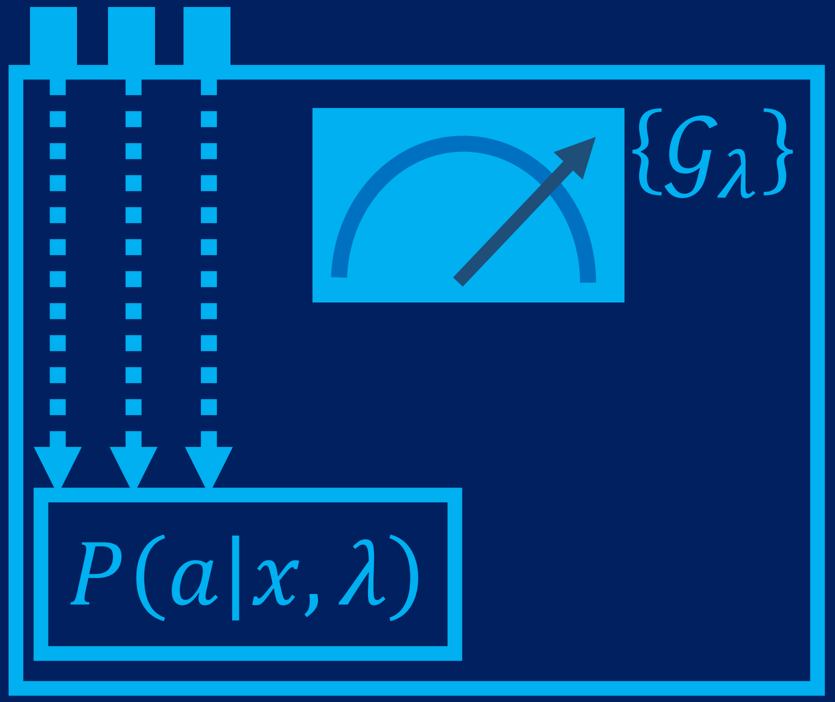
Appendix
Compatible Instruments

Technical Details

$\{\mathcal{G}_\lambda\}$

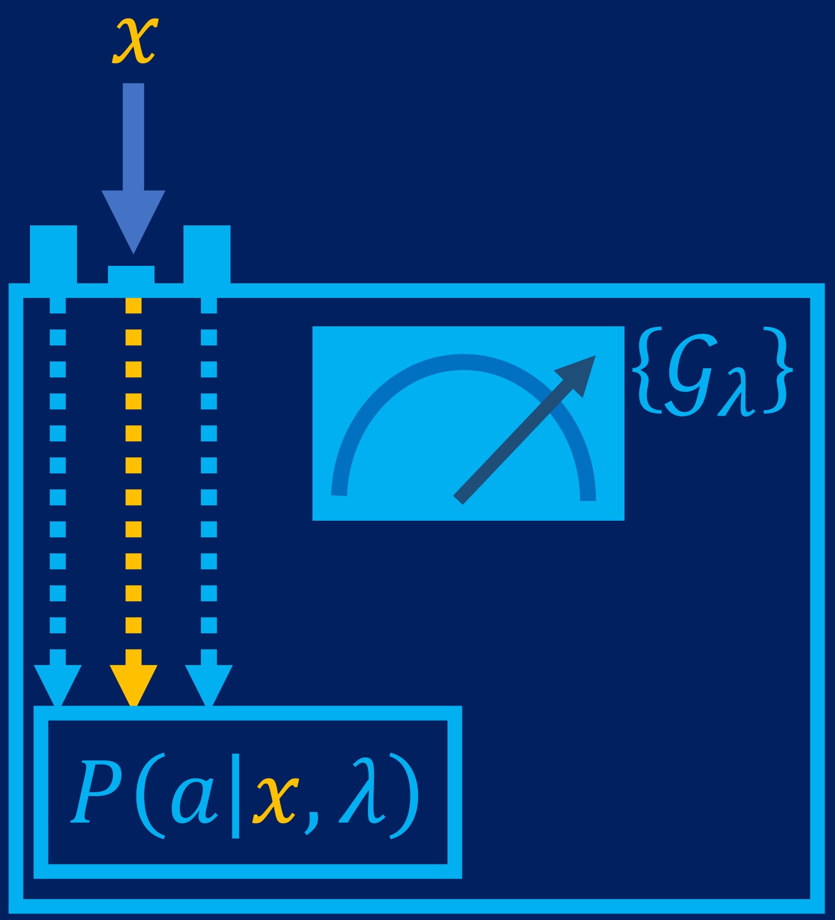
$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument

Technical Details



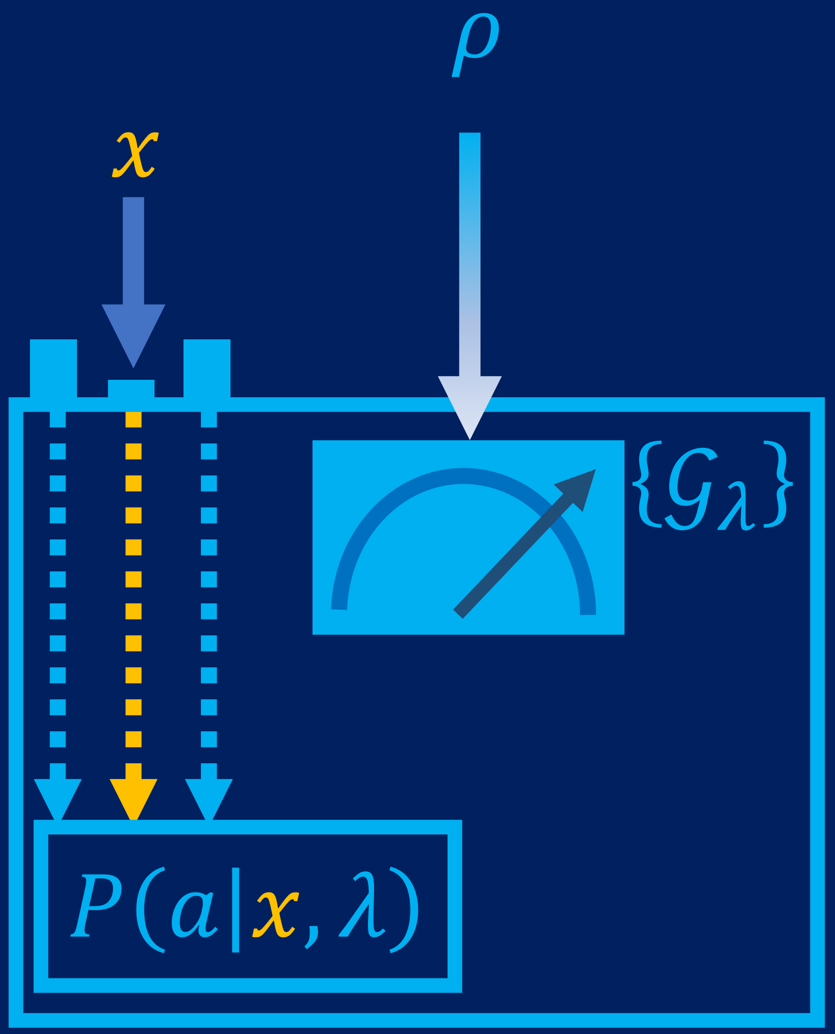
$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument

Technical Details



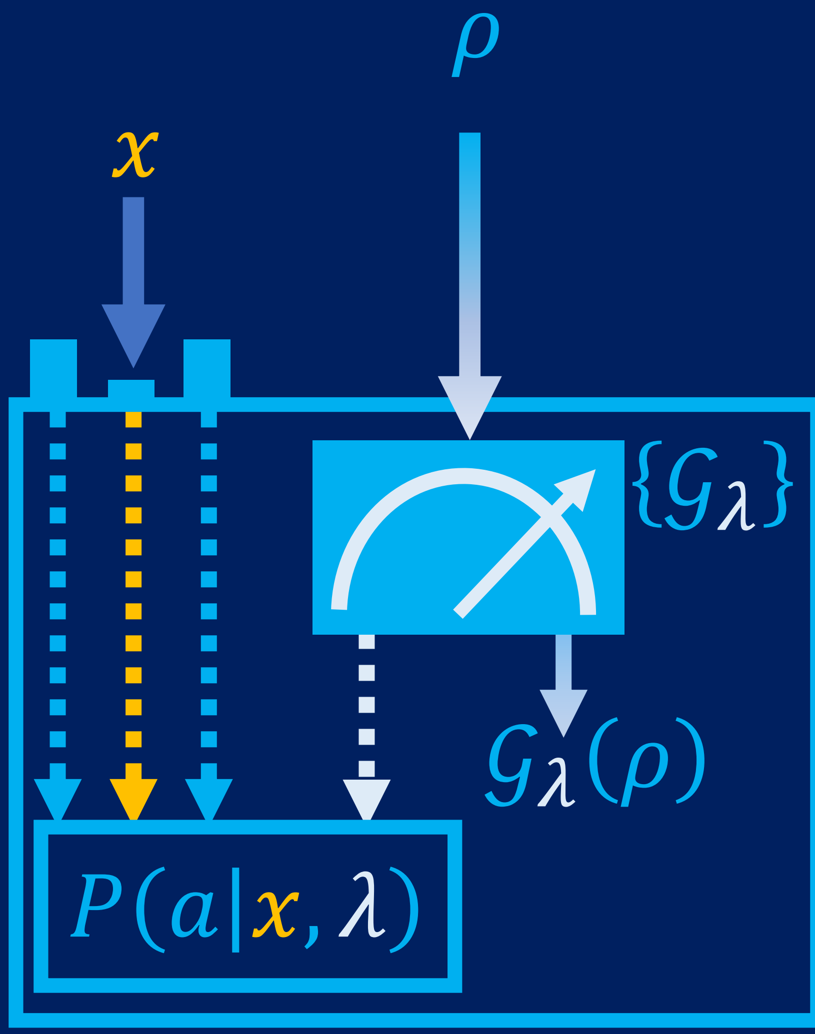
$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument

Technical Details



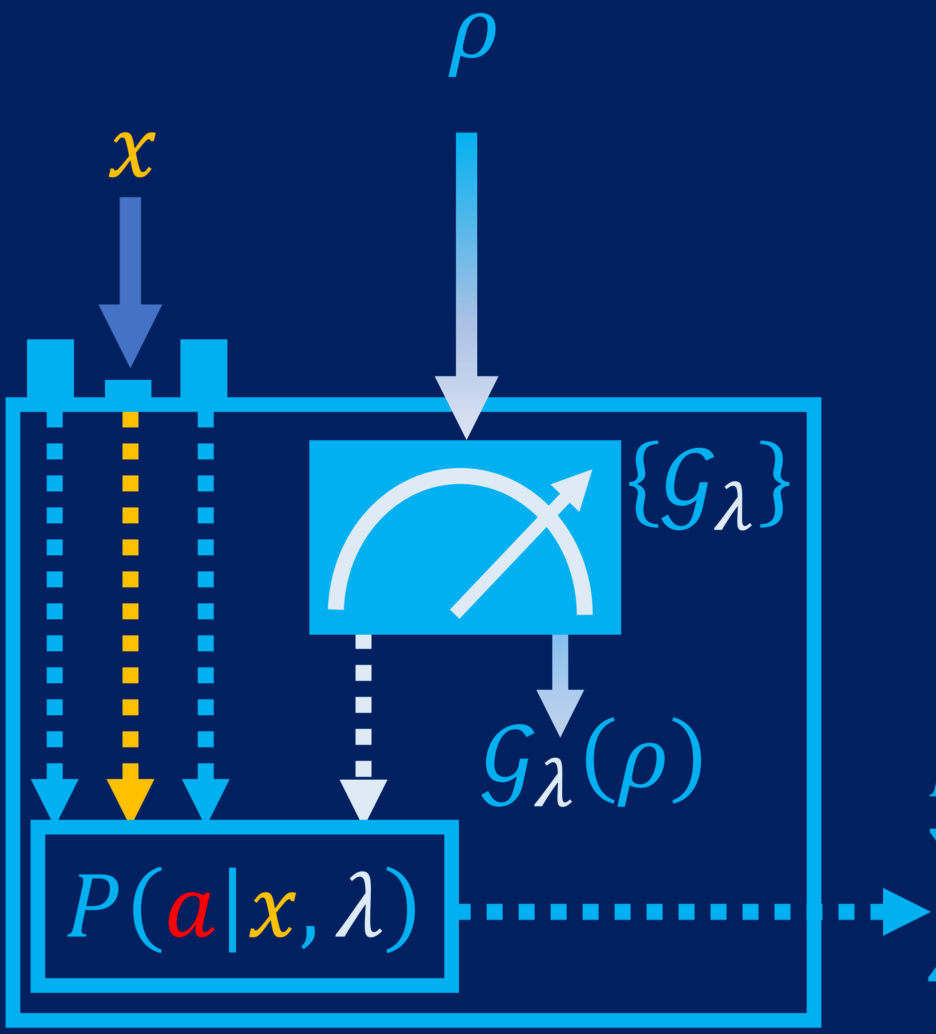
$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument

Technical Details



$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument

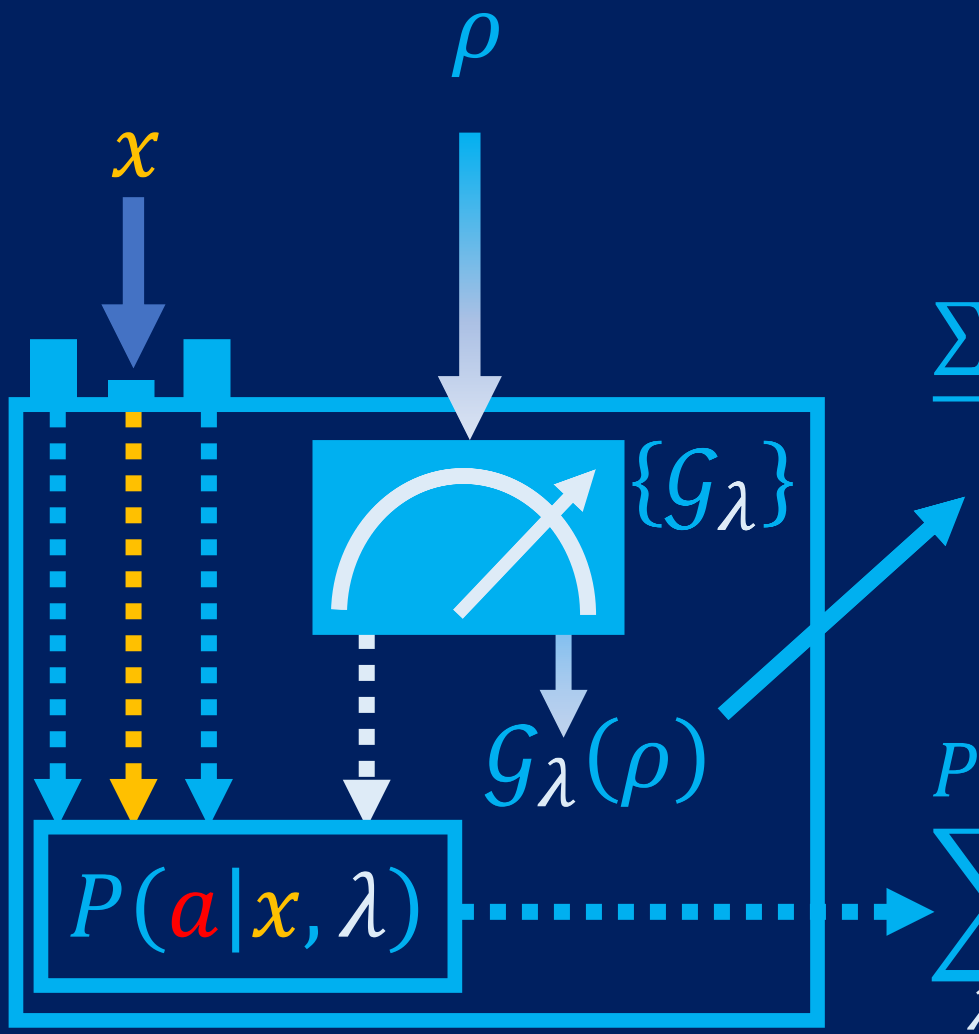
Technical Details



$$P(a|x) = \sum_\lambda P(a|x,\lambda)\mathrm{tr}[\mathcal{G}_\lambda(\rho)]$$

$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument

Technical Details



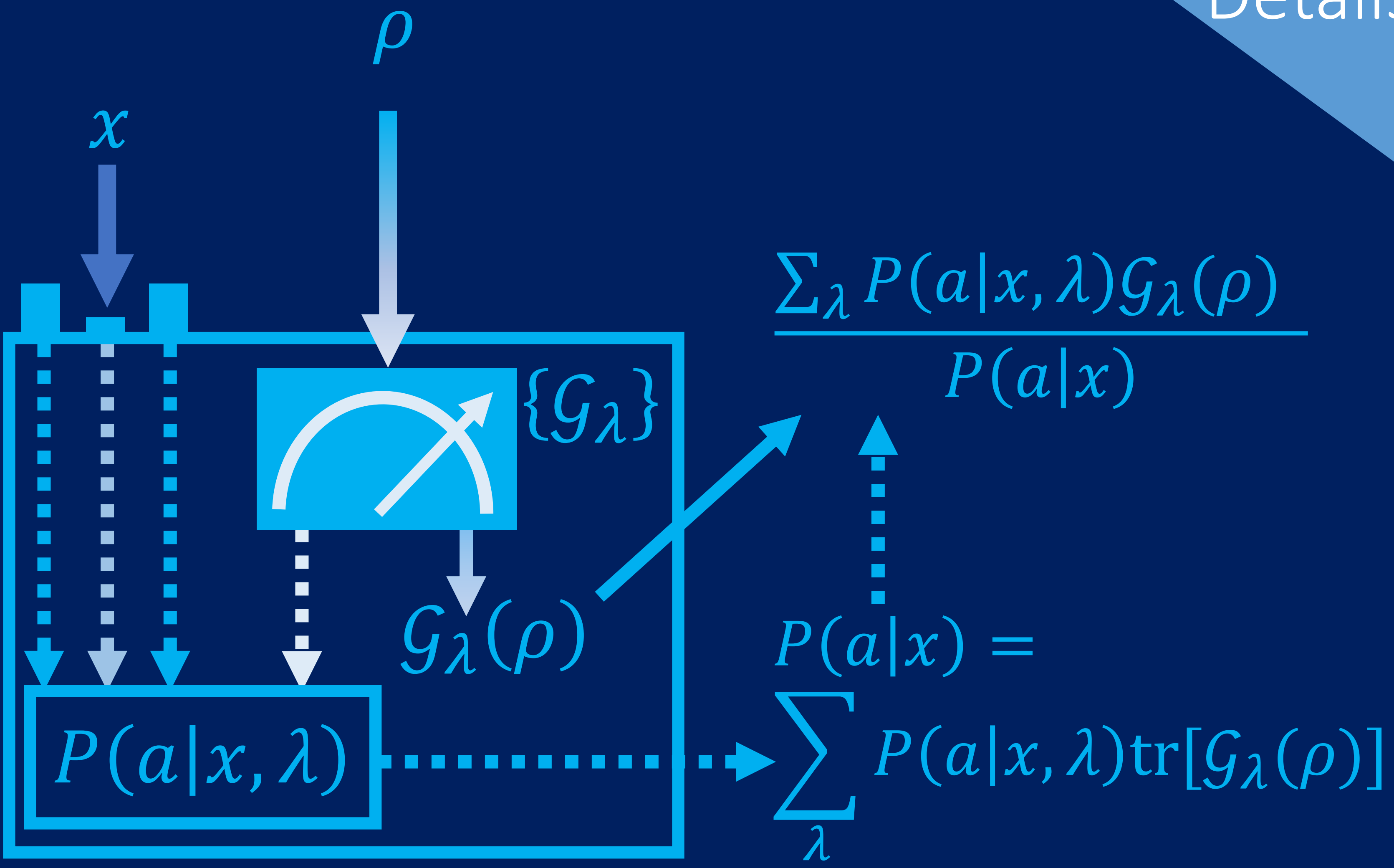
$$\frac{\sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda(\rho)}{P(a|x)}$$

$$P(a|x) = \sum_\lambda P(a|x,\lambda)\mathrm{tr}[\mathcal{G}_\lambda(\rho)]$$

$\{\mathcal{G}_\lambda\}_\lambda$: a single instrument

Technical Details

$$\mathcal{E}_{a|x} = \sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda$$

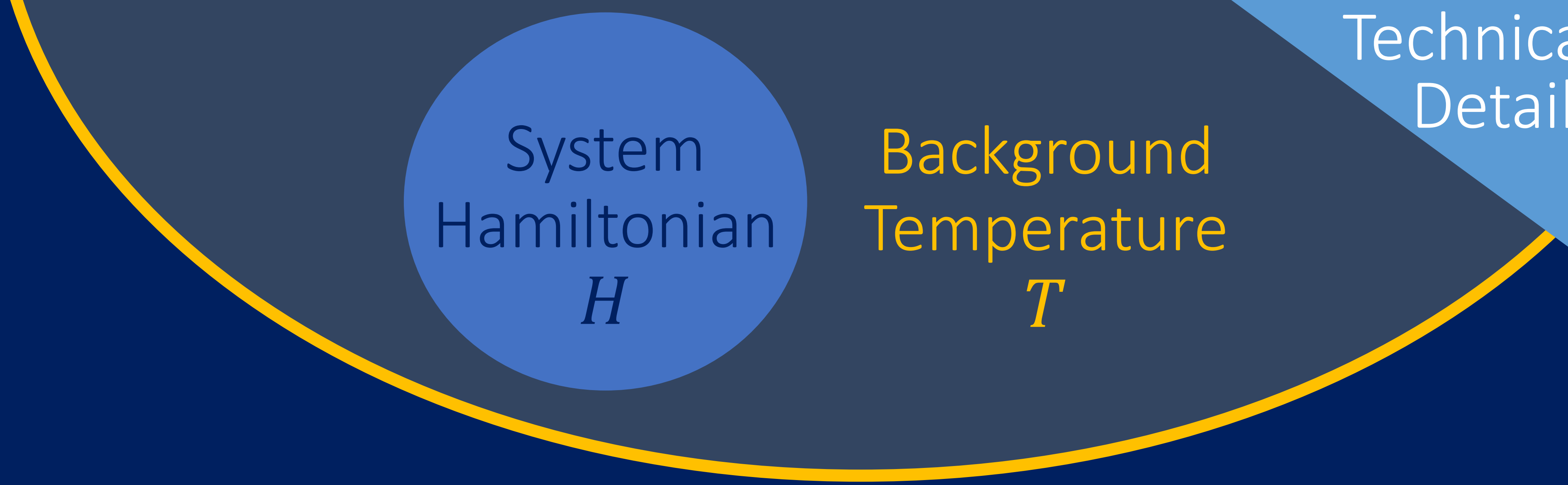


Compatible Instruments

Appendix
Quantifying Incompatible Instruments by Thermalisation

Technical Details

System



Technical Details

Technical Details

When system thermalises …

Technical Details

System Hamiltonian H

Background Temperature T

When system thermalises …

$$\rho \longrightarrow \mathcal{D}_t^{(h)}(\rho) = h(t)\rho + [1 - h(t)]\gamma$$

Thermalisation Model

h is strictly decreasing, $h(0) = 1, \lim_{t\to\infty} h(t) = 0$

At $t = \infty$ it reaches

Thermal Equilibrium

$$\gamma = \frac{\left(e^{-H/k_B T}\right)}{\mathrm{tr}\left(e^{-H/k_B T}\right)}$$

Technical Details

System

Technical Details

Thermal Equilibrium
γ

Technical Details

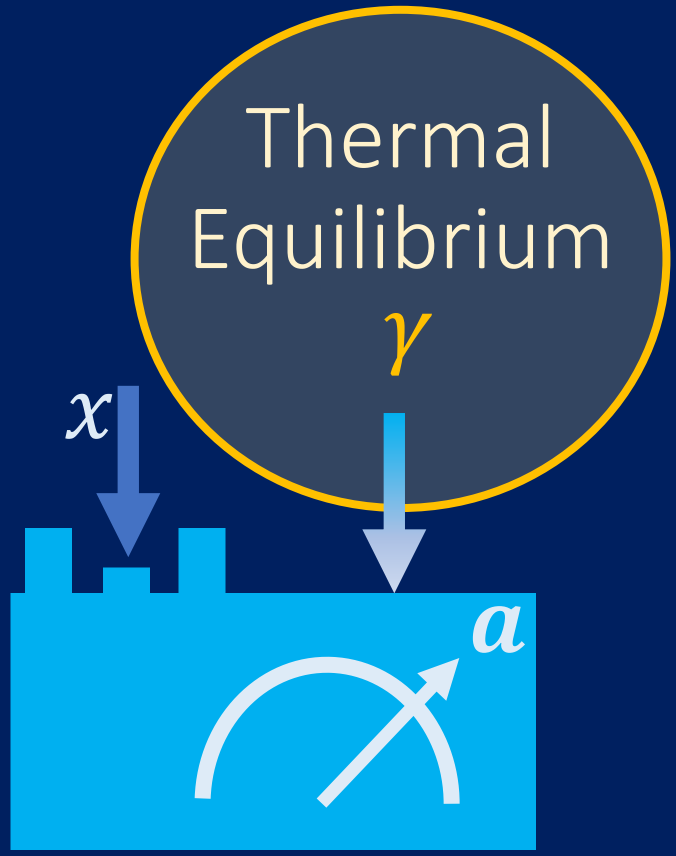
Thermal Equilibrium

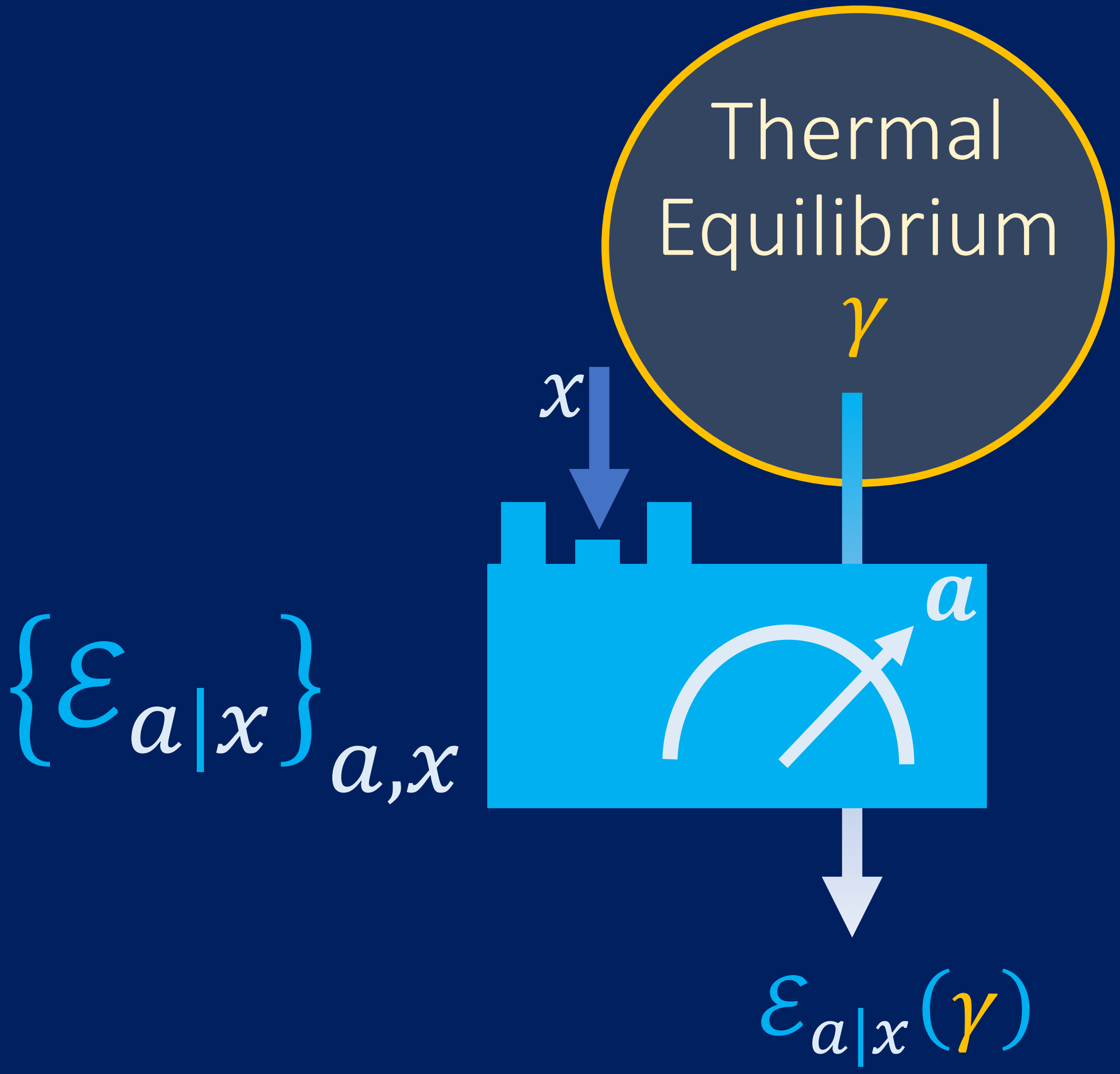
γ

$\{\mathcal{E}_{a|x}\}_{a,x}$

Technical Details

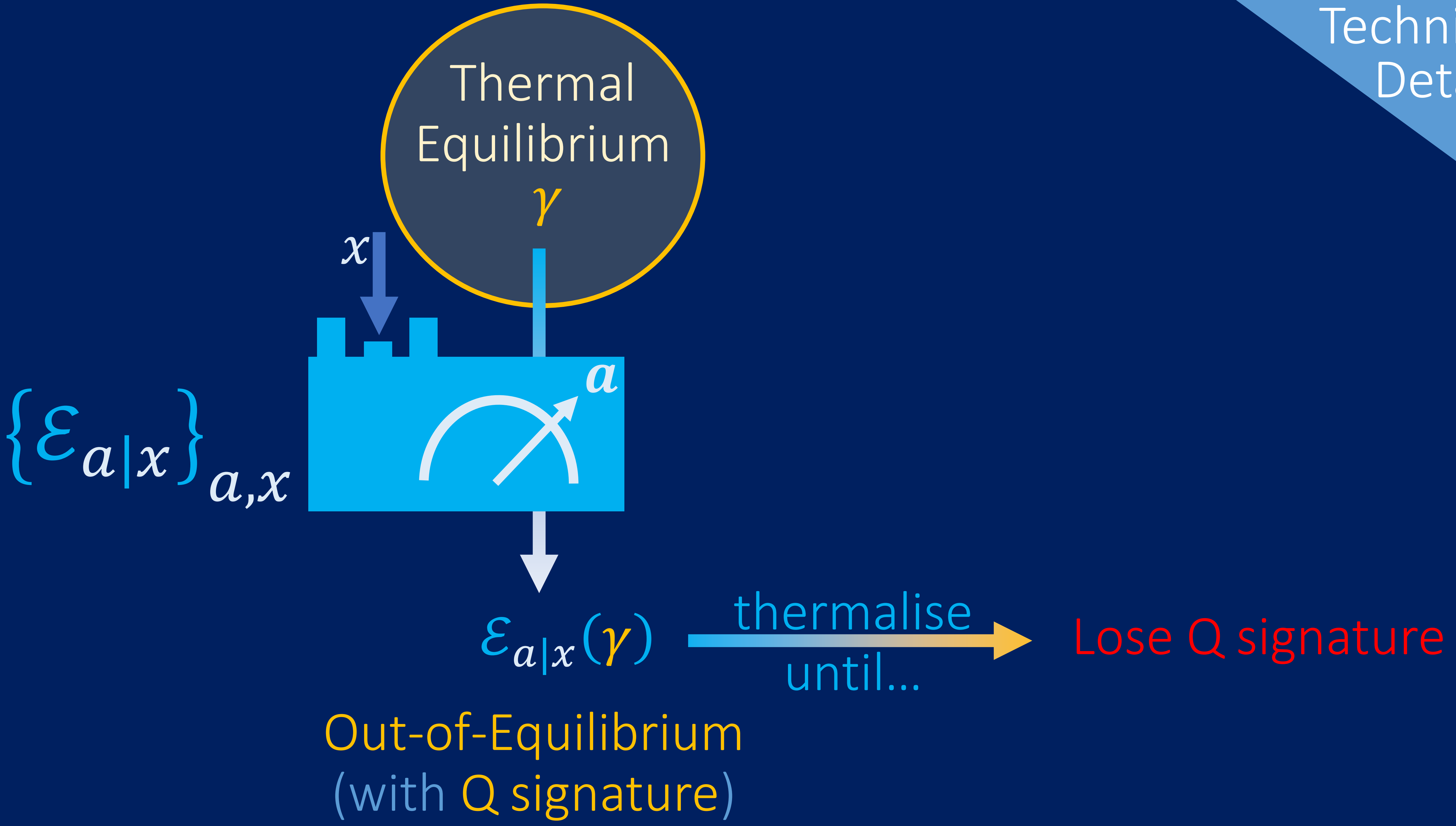
$$\{\mathcal{E}_{a|x}\}_{a,x}$$



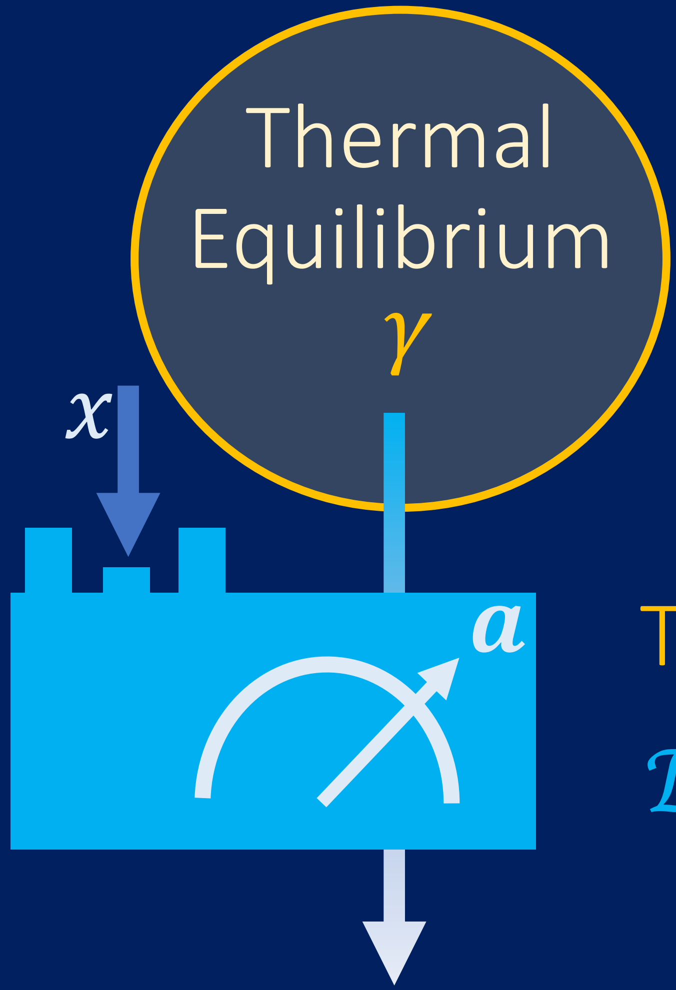


Out-of-Equilibrium
(with Q signature)

Technical Details



$$\{\mathcal{E}_{a|x}\}_{a,x}$$



Thermalisation Model

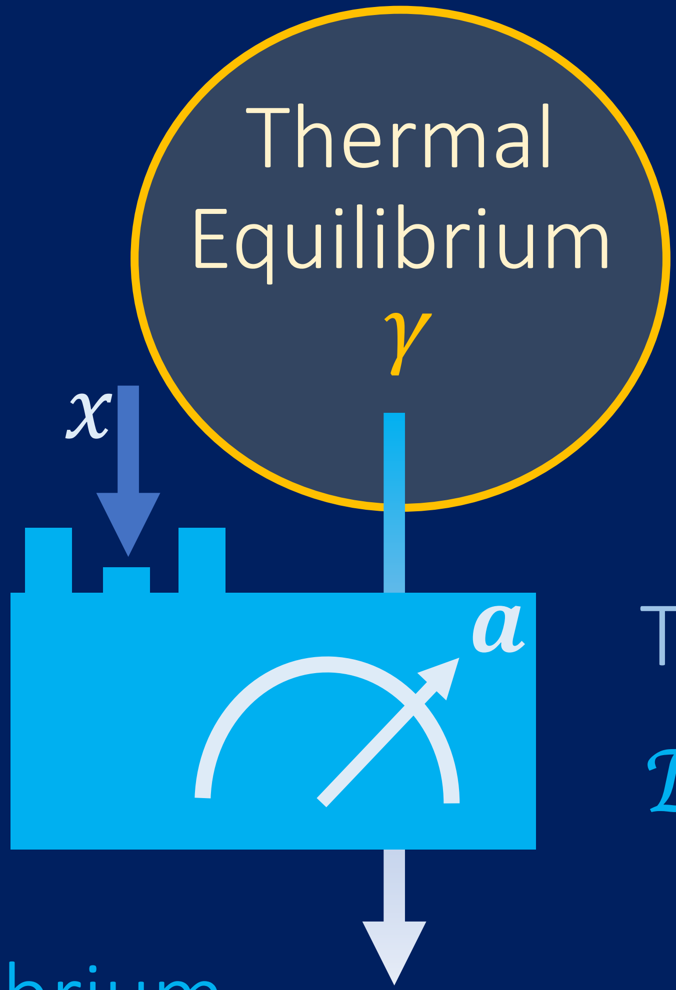
$$\mathcal{D}_t^{(h)}(\rho) = h(t)\rho + [1 - h(t)]\gamma$$

$$\mathcal{E}_{a|x}(\gamma) \xrightarrow{\text{thermalise}} \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)]$$

Out-of-Equilibrium
(with Q signature)

Technical Details

$\{\mathcal{E}_{a|x}\}_{a,x}$



Thermalisation Model

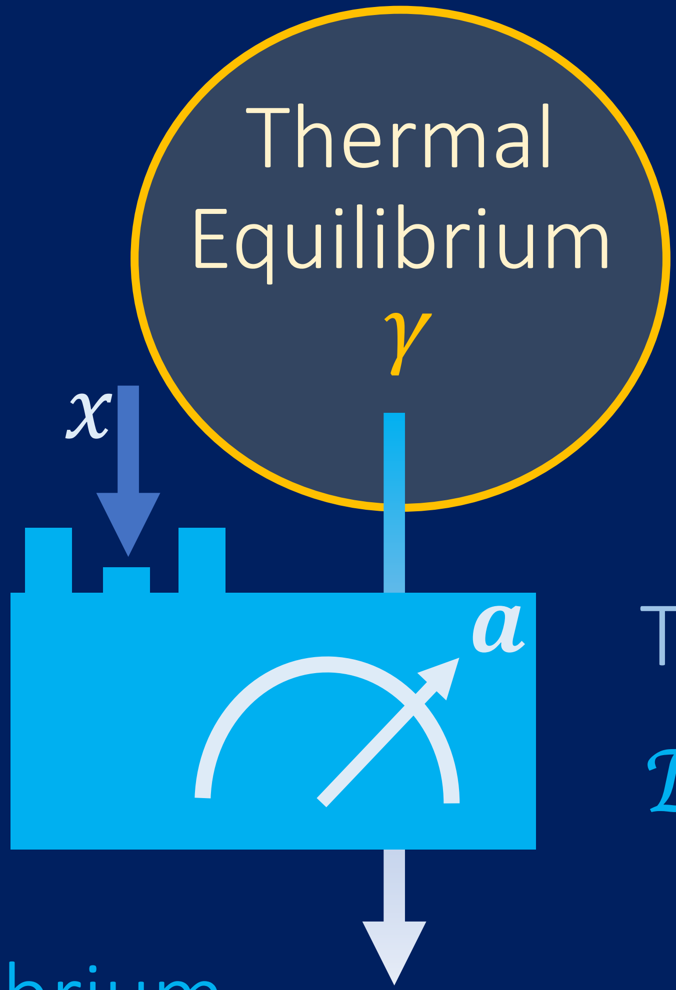
$$\mathcal{D}_t^{(h)}(\rho) = h(t)\rho + [1 - h(t)]\gamma$$

Out-of-Equilibrium (with Q signature)

$$\mathcal{E}_{a|x}(\gamma) \xrightarrow{\text{thermalise}} \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)]$$

$$t_{\min}(\{\mathcal{E}_{a|x}\}) := \min \quad t$$

$$\text{s.t.} \quad \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)] = \sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda(\gamma)$$



$\{\mathcal{E}_{a|x}\}_{a,x}$

Assumption

$$\sum_a \mathcal{E}_{a|x}(\gamma) = \gamma$$

Thermalisation Model

$$\mathcal{D}_t^{(h)}(\rho) = h(t)\rho + [1 - h(t)]\gamma$$

Out-of-Equilibrium (with Q signature)

$$\mathcal{E}_{a|x}(\gamma) \xrightarrow{\text{thermalise}} \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)]$$

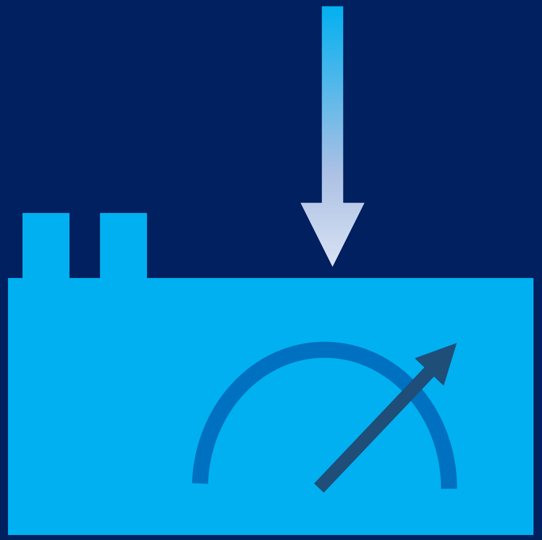
$$t_{\min}(\{\mathcal{E}_{a|x}\}) \coloneqq \min \quad t$$

$$\text{s.t.} \quad \mathcal{D}_t^{(h)}[\mathcal{E}_{a|x}(\gamma)] = \sum_\lambda P(a|x,\lambda)\mathcal{G}_\lambda(\gamma)$$

Appendix
Work Extraction Game for Incompatible Instruments

Technical Details

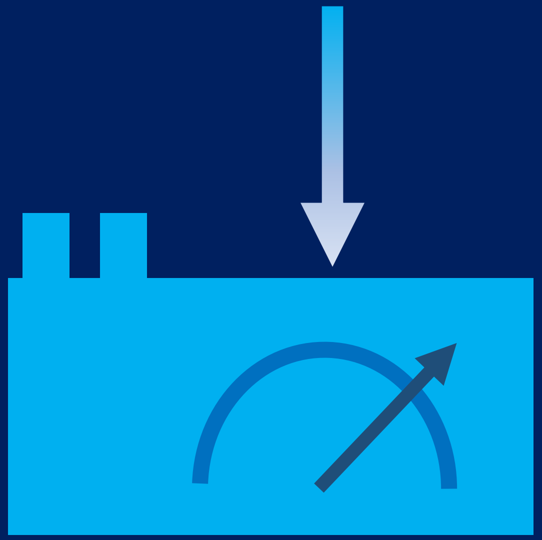
$$\frac{\mathbb{I}}{2}$$ (maximally mixed)



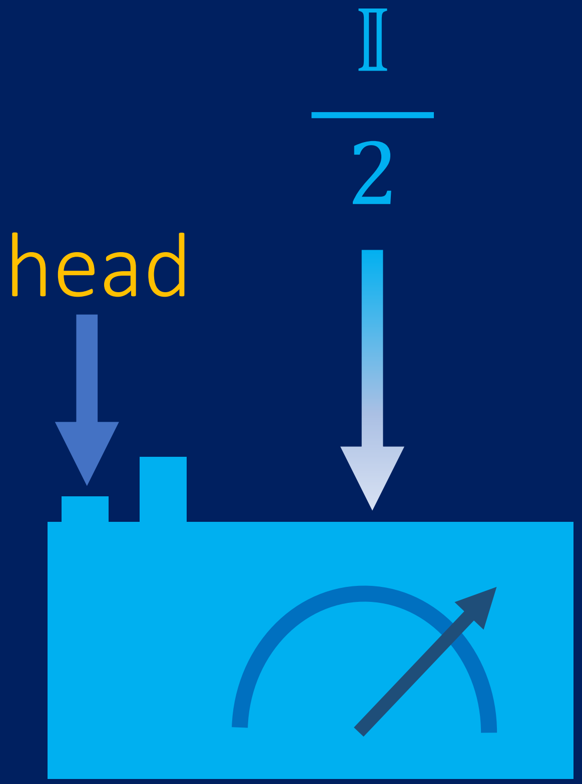
Technical Details

$$\frac{\mathbb{I}}{2}$$

Flip a fair coin...



Flip a fair coin...

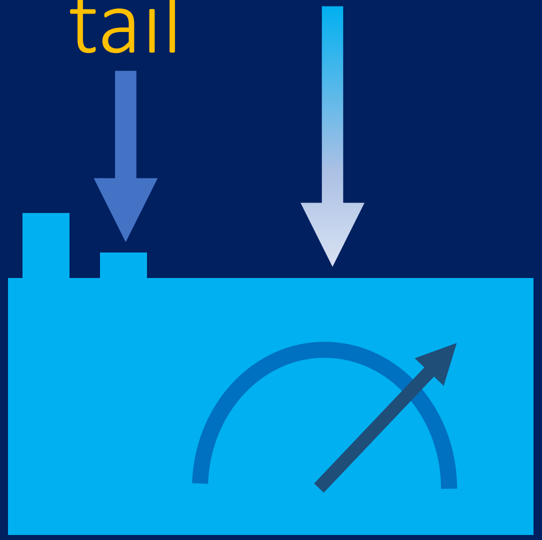


measure x-directional spin

Pauli X $|+\rangle\langle+|$ $|-\rangle\langle-|$

Technical Details

Flip a fair coin...



measure z-directional spin

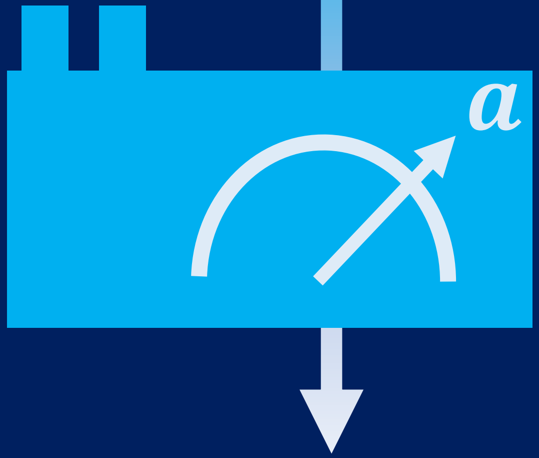
Pauli Z $|0\rangle\langle 0|$ $|1\rangle\langle 1|$

Technical Details

$$\frac{\mathbb{I}}{2}$$

head/tail

Flip a fair coin...



measure x/z-direction spin

Pauli X/Z

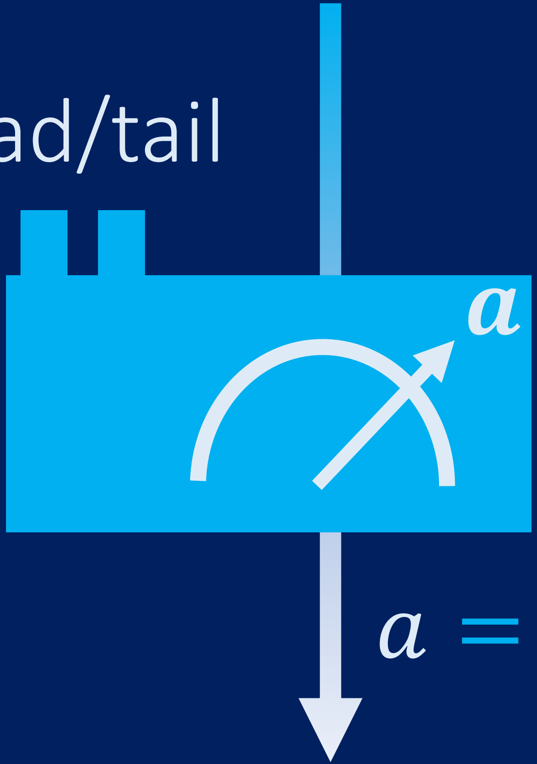
50% chance a = spin up

50% chance a = spin down

$\frac{\mathbb{I}}{2}$

head/tail

Flip a fair coin...



measure x/z-direction spin

Pauli X/Z

a = spin up/down

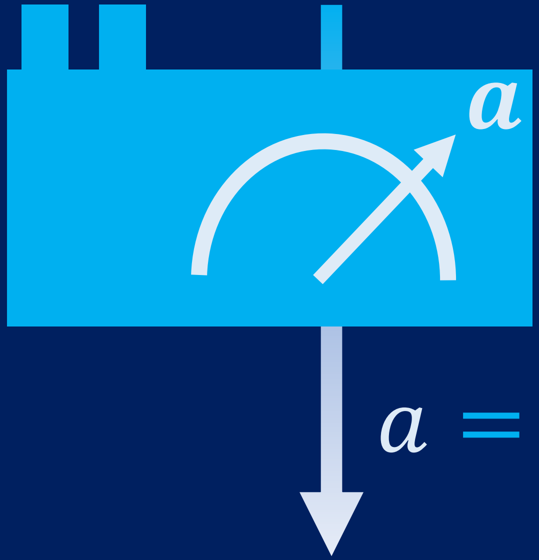
Work extraction task, extracting W_{Incomp} energy

Technical Details

W_{comp} highest extractible energy from compatible measurements

Flip a fair coin...

head/tail $\mathbb{I}/2$



measure x/z-direction spin

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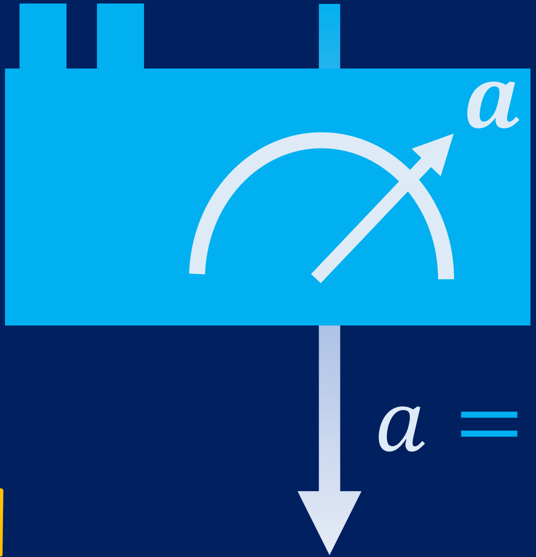
Work extraction task, extracting W_{Incomp} energy

C.-Y. Hsieh, M. Gessner, arXiv:2403.18753

C.-Y. Hsieh, S.-L. Chen, *Phys. Rev. Lett.* **133**, 170401 (2024)

W_{comp} highest extractible energy from compatible measurements

head/tail $\mathbb{I}/2$



measure x/z-direction spin

Pauli X/Z

a = spin up/down

$W_{\text{Incomp}} > W_{\text{comp}}$

Advantage in extracting energy!

Work extraction task, extracting W_{Incomp} energy

C.-Y. Hsieh, M. Gessner, arXiv:2403.18753
C.-Y. Hsieh, S.-L. Chen, *Phys. Rev. Lett.* **133**, 170401 (2024)