Quantum theory of dark matter scattering

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Based on
AK, Hee Jung Kim and Takumi Kuwahara, JHEP, 2020
AK, Takumi Kuwahara and Ami Patel, JHEP, 2023
AK, Shigeki Matsumoto and Yuki Watanabe, in progress

June 19, 2025 @ Valencia Workshop

Contents

Long-range force

- self-scattering and Sommerfeld enhancement
- tight correlation

Scattering theory in quantum mechanics

- Jost function
- Omnès solution
- violation of partial-wave Unitarity on zero-energy resonances

Unitarization

- contact interaction
- bound state with decay width

Long-range force

Light mediator

talk by Xiaoyong Chu

- mediator lighter than the dark matter
- electroweak-scale or lighter dark matter
  - new dark force (e.g., dark photon)
- TeV-scale dark matter (e.g., weak multiplet)
  - weak force

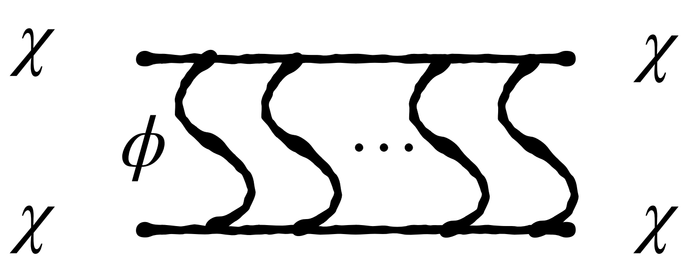
$$V = -\frac{\alpha_\chi}{r} e^{-m_\phi r}$$

- (attractive) Yukawa potential

Self-scattering

- velocity dependent and large scattering cross section

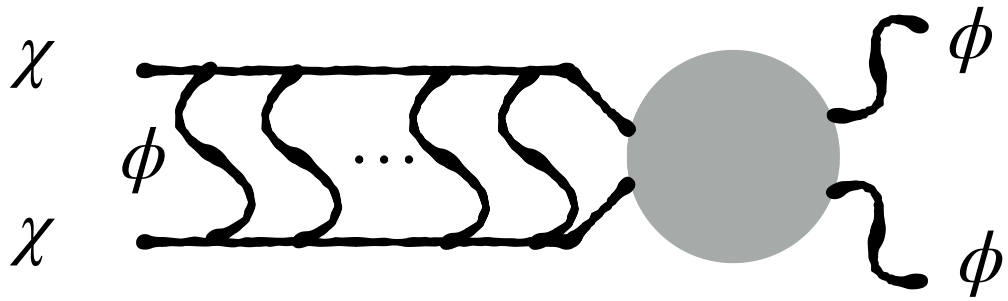
- non-perturbative (infinite exchanges of a mediator) when the distortion of wave function is significant



Sommerfeld enhancement

Enhanced annihilation

- enlarge probability of finding two particles at the same position



- annihilation cross section is enhanced

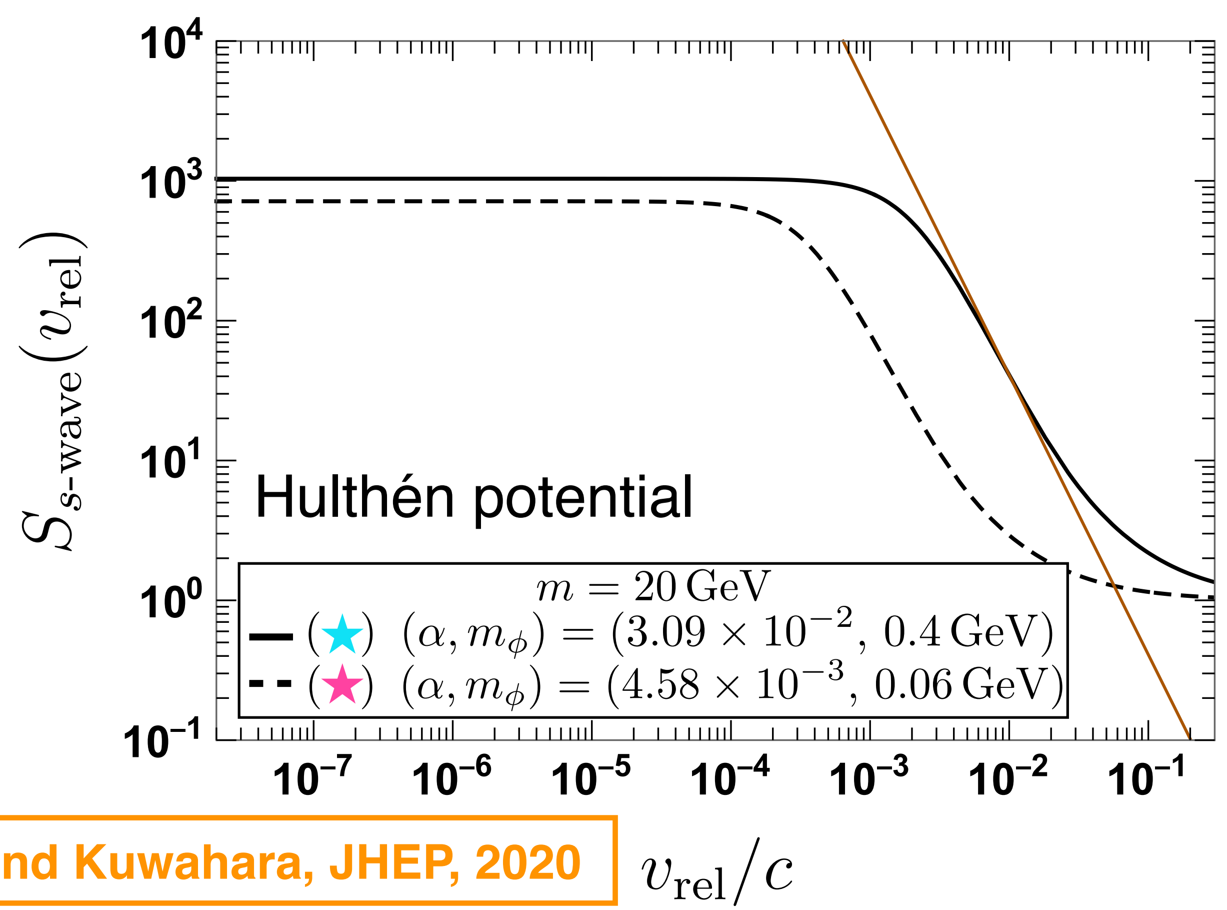
$$(\sigma_{\rm ann} v_{\rm rel}) = S(\sigma^{(0)}_{\rm ann} v_{\rm rel})$$

- without potential

- Sommerfeld enhancement factor

- velocity dependent

- larger cross section in the late Universe than the thermal one



AK, Kim and Kuwahara, JHEP, 2020

Indirect detection

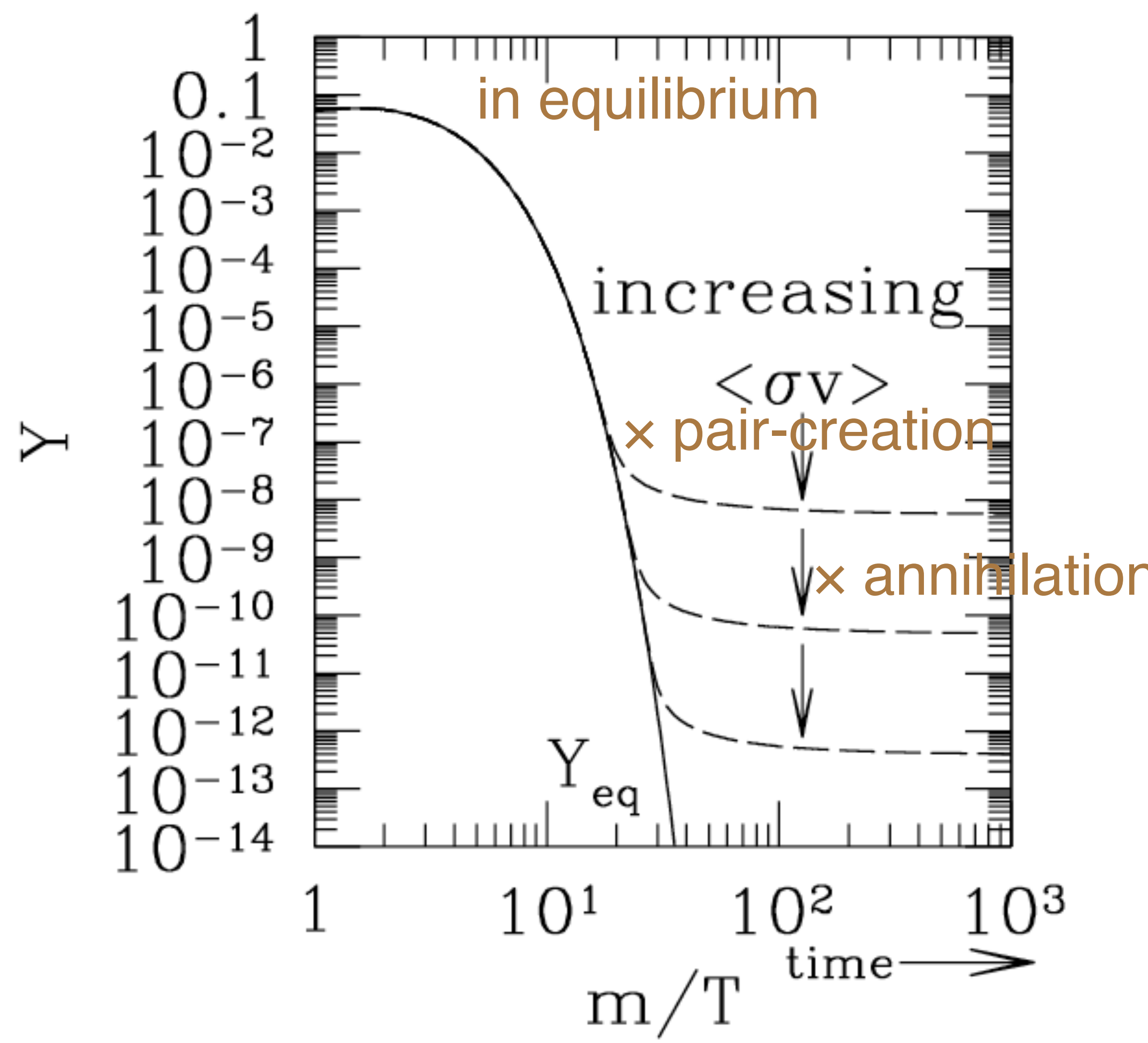
Canonical cross section

- thermal freeze-out (annihilation in the early Universe) $v_{\rm rel} \simeq 1/2$

$$\Omega h^2 = 0.1 \times \frac{3\times 10^{-26}\,{\rm cm^3/s}}{\langle \sigma_{\rm ann} v\rangle}$$

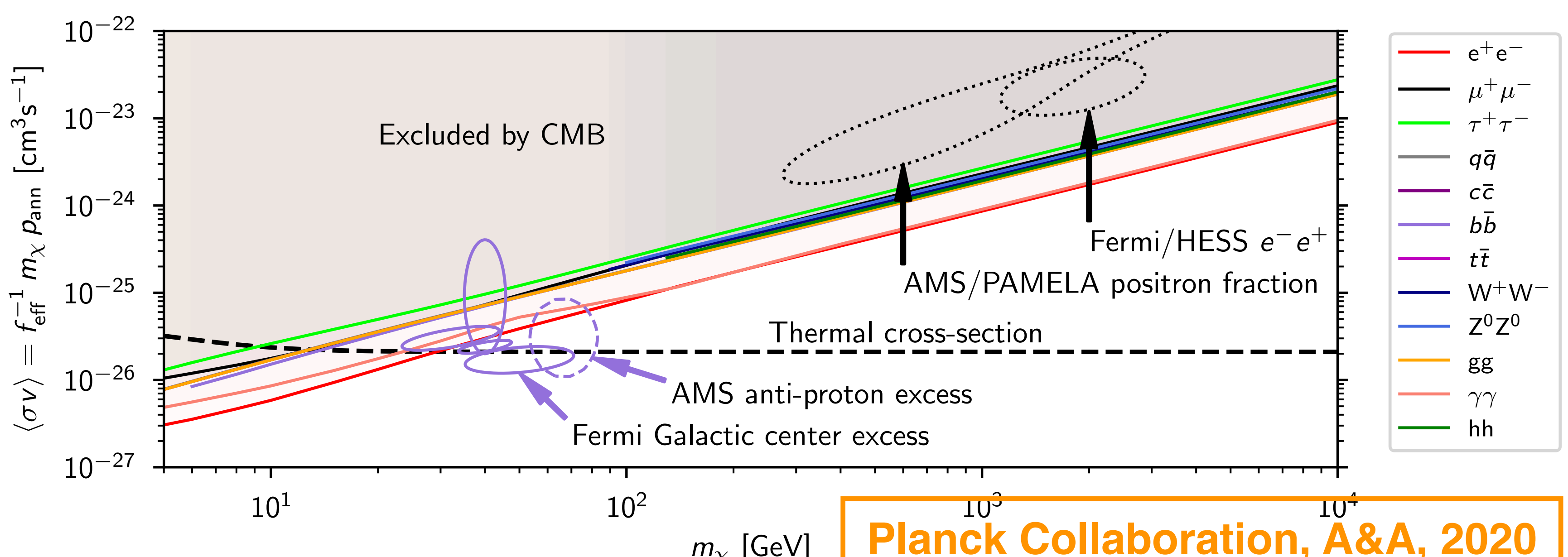
- requires a weak-scale annihilation cross section

$$\langle \sigma_{\rm ann} v\rangle \simeq 1\,{\rm pb} \times c$$



CMB constraints

- energy deposit around the last scattering



Planck Collaboration, A&A, 2020

Sommerfeld enhancement and self-scattering

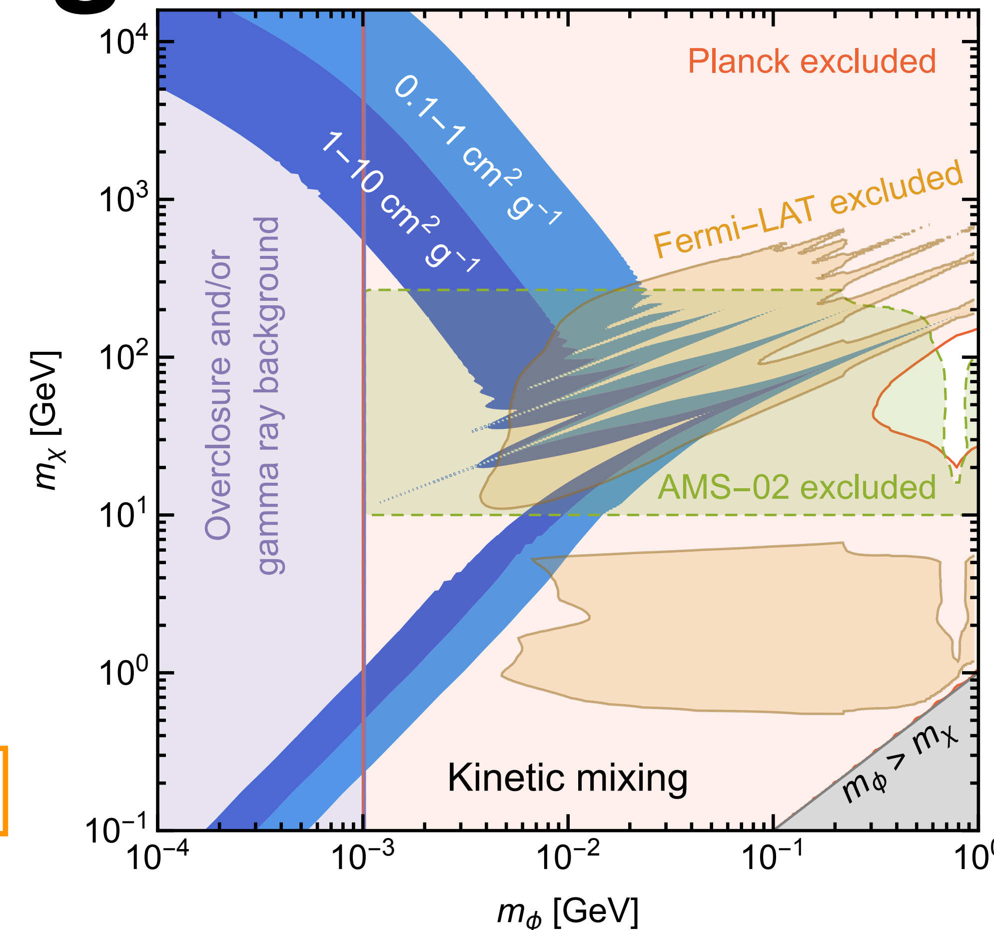
Tight correlation

- resonant enhancement occurs at the same parameter point

- zero-energy resonances (shallow bound states)

- main obstacle in SIDM model building

talk by Lorenzo De Ros

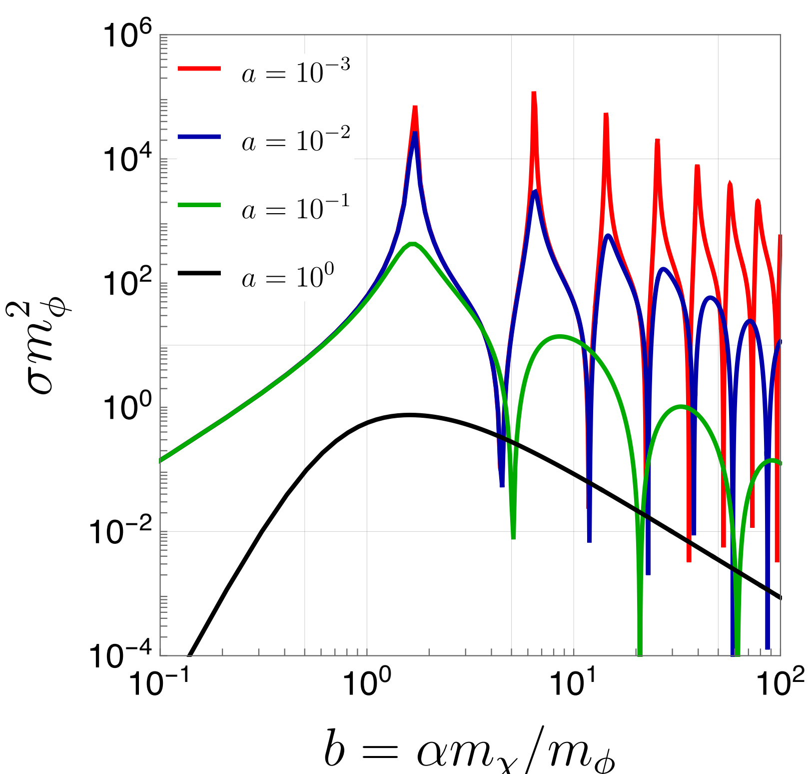


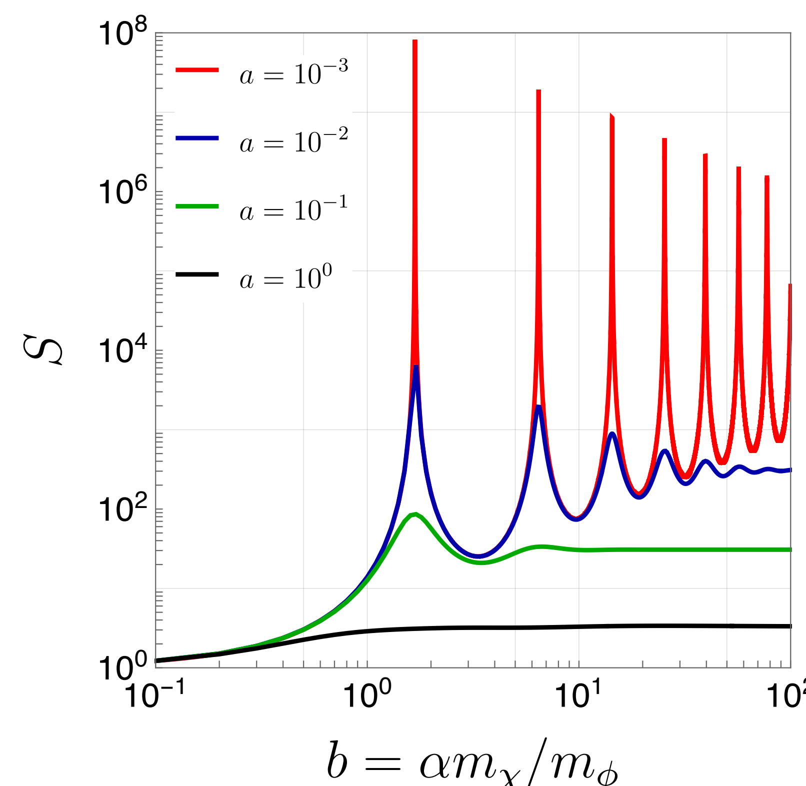
- dark photon

Bringmann, Kahlhoefer, Schmidt-Hoberg and Walia, JHEP, 2020

$$a = \frac{v_{\rm rel}}{2\alpha_\chi}$$

$$b = \frac{\alpha_\chi m_\chi}{m_\phi}$$





AK, Kuwahara and Patel, JHEP, 2023

Contents

Scattering theory in quantum mechanics

- Jost function
- Omnès solution
- violation of partial-wave Unitarity on zero-energy resonances

Unitarization

- contact interaction
- bound state with decay width

Scattering in quantum mechanics

Schrödinger equation

Weinberg, "Lectures on Quantum Mechanics"

$$\left[-\frac{1}{2\mu}\nabla^2 + V(r)\right]\psi_k(\vec{x}) = E\psi_k(\vec{x}) \qquad E = \frac{k^2}{2\mu} \qquad k = \mu v_{\rm rel}$$

- potential from long-range force

- reduced mass ($\mu = m/2$ for identical particle)

- scattering state (energy eigenstate)

$$\psi_k(\vec{x}) \to e^{ikz} + f(k,\theta)\frac{e^{ikr}}{r} \qquad r \to \infty$$

- (initial) plane wave

- scattering amplitude

- out-going spherical wave

Partial-wave decomposition

- motivated by $e^{ikz} = \sum_{\ell=0}^{\infty}(2\ell+1)e^{i\frac{1}{2}\ell\pi}j_\ell(kr)P_\ell(\cos\theta)$

$$\psi_k(\vec{x}) = \sum_{\ell=0}^{\infty}(2\ell+1)e^{i\left(\frac{1}{2}\ell\pi + \delta_\ell(k)\right)}\frac{1}{k}R_{k,\ell}(r)P_\ell(\cos\theta)$$

- phase shift

Sommerfeld enhancement and self-scattering

Scattering phase shift

- radial wave function at infinity

$$R_{k,\ell}(r) \to \frac{\sin(kr - \frac{1}{2}\ell\pi + \delta_\ell(k))}{r} \quad r \to \infty$$

$$f(k,\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta) \qquad f_\ell(k) = \frac{e^{2i\delta_\ell(k)} - 1}{2ik}$$

$$\sigma = \sum_{\ell=0}^{\infty}\sigma_\ell \qquad \sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)\sin^2\delta_\ell(k)$$

- diagonalized S-matrix

$$S_\ell(k) = e^{2i\delta_\ell(k)}$$

Sommerfeld enhancement

Iengo, JHEP, 2009 | Cassel, J.Phys.G, 2010

- radial wave function around the origin

  - annihilation through the contact interaction (delta function potential)

$$\mathrm{S}_\ell(k) = \left|\frac{R_{k,\ell}(r)}{R^{(0)}_{k,\ell}(r)}\right|^2 \quad r \to 0$$

- without potential

Jost function

How to find $R_{k,\ell}(r)$ in practice?

AK, Kuwahara and Patel, JHEP, 2023

AK, Matsumoto and Watanabe, in progress

- "initial" condition given at the origin (regularity)

$$\mathscr{R}_{k,\ell}(r) \to k j_\ell(kr) \approx k \frac{(kr)^\ell}{(2\ell+1)!!} \qquad r \to 0$$

- radial Schrödinger equation

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2} - 2\mu V(r)\right]\mathscr{R}_{k,\ell}(r) = 0$$

- asymptotic behavior of solution

$$\mathscr{R}_{k,\ell}(r) \to \frac{i}{2r}\left[\mathscr{J}_\ell(k)e^{-i\left(kr-\frac{1}{2}\ell\pi\right)} - \mathscr{J}_\ell(-k)e^{i\left(kr-\frac{1}{2}\ell\pi\right)}\right] \quad r \to \infty$$

- Jost function

- by comparing asymptotic behavior

$$R_{k,\ell}(r) = \frac{1}{|\mathscr{J}_\ell(k)|}\mathscr{R}_{k,\ell}(r)$$

Jost function

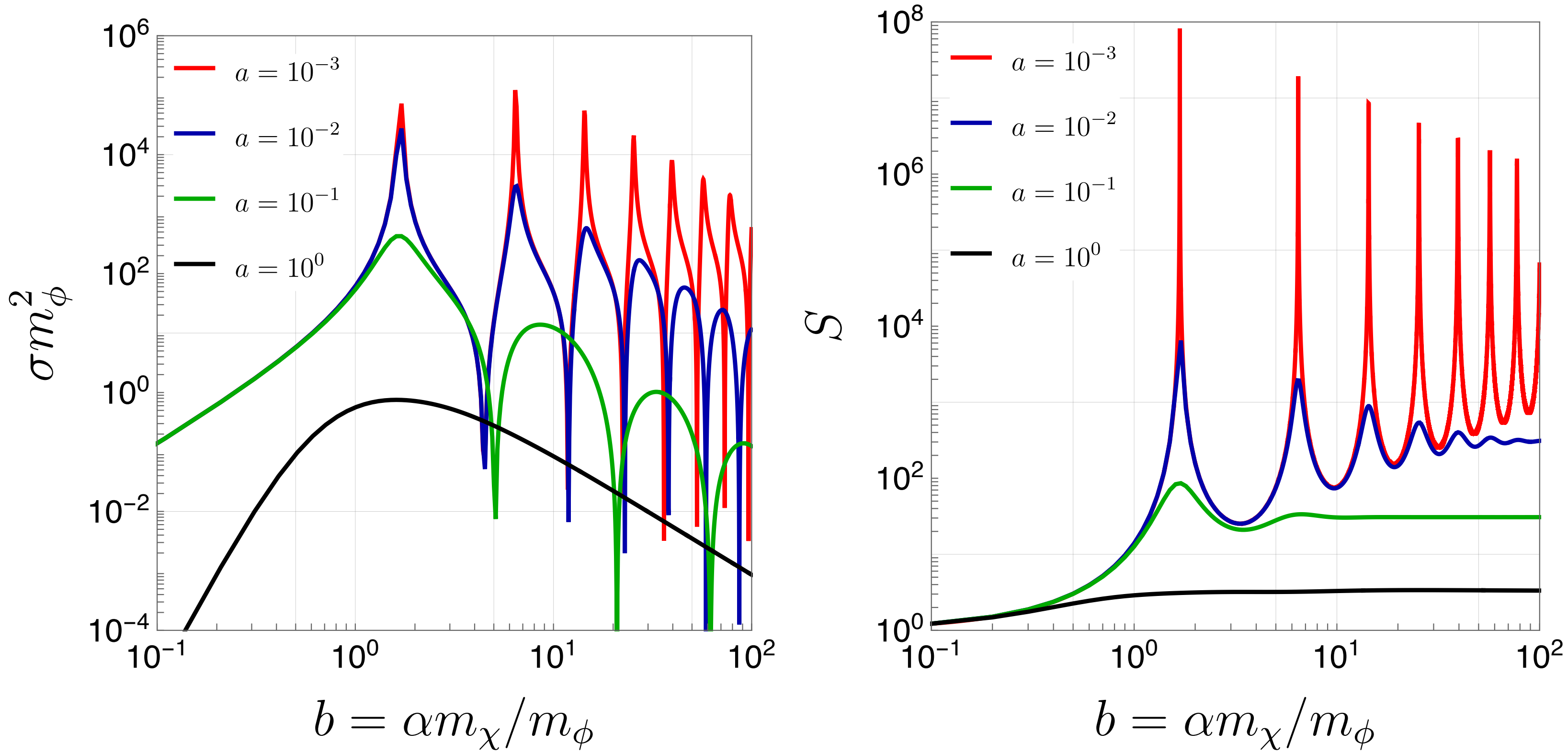
Sommerfeld enhancement and self-scattering

AK, Kuwahara and Patel, JHEP, 2023

AK, Matsumoto and Watanabe, in progress

- Jost function determines both

$$S_\ell(k) = e^{2i\delta_\ell(k)} = \frac{\mathcal{J}_\ell(-k)}{\mathcal{J}_\ell(k)} \qquad \mathrm{S}_\ell(k) = \frac{1}{|\mathcal{J}_\ell(k)|^2}$$



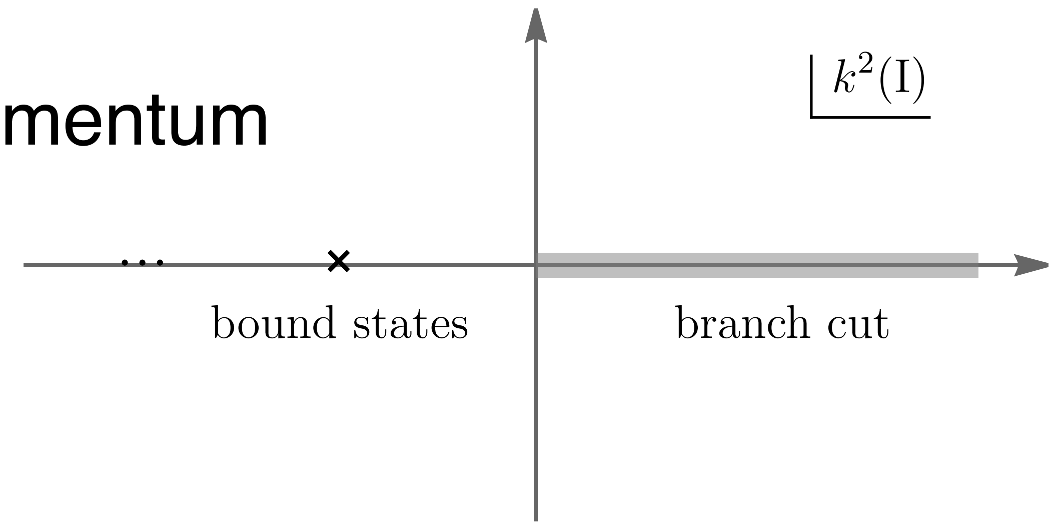
Omnès solution

AK, Kuwahara and Patel, JHEP, 2023

Inverse Jost function

- analytic continuation to complex momentum

$$\frac{1}{\mathcal{J}_\ell(k^2)} = \Omega_\ell(k^2) F_\ell(k^2)$$



- 1st Riemann sheet $\mathrm{Im}(k) > 0$

$$\Omega_\ell(k^2) = \exp[\omega_\ell(k^2)]$$

$$\omega_\ell(k^2) = \frac{1}{\pi}\int_0^\infty dq^2 \frac{\delta_\ell(q)}{q^2 - k^2}$$

- reproducing the brunch cut

$$F_\ell(k^2) = \prod_{b_\ell} \frac{k^2}{k^2 + \kappa_{b,\ell}^2}$$

- rational function reproducing bound-state poles
- numerator is chosen so that no singularity at $k \to 0$
  - discussed next

Omnès solution

Levinson theorem

Weinberg, "Lectures on Quantum Mechanics"

- # of bound states is given by phase shift

$$\delta_\ell(k \to 0) - \delta_\ell(k \to \infty) = \left[\#b_\ell \left(+\frac{1}{2}\right)\right] \pi$$

- excluding virtual levels
- zero in our normalization
- only for s-wave zero-energy resonances

$k \to 0$ behavior

- Omnès function is singular with the power of # of bound states

$$\mathrm{Re}[\omega_\ell(k^2 + i\epsilon)] = \frac{1}{\pi} \int_0^\infty dq^2 \frac{\delta_\ell(q)}{q^2 - k^2} \quad \to -\left[\#b_\ell \left(+\frac{1}{2}\right)\right] \ln(k^2/\Lambda^2) \quad k \to 0$$

$$\frac{1}{J_\ell(k^2 + i\epsilon)} = \exp[\omega_\ell(k^2 + i\epsilon)] F_\ell(k^2) \quad \propto \frac{F_\ell(k^2)}{k^{2\#b_\ell(+1)}} \quad k \to 0$$

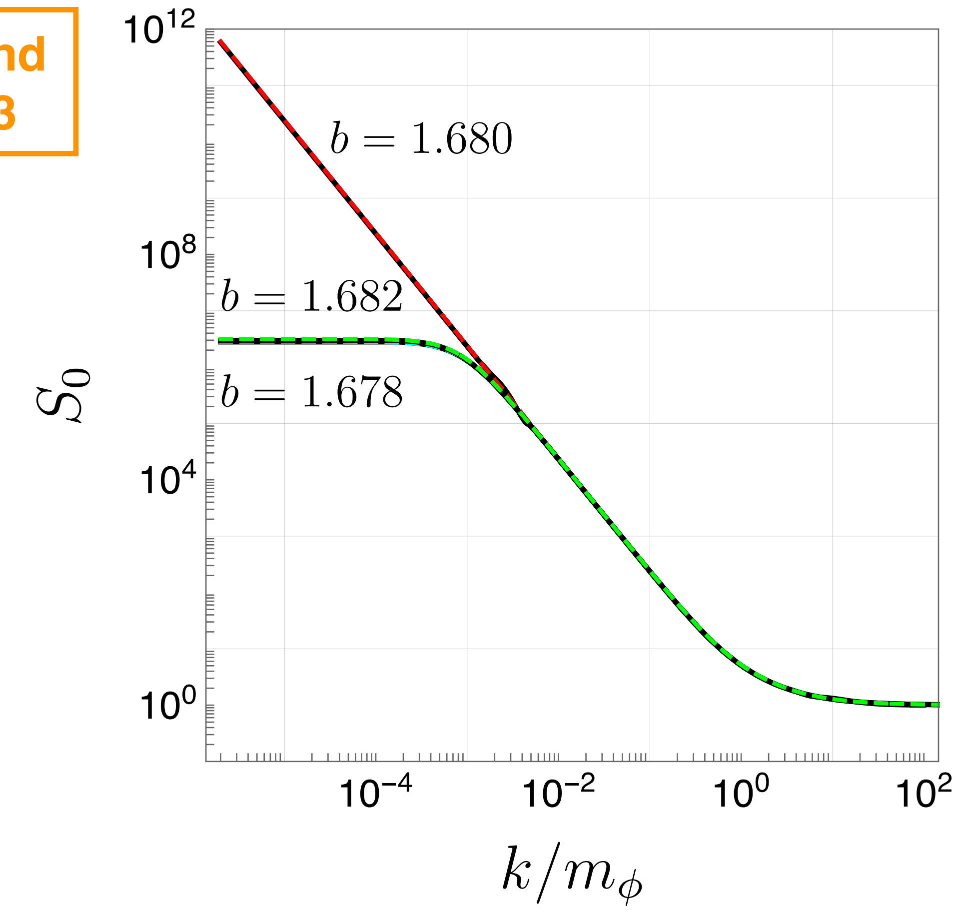
$$F_\ell(k^2) = \prod_{b_\ell} \frac{k^2}{k^2 + \kappa_{b,\ell}^2} \propto k^{2\#b_\ell} \quad k \to 0$$

Omnes solution vs direct computation

AK, Kuwahara and Patel, JHEP, 2023

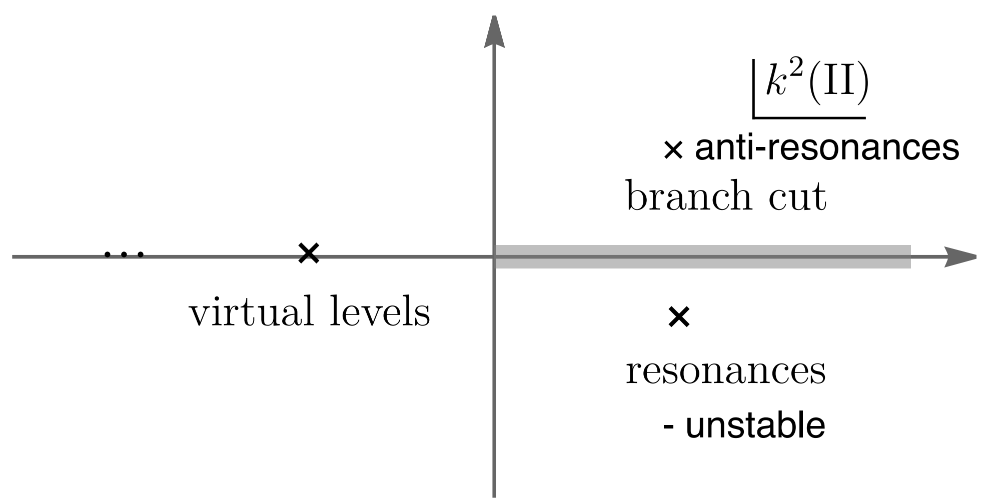
s-wave

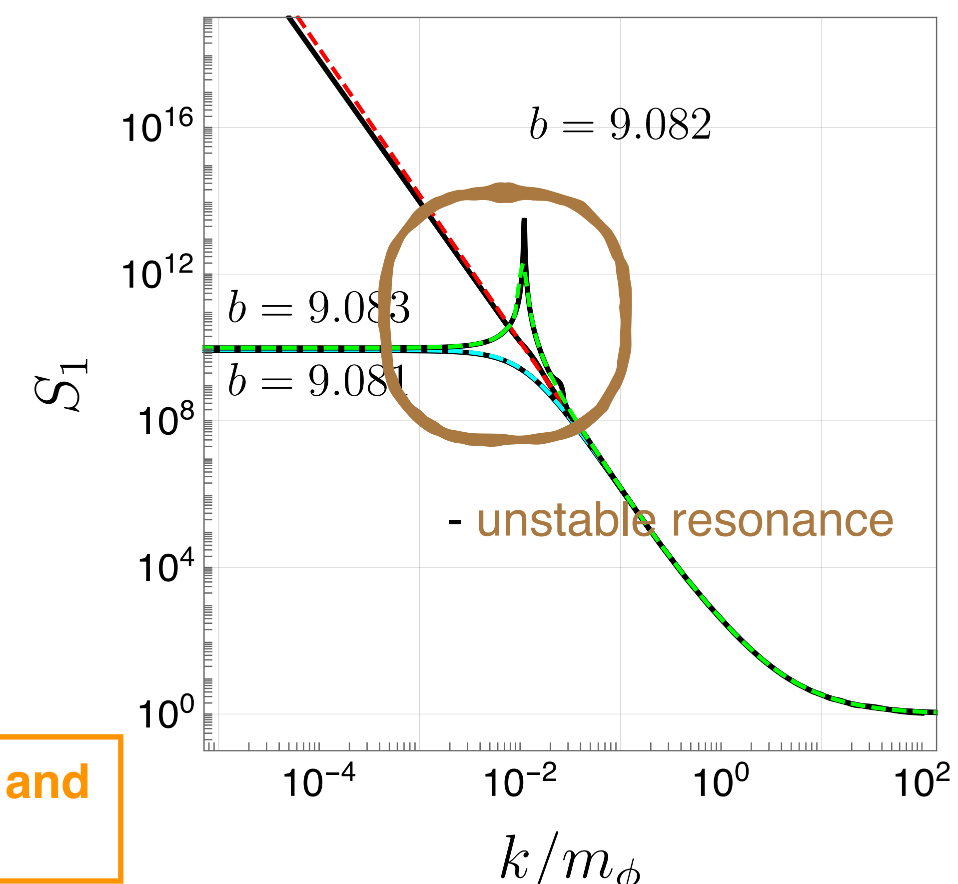
- Omnès solution agrees with direct computation from scattering state



p-wave

- reproduce unstable (positive energy) resonance as well





- 2nd Riemann sheet $\mathrm{Im}(k) < 0$

Chu, Garcia-Cely and Murayama, JCAP, 2020

Beneke, Binder, De Ros and Grany, JHEP 2024

Zero-energy resonances

No cancellation

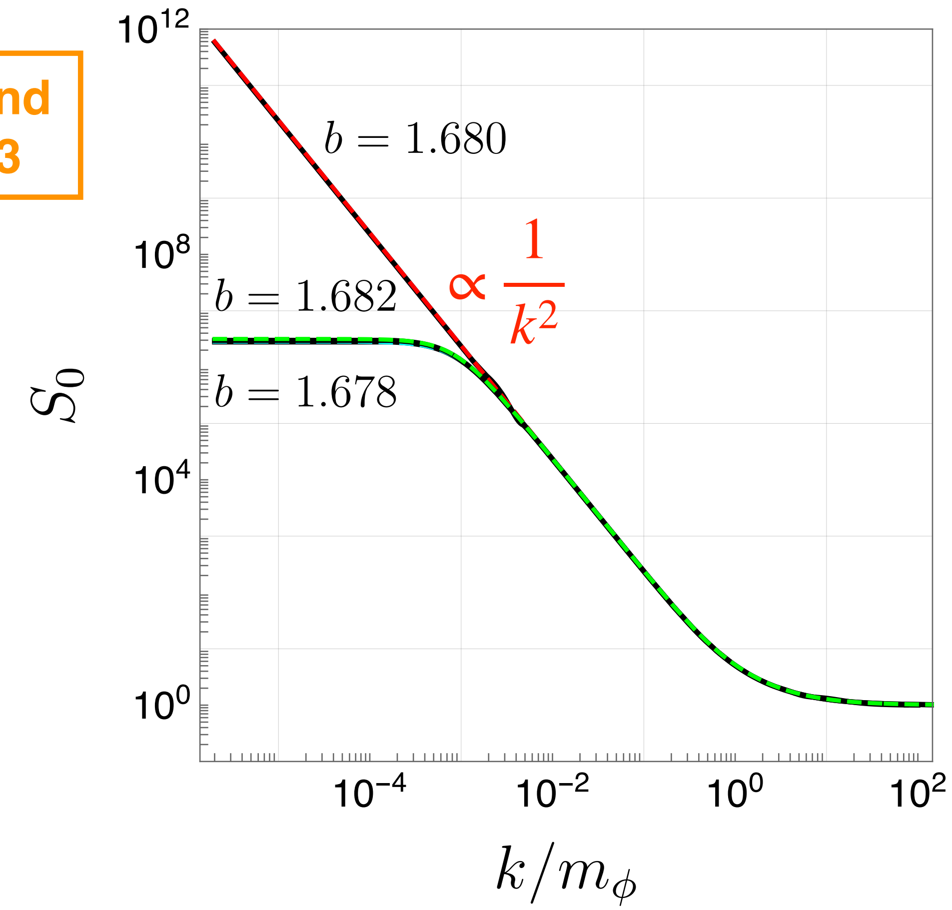
AK, Kuwahara and Patel, JHEP, 2023

- s-wave 1st resonance $\#b_0 = 0$

$$k \to 0 \quad \mathrm{Re}[\omega_0(k^2 + i\epsilon)] \to -\frac{1}{2}\ln(r_{e,0}^2 k^2)$$

- only zero energy "virtual" level $F_0(k^2) = 1$

$$k \to 0 \quad \frac{1}{J_0(k^2)} = \exp[\omega_0(k^2)]F_0(k^2) \propto \frac{1}{k}$$

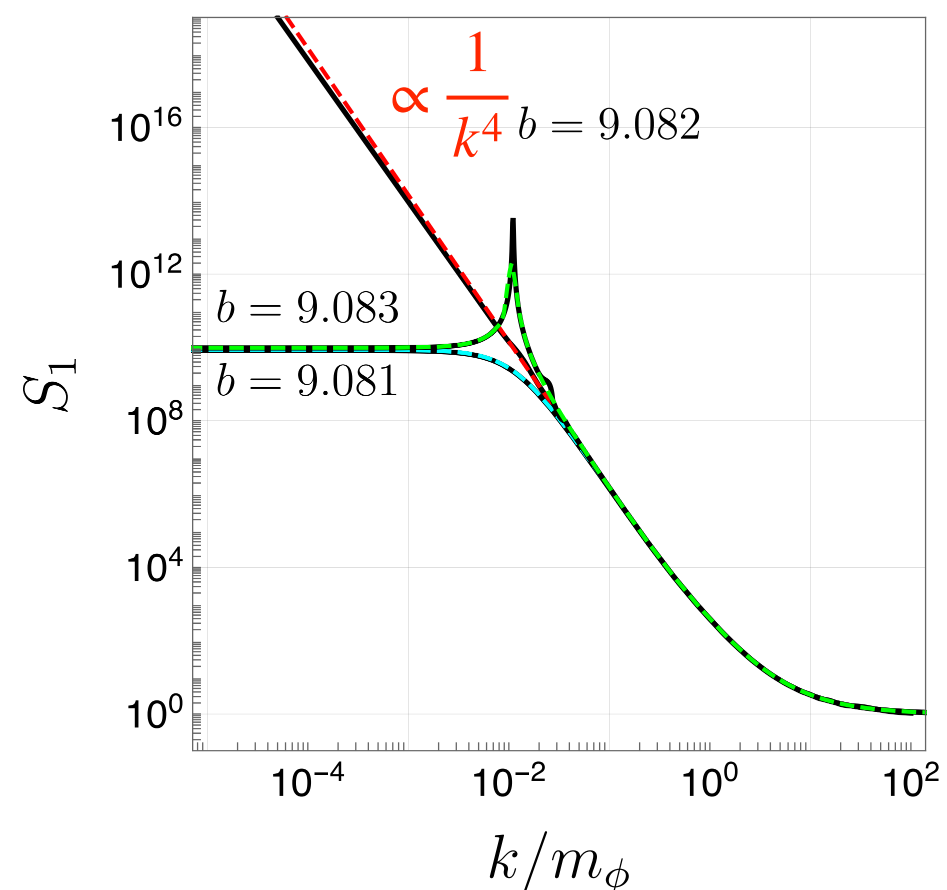


- p-wave 1st resonance $\#b_1 = 1$

$$k \to 0 \quad \mathrm{Re}[\omega_1(k^2)] \to -\ln(r_{e,1}^2 k^2)$$

- zero energy bound state $F_1(k^2) = \frac{k^2}{k^2} = 1$

$$k \to 0 \quad \frac{1}{J_1(k^2)} = \exp[\omega_1(k^2)]F_1(k^2) \propto \frac{1}{k^2}$$



Contents

Unitarization

talk by Kalliopi Petraki

- contact interaction
- bound state with decay width

Zero-energy resonances

Unitarity violation

- on s- and p-waves zero-energy resonances, partial-wave Unitarity is violated at low velocity

$$(\sigma_{\ell,\mathrm{ann}} v_{\mathrm{rel}}) = \mathrm{S}_\ell (\sigma^{(0)}_{\ell,\mathrm{ann}} v_{\mathrm{rel}}) \qquad (\sigma^{(0)}_{\ell,\mathrm{ann}} v_{\mathrm{rel}}) \propto k^{2\ell} \qquad (\sigma^{\mathrm{Uni}}_{\ell,\mathrm{ann}} v_{\mathrm{rel}}) = \frac{\pi}{\mu k}$$

$$\mathrm{S}_0(k^2) \propto \frac{1}{k^2} \qquad \mathrm{S}_{\ell\geq 1}(k^2) \propto \frac{1}{k^4}$$

- because we ignored a contact interaction including $V \supset u\delta^3(\vec{x})$ annihilation when solving the Schrödinger equation

Self-consistent solution

Blum, Sato and Slatyer, JHEP, 2016

Parikh, Sato and Slatyer, arXiv:2410.18168

AK, Matsumoto and Watanabe, in progress

- incorporating contact interaction is not as easy as one expects

- mathematical fact: there is no bounded wave function if a potential is singular than the centrifugal one

Full scattering state

AK, Matsumoto and Watanabe, in progress

S-matrix

$$S_\ell(k) = \frac{\mathcal{J}_\ell(-k)}{\mathcal{J}_\ell(k)} \frac{p_\ell(k) - k^{2\ell+1}\dfrac{i\mathcal{K}_\ell(k)}{\mathcal{J}_\ell(k)} - k^{2\ell+1}\dfrac{i\mathcal{K}_\ell(-k)}{\mathcal{J}_\ell(-k)}}{p_\ell(k)}$$

- K is determined by singular solution

$$\mathcal{S}_{k,\ell}(r) \to k y_\ell(kr) \approx -k\frac{(2\ell-1)!!}{(kr)^{\ell+1}} \qquad r \to 0$$

$$\mathcal{S}_{k,\ell}(r) \to -\frac{1}{2r}\left[\mathcal{K}_\ell(k) e^{-i\left(kr - \frac{1}{2}\ell\pi\right)} + \mathcal{K}_\ell(-k) e^{i\left(kr - \frac{1}{2}\ell\pi\right)}\right] \qquad r \to \infty$$

- p(k) represents the contact interaction
    - k-dependence is determined by

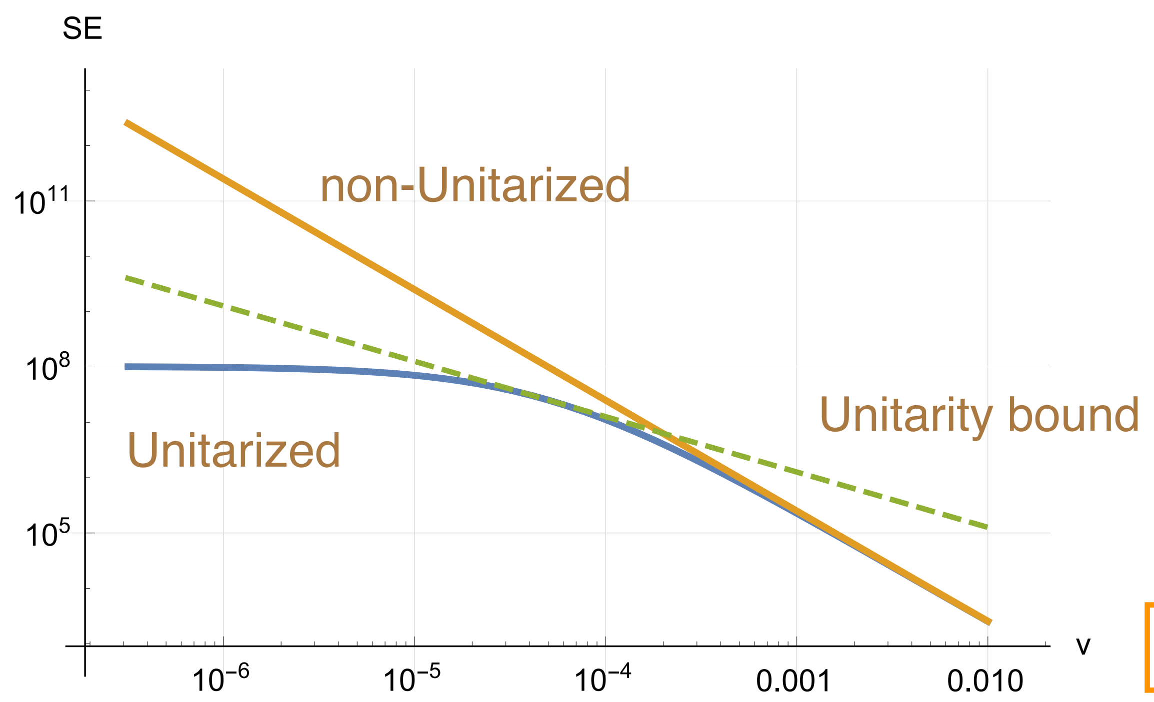
$$p_\ell(k) - k^{2\ell+1}\frac{i\mathcal{K}_\ell(k)}{\mathcal{J}_\ell(k)} + \frac{d^{2\ell+1}}{dr^{2\ell+1}}\left[\frac{(kr)^\ell}{(2\ell)!!} r\mathcal{S}_{k,\ell}\right](0) = p_\ell(k_0) - k_0^{2\ell+1}\frac{i\mathcal{K}_\ell(k_0)}{\mathcal{J}_\ell(k_0)} + \frac{d^{2\ell+1}}{dr^{2\ell+1}}\left[\frac{(k_0 r)^\ell}{(2\ell)!!} r\mathcal{S}_{k_0,\ell}\right](0)$$

- large-k value is determined by UV cross section

$$\frac{4\pi}{|p_\ell(k)|^2} = \frac{\sigma^\ell_{\mathrm{sc},0}}{(2\ell+1)k^{4\ell}} \qquad \mathrm{Im}\frac{4\pi}{p_\ell(k)} \approx -\frac{\sigma^\ell_{\mathrm{ann},0}}{(2\ell+1)k^{2\ell-1}}$$

Full scattering state

Unitarized Sommerfeld enhancement factor



Blum, Sato and Slatyer, JHEP, 2016

Bound state with decay width

AK, Matsumoto and Watanabe, in progress

- bound state is a pole of S-matrix

- one can find decay width as

$$\mathrm{Im}E_B = -\frac{1}{2(4\pi)}\frac{\sigma^{\ell}_{\mathrm{ann},0}v}{(2\ell+1)p^{2\ell}}\left|\frac{(2\ell+1)!!}{\ell\,!}\frac{d^{\ell}R^{\ell}_{B}}{dr^{\ell}}(0)\right|^2$$

Summary

Long-range force of dark matter

- Sommerfeld enhancement and self-scattering cross section
  - indirect detection and structure formation
  - correlated

This talk

- they are determined by Jost function
  - Omnès solution reproduces (inverse) Jost function
- zero-energy resonances lead to violation of partial-wave Unitarity for s- and p-waves
- Unitarized S-matrix has a bound state pole with decay width

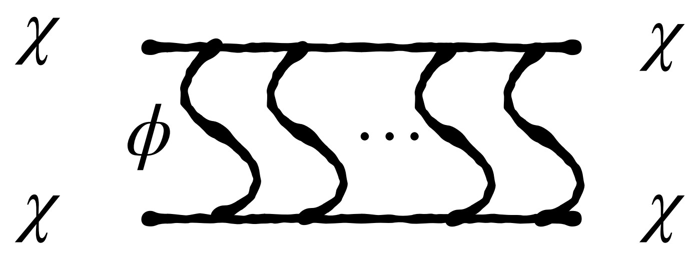
Thank you

Self-scattering

The same light mediator

- non-perturbative (infinite exchanges) when the distortion of wave function is significant

- again described by the Schrödinger equation (later)

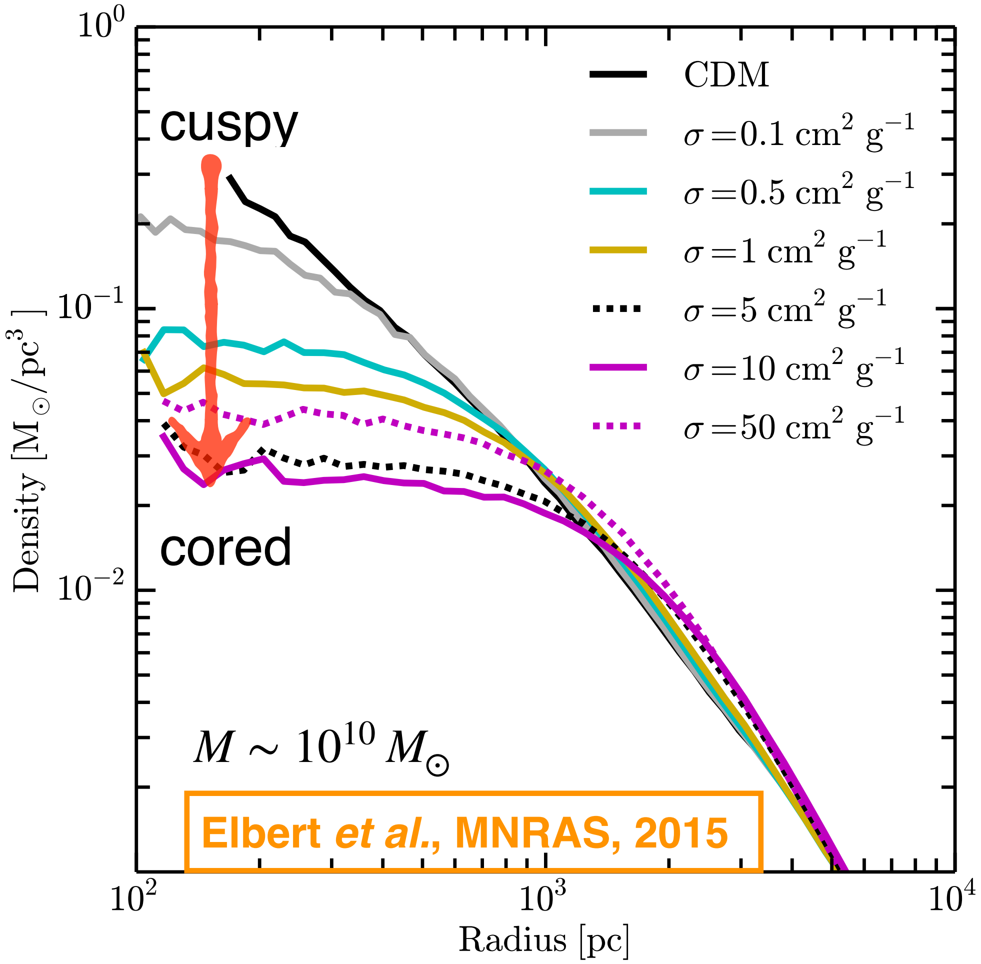


Self-interacting dark matter

- interactions among dark matter particles

$$\sigma/m \sim 1\,\mathrm{cm}^2/\mathrm{g} \sim 1\,\mathrm{barn/GeV}$$

- dark matter density profile inside a halo turns from cuspy to cored



Velocity dependence

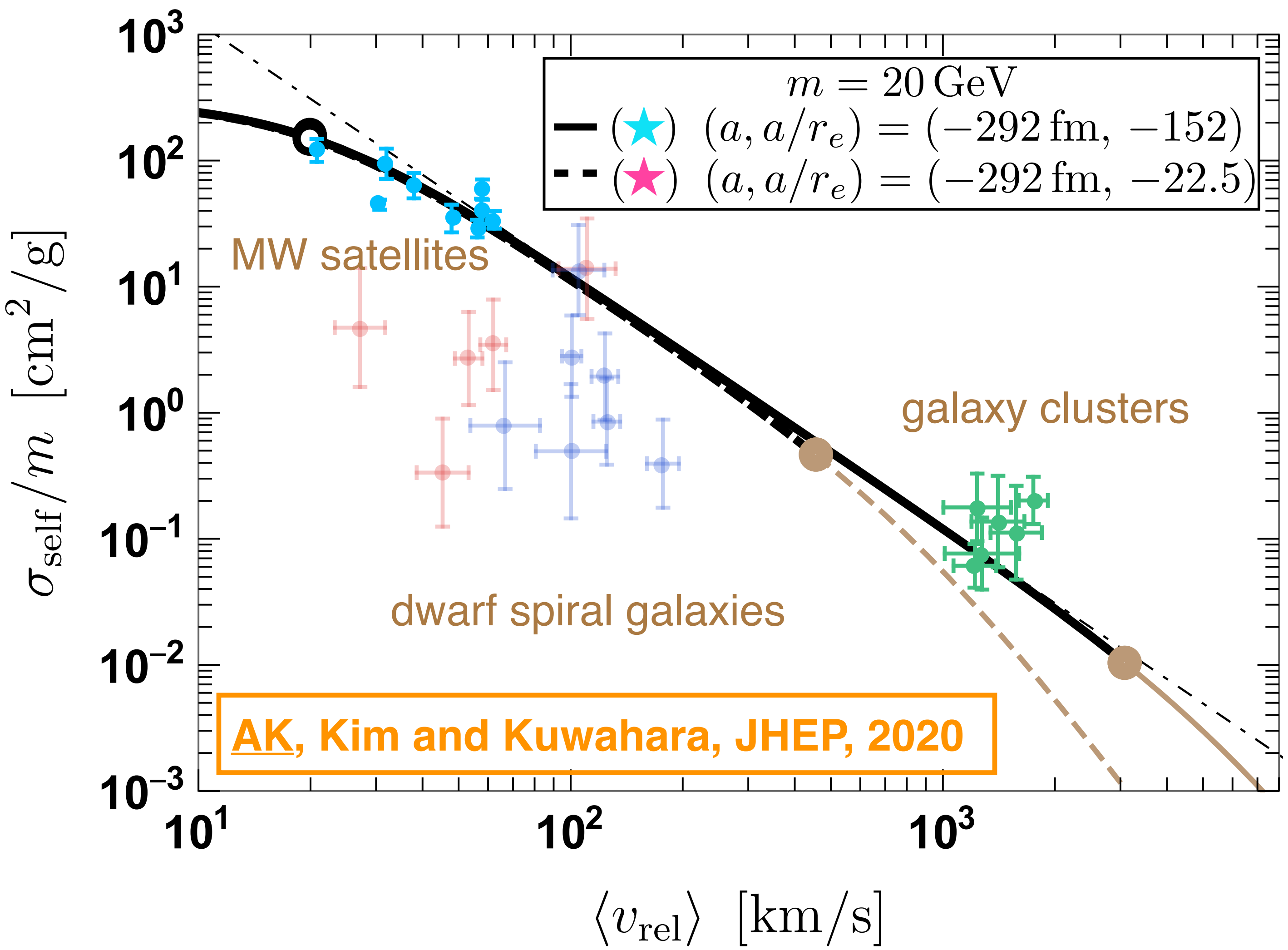
Self-interacting dark matter

- cored profile "appear to" provide better fit to astronomical data

- "data" points from astrophysical observations of various size halos

Light mediator

- introduce a velocity dependence, which is compatible with "data"



Data points

Overview

- cores in various-size halos

- MW satellite (Draco)

$M_{\mathrm{infall}} \sim 10^9\, M_\odot$

- dwarf spiral galaxy (IC 2574) $M \sim 10^{11}\, M_\odot$

- galaxy cluster (Abell 2744)

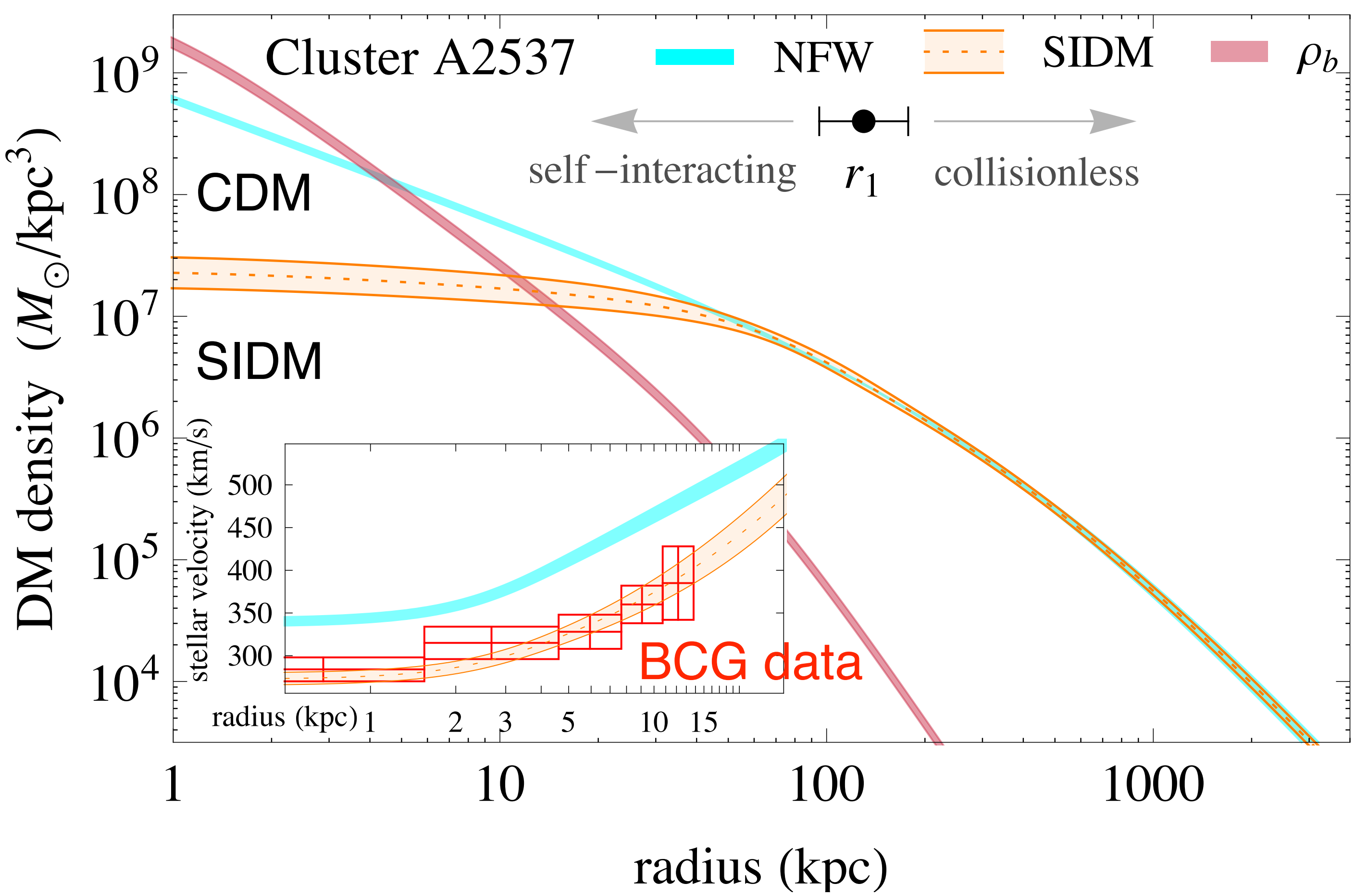
$M \sim 10^{14}\, M_\odot$

Data points

Galaxy clusters

- mass distribution in the outer region is determined by strong/weak gravitational lensing

- stellar kinematics in the central region (brightest cluster galaxies) prefer cored SIDM profile



$\sigma_{\rm self}/m \sim 0.1\,\mathrm{cm}^2/\mathrm{g}$

$\langle v_{\rm rel}\rangle \sim 10^3\,\mathrm{km/s}$

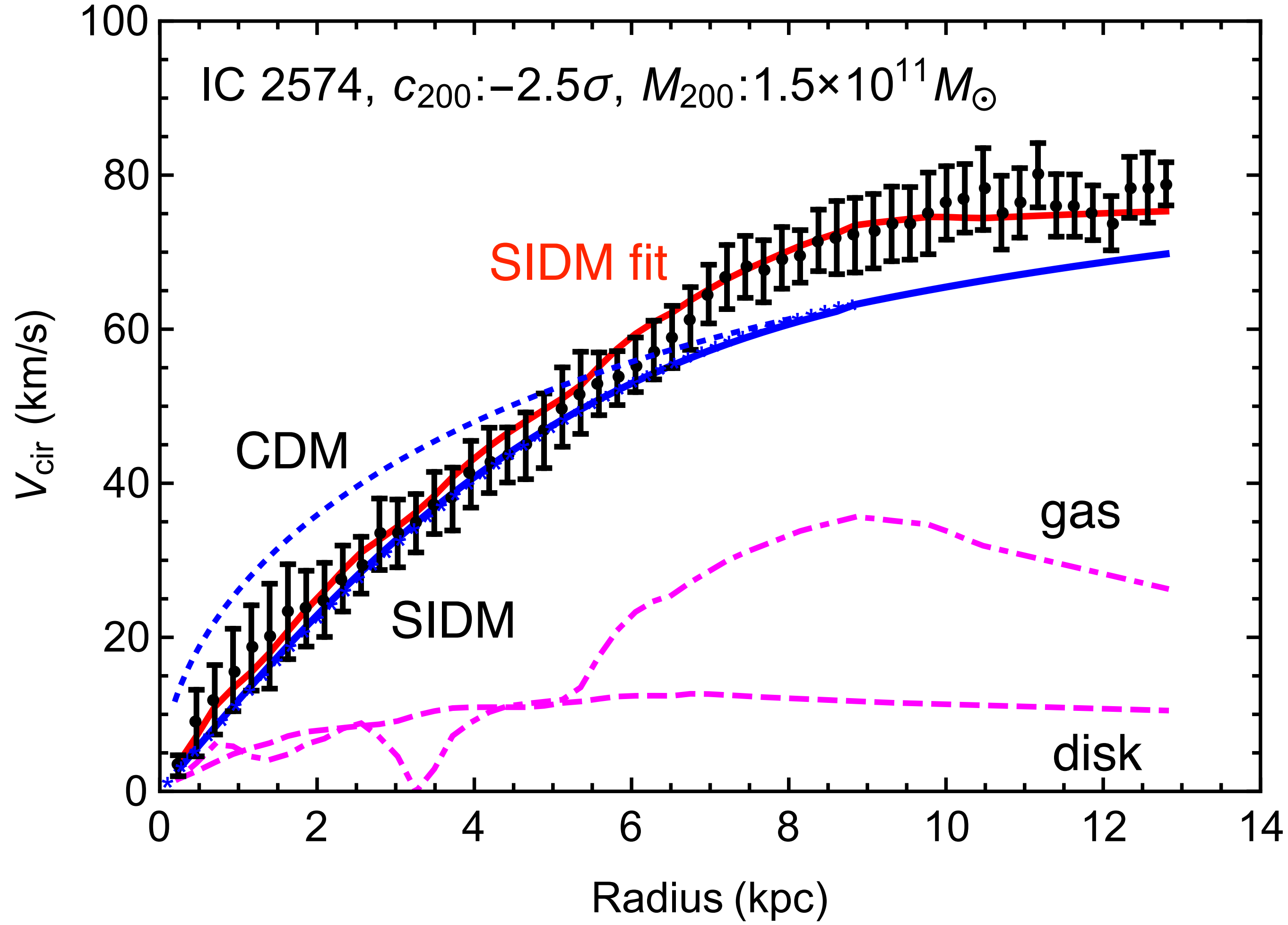
Kaplinghat, Tulin and Yu, PRL, 2016

Data points

Dwarf spiral galaxies

- mass distribution is broadly determined by rotation curves

- rotation velocity in central region (of some galaxies) prefer cored SIDM profile



$\sigma_{\rm self}/m \sim 1\,{\rm cm}^2/{\rm g}$

$\langle v_{\rm rel}\rangle \sim 10^2\,{\rm km/s}$

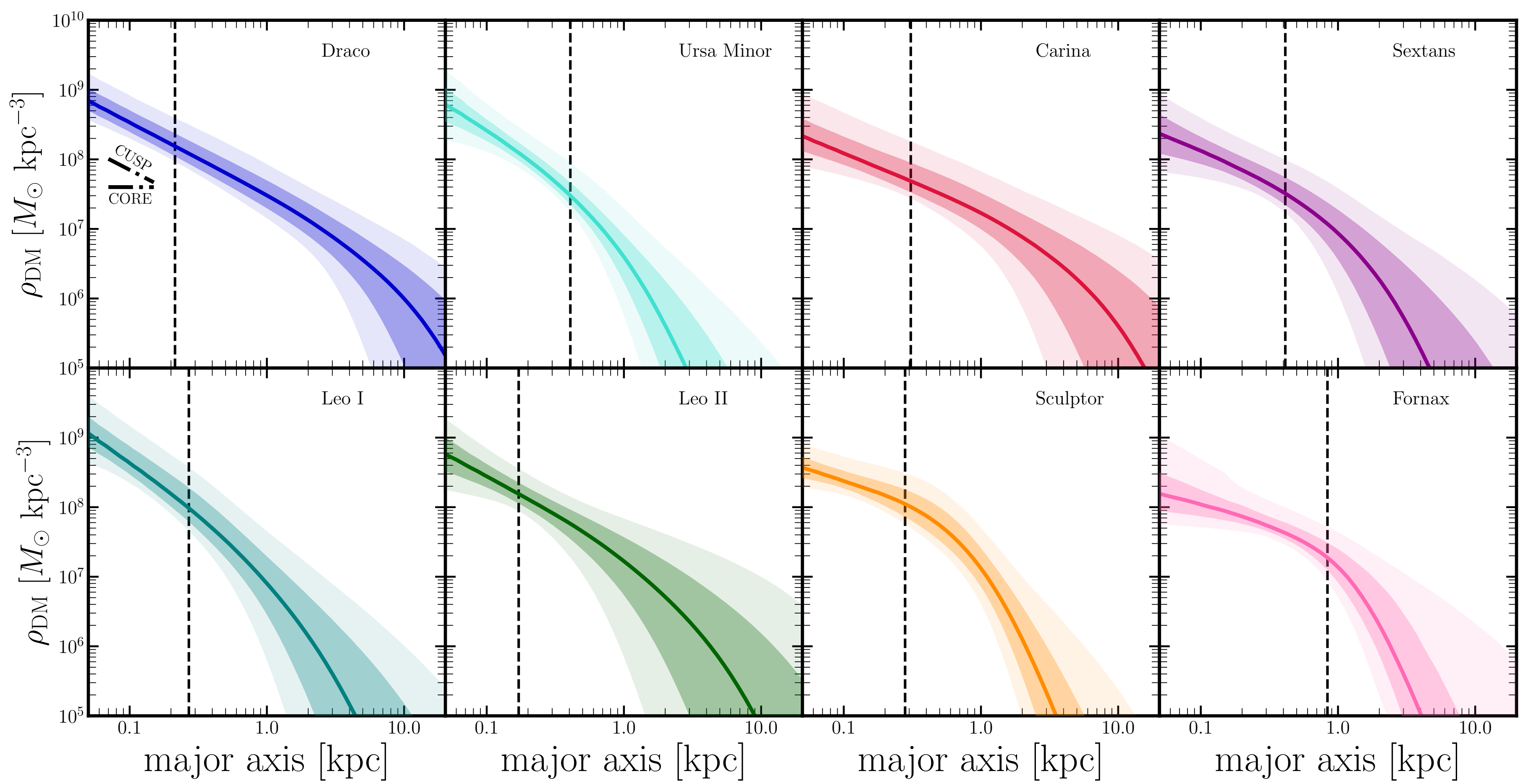
AK, Kaplinghat, Pace and Yu, PRL, 2017

Data points

MW satellites

- mass distribution is determined by stellar kinematics

- stellar kinematics in the central region (of some satellites) prefer cuspy CDM profile



Hayashi, Chiba and Ishiyama, ApJ, 2020

Data points

MW satellites

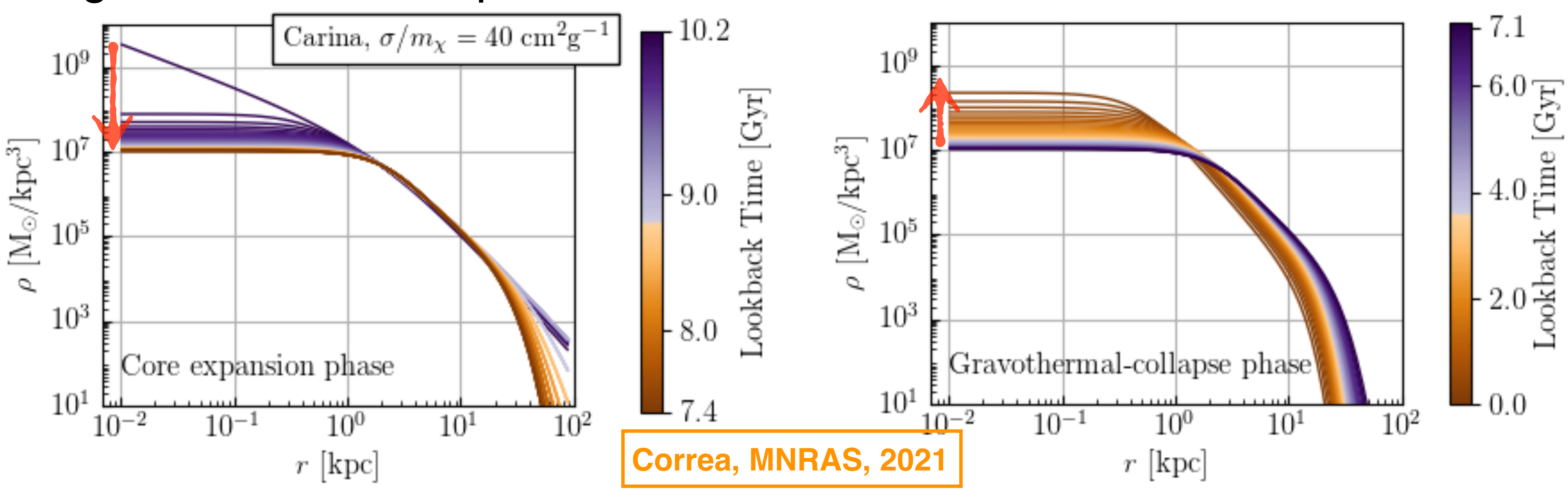
- one possibility is to take as a tiny cross section as $\sigma_{\rm self}/m \simeq 0.01\,{\rm cm^2/g}$

$$\langle v_{\rm rel}\rangle \sim 30\,{\rm km/s}$$

  - resonance? **Chu, Garcia-Cely and Murayama, PRL, 2019**

- another possibility is to take as a large cross section as $\sigma_{\rm self}/m \sim 40\,{\rm cm^2/g}$ $\langle v_{\rm rel}\rangle \sim 30\,{\rm km/s}$

- gravothermal collapse

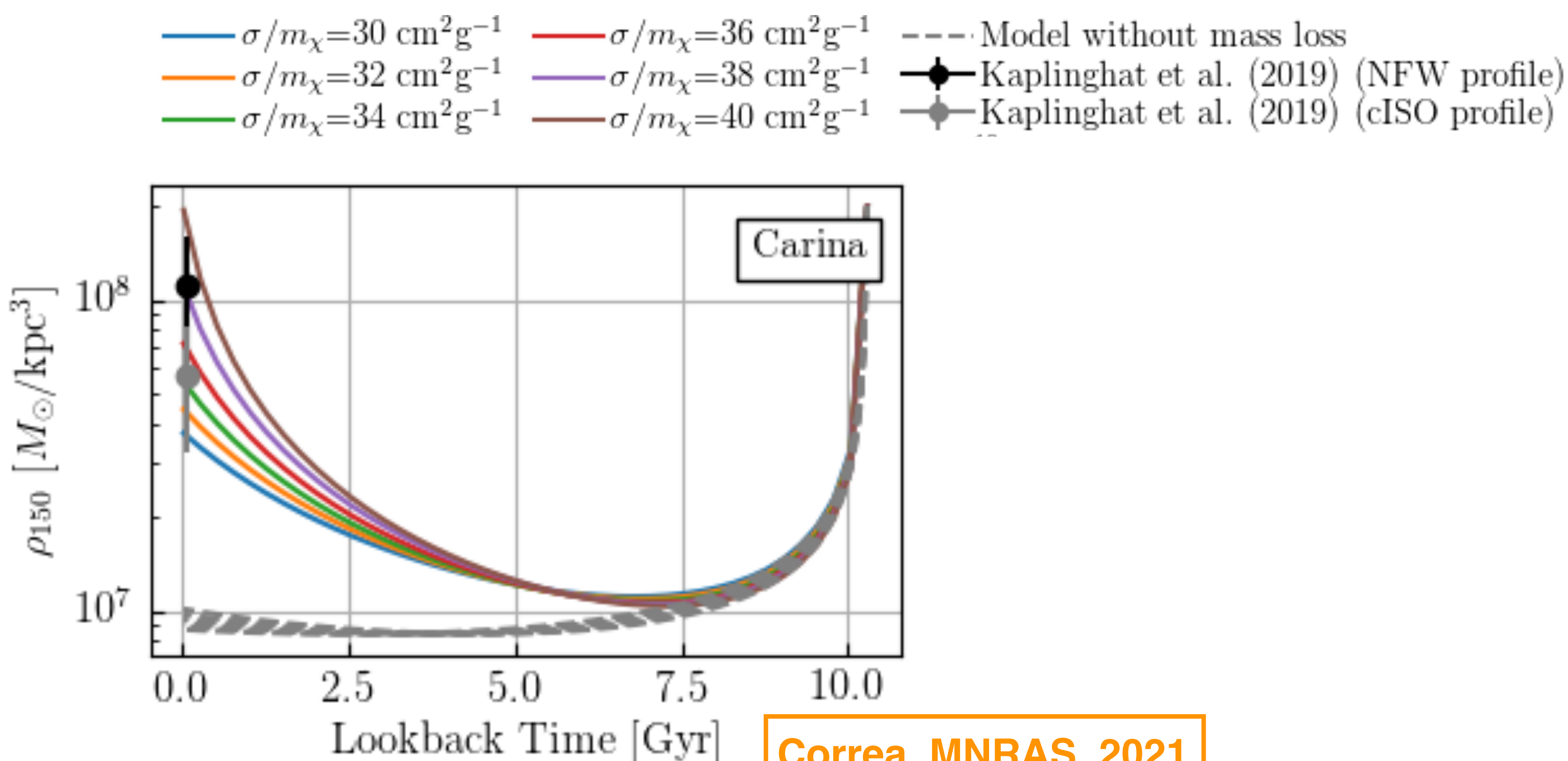


Correa, MNRAS, 2021

Data points

MW satellites

- gravothermal collapse
  - core shrinks and central density gets higher
  - central density at present is very sensitive to the cross section

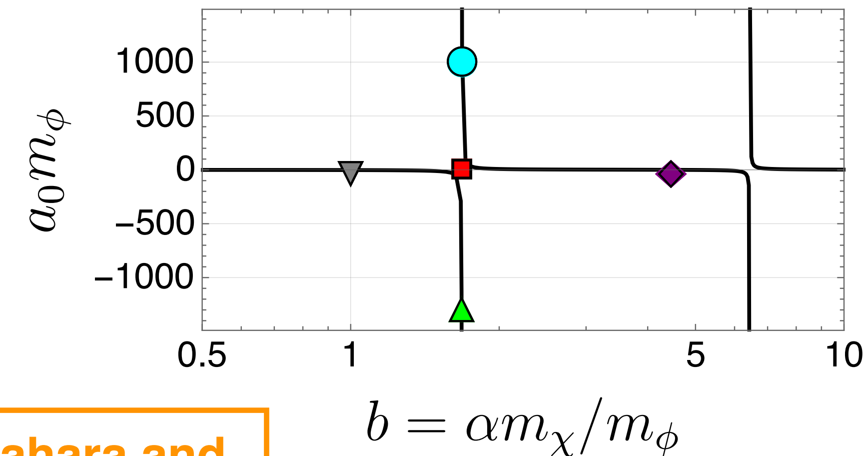


Correa, MNRAS, 2021

Correlation

Zero-energy resonances

- resonant enhancement occur



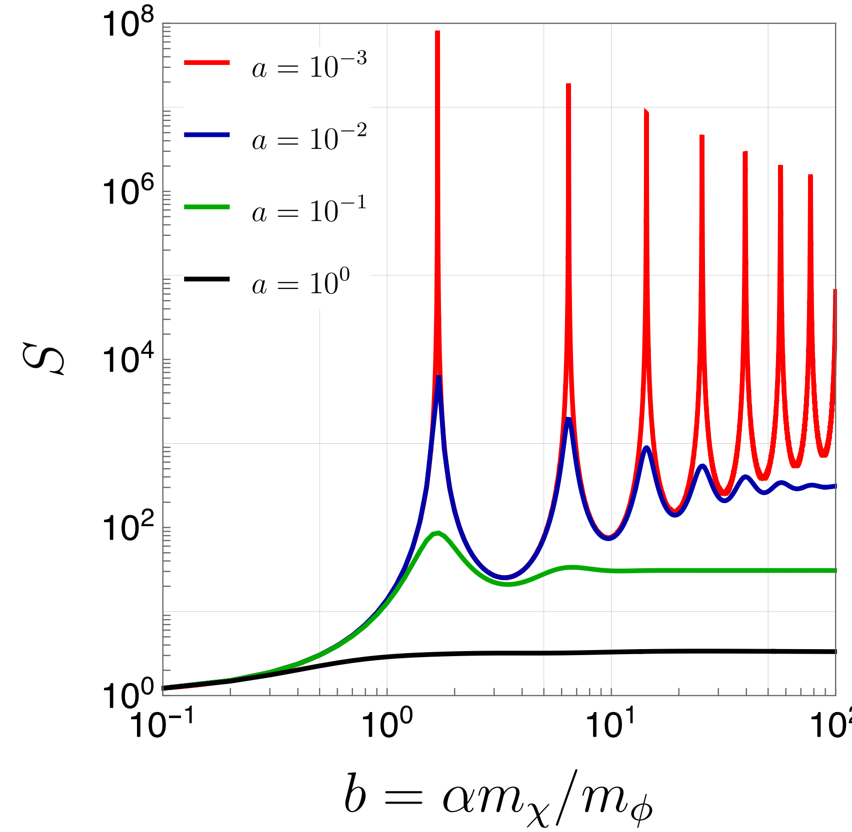
AK, Kuwahara and Patel, JHEP, 2023

Effective range theory

$$k^{2\ell+1} \cot \delta_\ell \to -\frac{1}{a_\ell{}^{2\ell+1}} + \frac{1}{2 r_{e,\ell}{}^{2\ell-1}} k^2$$

- scattering length
- effective range

- on resonance $a_0 \to \infty$
- shallow virtual level
  - non-normalizable
- shallow bound state
  - pole of scattering amplitude



Analytic property

Complex momentum squared

$$\Gamma_\ell(k^2) = \frac{1}{\mathcal{J}_\ell(k^2)}$$

- “real” complex function

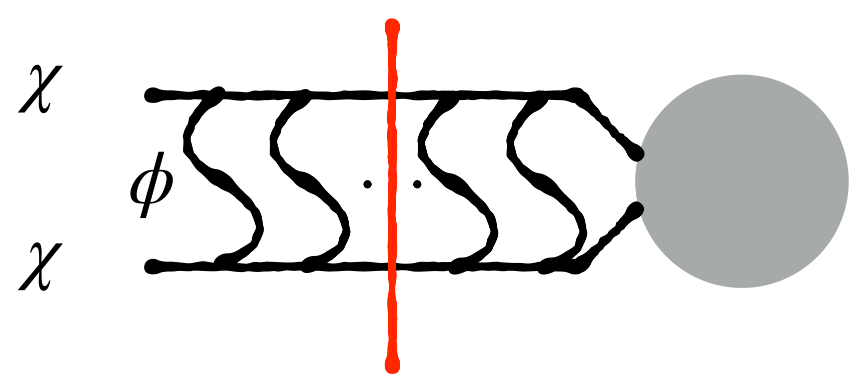
$$\Gamma_\ell^*(k^2) = \Gamma_\ell(k^{2*})$$

- brach cut along real (physical) axis

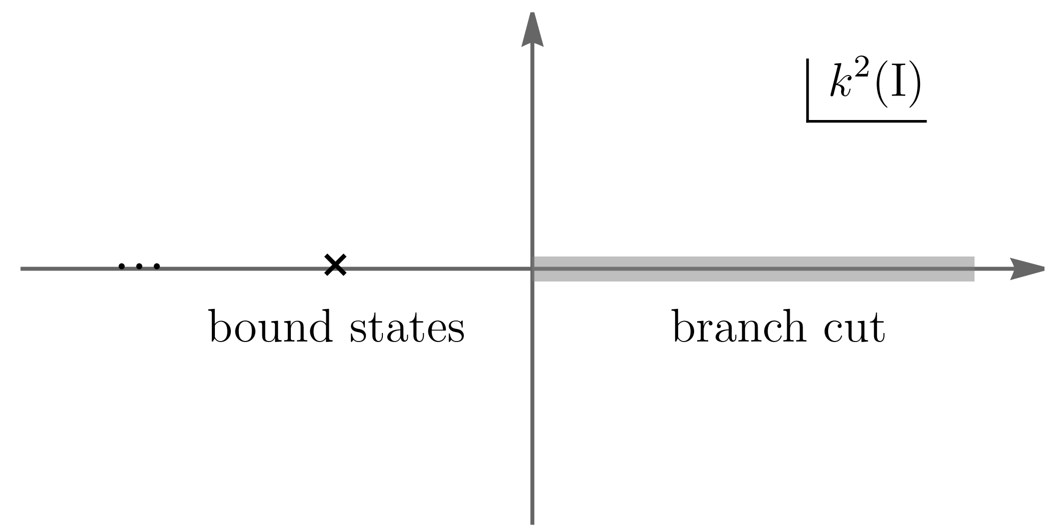
$$\Gamma_\ell(k^2 + i\epsilon) = e^{2i\delta_\ell(k)}\Gamma_\ell(k^2 - i\epsilon)$$

- real k^2

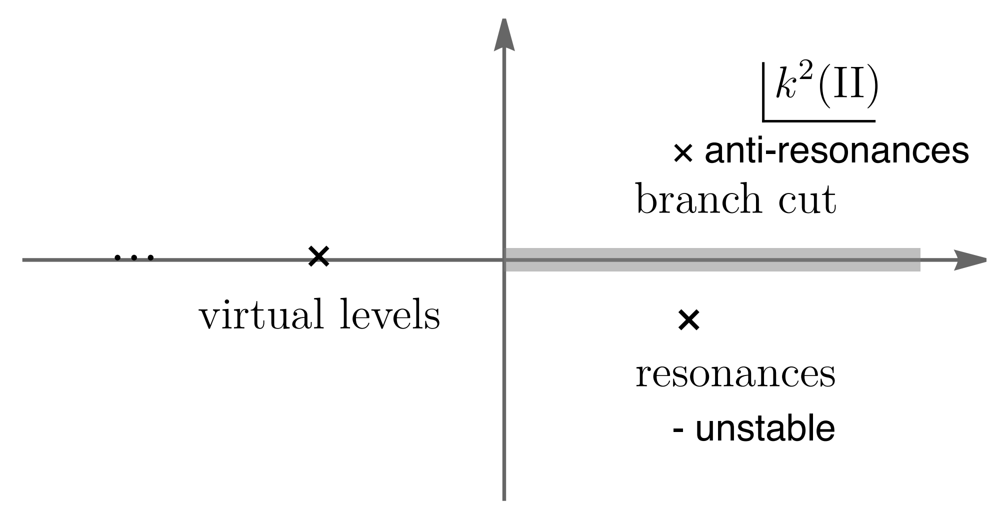
- known as Watson theorem (kind of optical theorem)



- bound states



- 1st Riemann sheet $\mathrm{Im}(k) > 0$



- 2nd Riemann sheet $\mathrm{Im}(k) < 0$

Omnès solution

Levinson theorem

Weinberg, "Lectures on Quantum Mechanics"

- # of bound states is given by phase shift

$$\delta_\ell(k \to 0) - \delta_\ell(k \to \infty) = \left[\#b_\ell \left(+\frac{1}{2}\right)\right]\pi$$

- excluding virtual levels
- zero in our normalization
- only for s-wave zero-energy resonances

- underlying idea
  - consider the system confined in a large sphere

$$R_{k\ell}(r) \to \frac{\sin(kr - \frac{1}{2}\ell\pi + \delta_\ell)}{r}$$

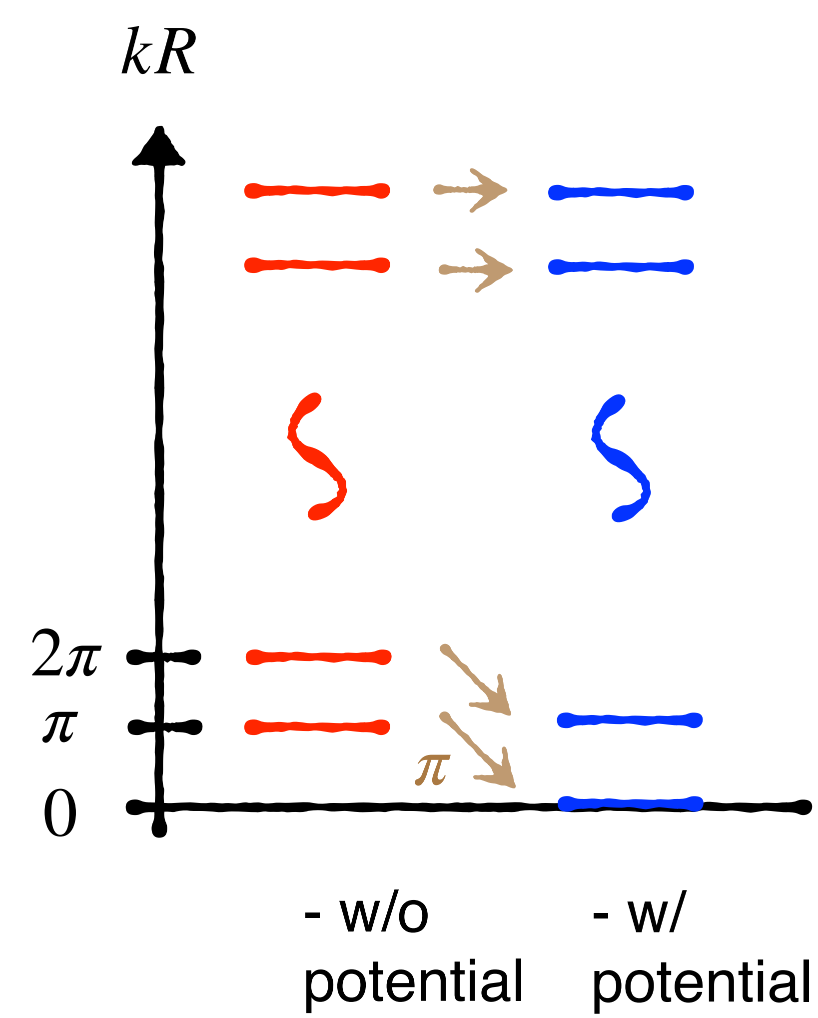
$r \to \infty$

$k > 0$

$$kR - \frac{1}{2}\ell\pi + \delta_\ell = n\pi$$

$n = 0, \pm 1, \pm 2 \ldots$

  - scattering states are discretized (countable infinity)
  - decrease in # of scattering states = # of bound states
    - total number does not change

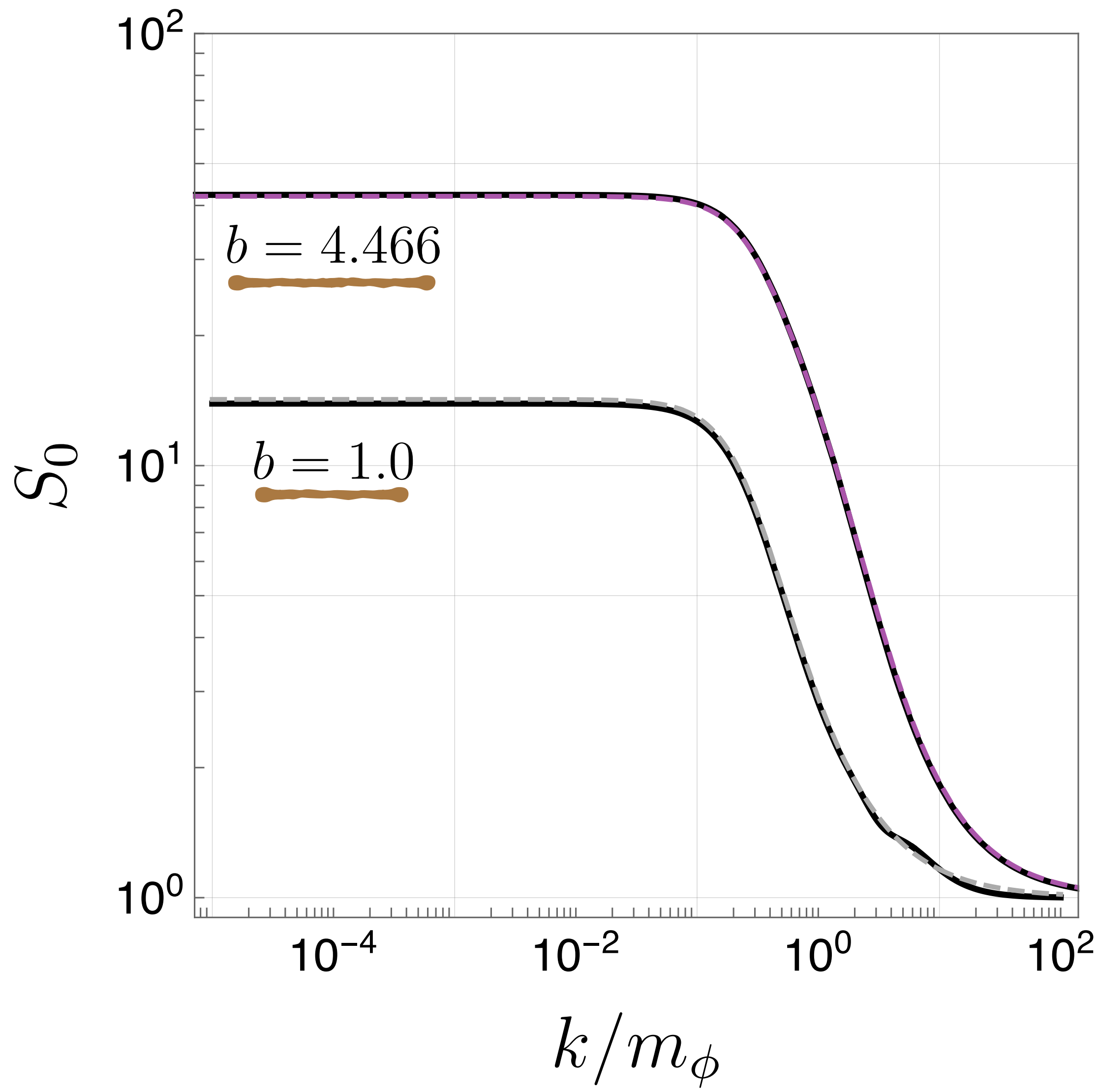


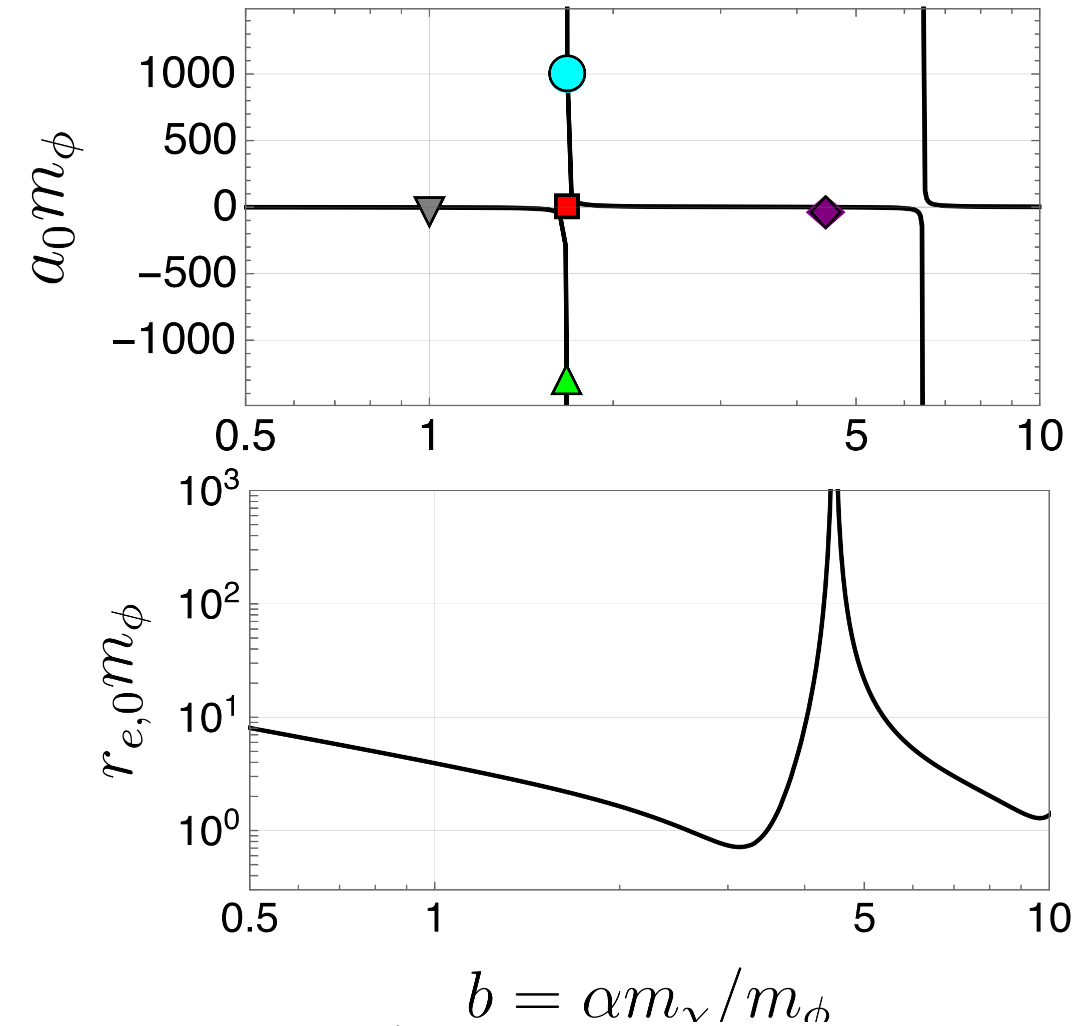
Omnès solution

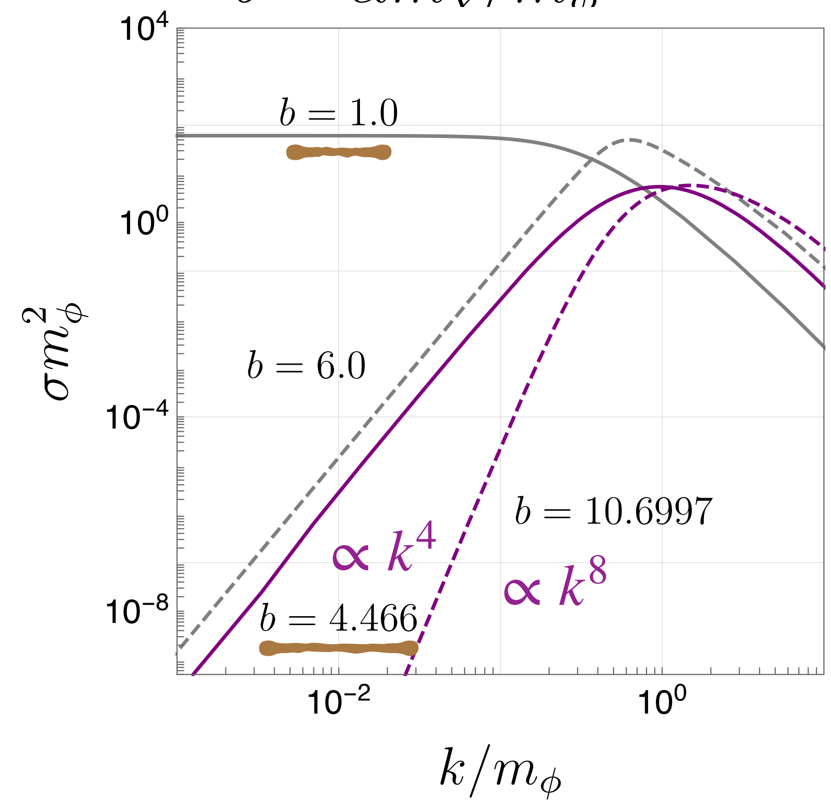
Yukawa potential

AK, Kuwahara and Patel, JHEP, 2023

- s-wave







- Omnès solution agrees with direct computation from scattering state

  - with proper $F_0(k^2)$

Around zero-energy resonances

S-wave

AK, Kuwahara and Patel, JHEP, 2023

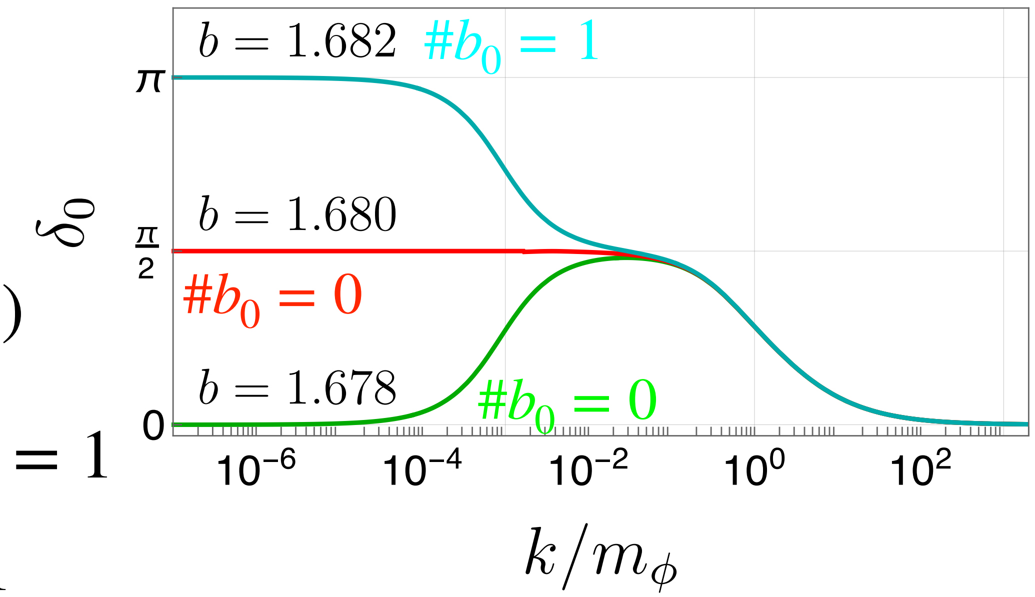
$$\delta_0(k \to 0) = \left[\#b_0 \left(+\frac{1}{2}\right)\right]\pi$$

- on 1st resonance $\#b_0 = 0$

$$k \to 0 \quad \mathrm{Re}[\omega_0(k^2 + i\epsilon)] \to -\frac{1}{2}\ln(r_{e,0}^2 k^2)$$

  - only zero energy "virtual" level $F_0(k^2) = 1$

$$k \to 0 \quad \Gamma_0(k^2) = \exp[\omega_0(k^2)]F_0(k^2) \propto \frac{1}{k}$$



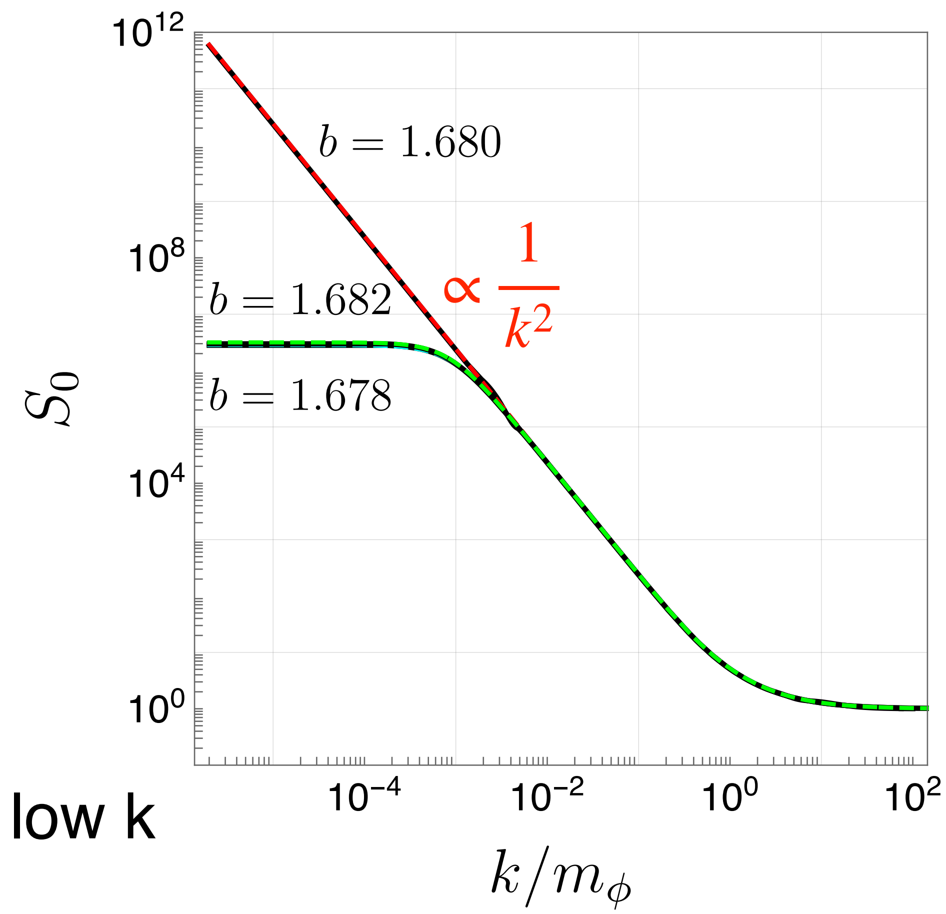
- slightly below the 1st resonance $\#b_0 = 0$
  - no bound state $F_0(k^2) = 1$
- slightly above the 1st resonance $\#b_0 = 1$

$$k \to 0 \quad \mathrm{Re}[\omega_0(k^2 + i\epsilon)] \to -\ln(r_{e,0}^2 k^2)$$

  - single bound state $F_0(k^2) = \frac{k^2}{k^2 + \kappa_{b,0}^2}$

$$k \to 0 \quad \Gamma_0(k^2) \propto \frac{1}{k^2 + \kappa_{b,0}^2}$$

- saturates at low k

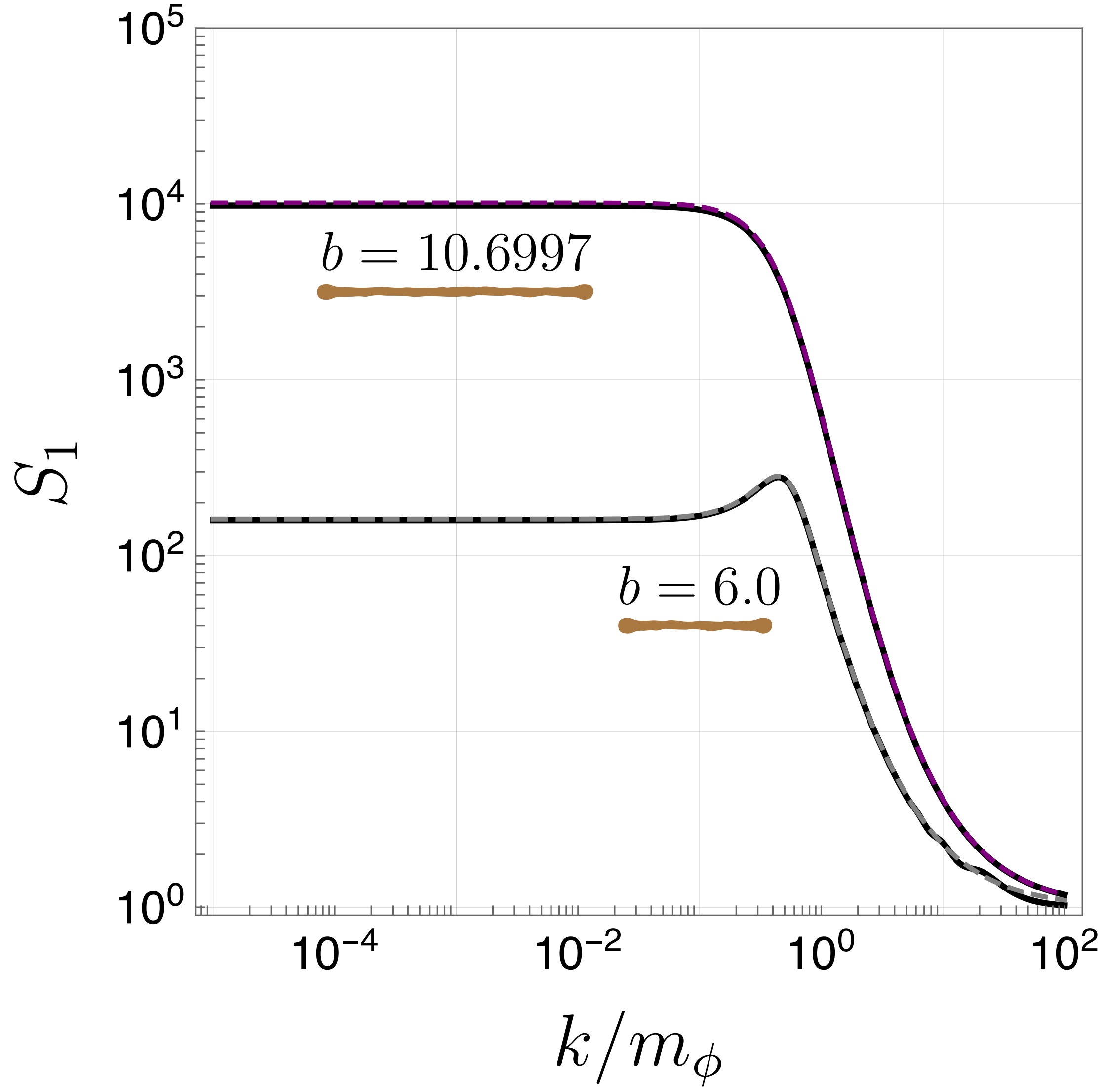


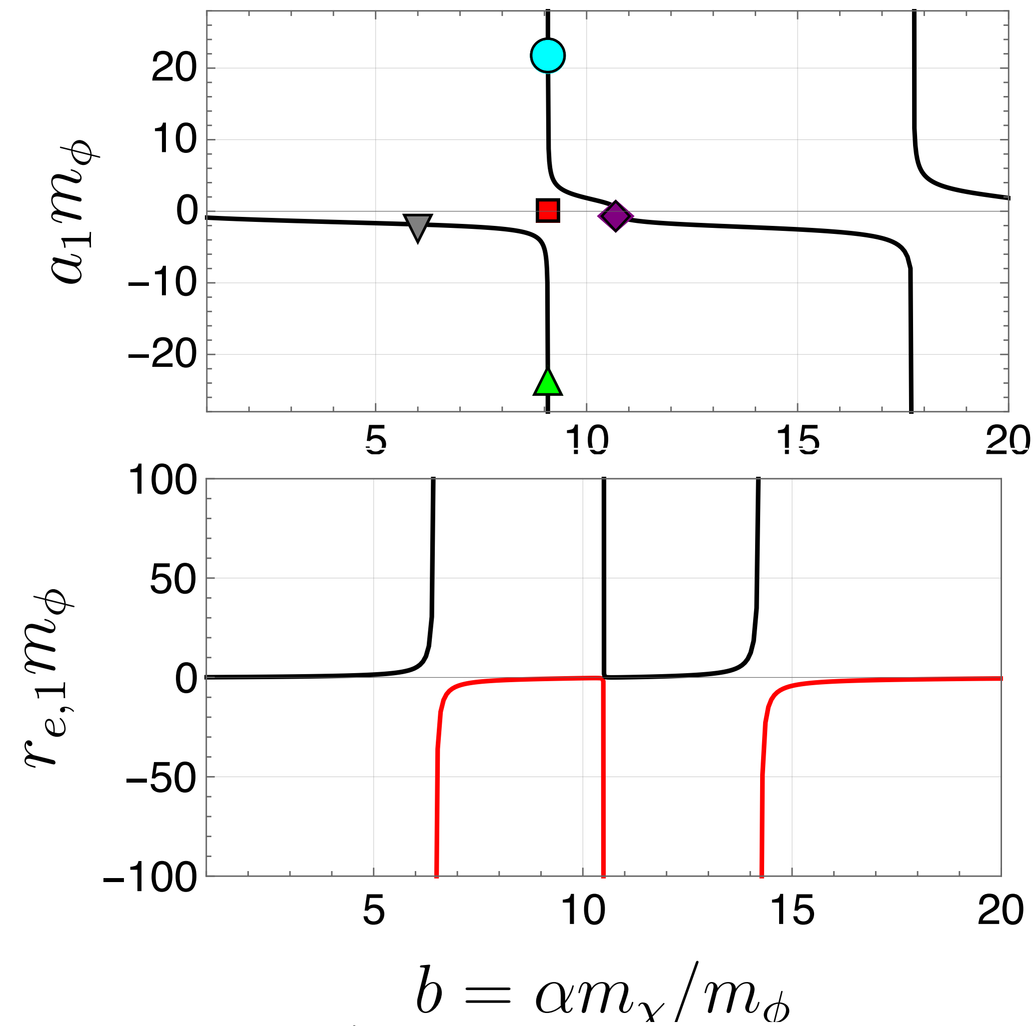
Omnès solution

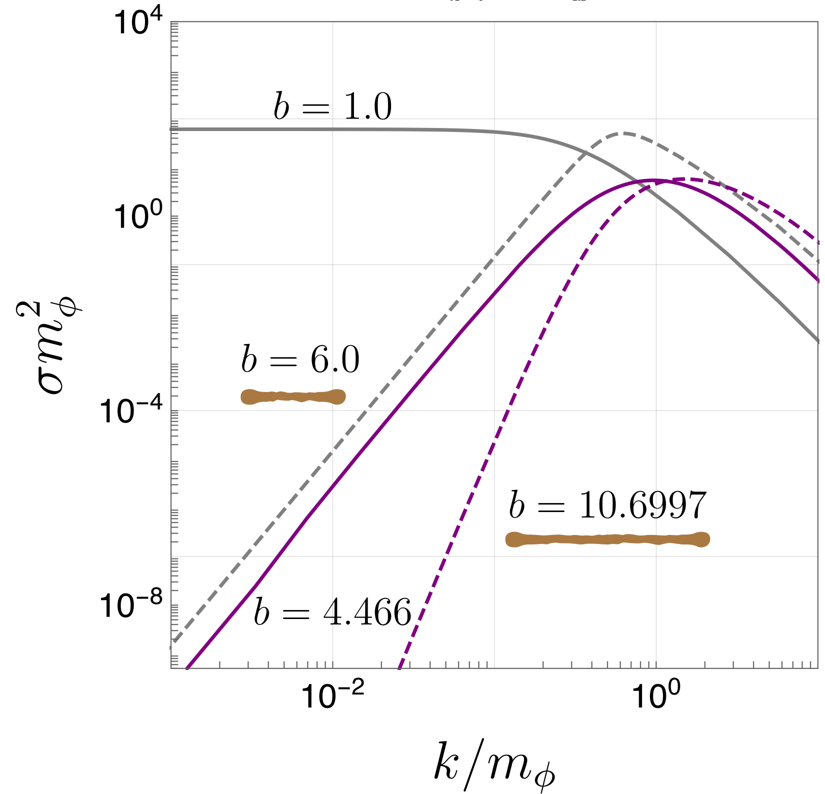
Yukawa potential

- p-wave

AK, Kuwahara and Patel, JHEP, 2023







Around zero-energy resonances

P-wave

$$\delta_1(k \to 0) = \#b_1 \pi$$

- on the 1st resonance $\#b_1 = 1$

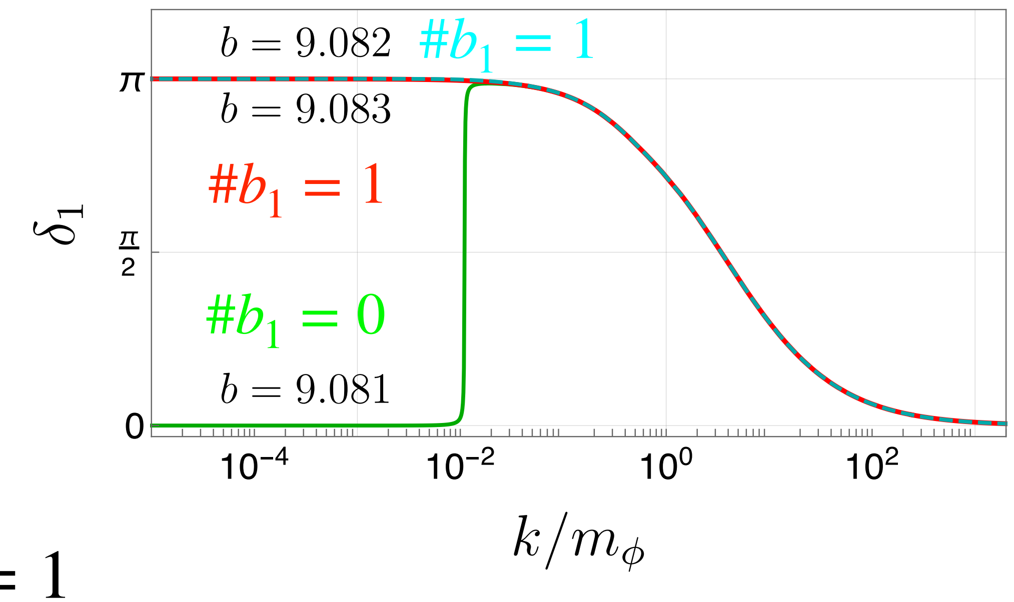
$$k \to 0 \quad \mathrm{Re}[\omega_1(k^2)] \to -\ln(r_{e,1}^2 k^2)$$

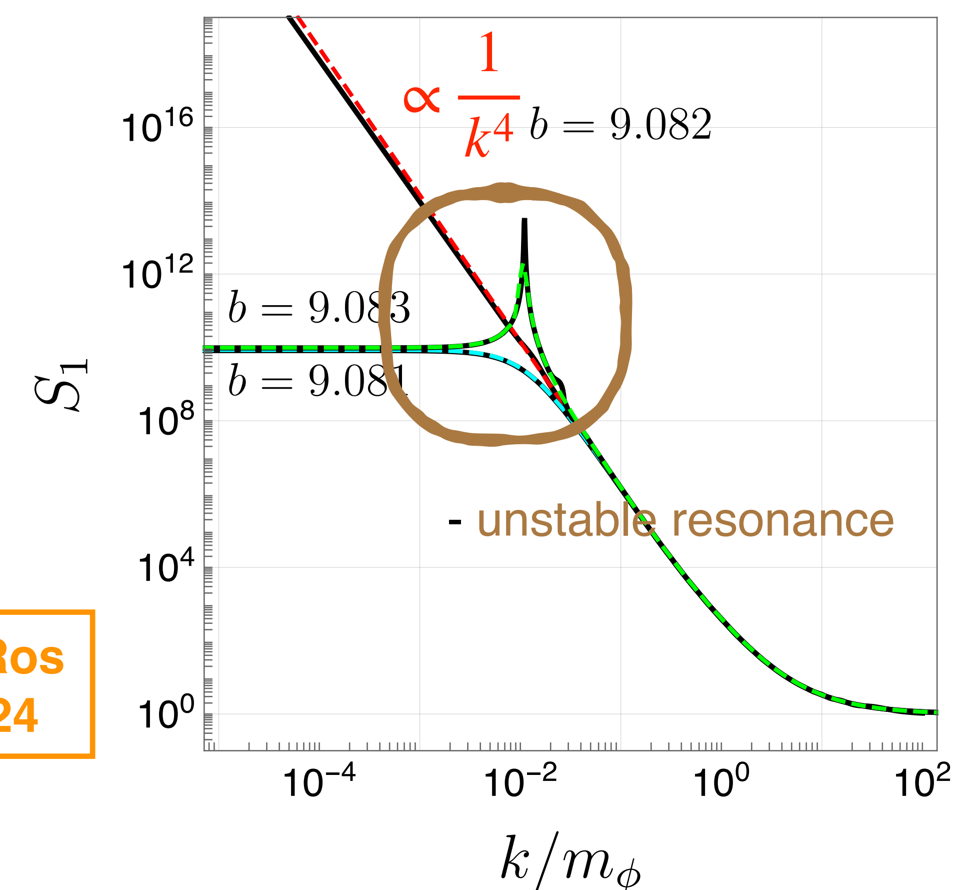
- zero energy bound state $F_1(k^2) = \dfrac{k^2}{k^2} = 1$

$$k \to 0 \quad \Gamma_1(k^2) = \exp[\omega_1(k^2)] F_1(k^2) \propto \frac{1}{k^2}$$

- slightly below/above the 1st resonance
  - similar to s-wave

AK, Kuwahara and Patel, JHEP, 2023





Beneke, Binder, De Ros and Grany, JHEP, 2024

Full scattering state

AK, Matsumoto and Watanabe, in progress

Linear combination of regular and singular solutions

$$\tilde{R}_{k,\ell}(r) = \mathscr{A}_\ell(k)\mathscr{R}_{k,\ell}(r) + \mathscr{B}_\ell(k)\mathscr{S}_{k,\ell}(r)$$ - valid except for the origin

- regular solution we discussed before

- singular solution we introduce now

$$\mathscr{S}_{k,\ell}(r) \to k y_\ell(kr) \approx -k\frac{(2\ell-1)!!}{(kr)^{\ell+1}} \qquad r \to 0$$

$$\mathscr{S}_{k,\ell}(r) \to -\frac{1}{2r}\left[\mathscr{K}_\ell(k)e^{-i\left(kr-\frac{1}{2}\ell\pi\right)} + \mathscr{K}_\ell(-k)e^{i\left(kr-\frac{1}{2}\ell\pi\right)}\right] \qquad r \to \infty$$

- one combination of two unknown coefficients is fixed by requirement of in-coming wave

$$\tilde{R}_{k,\ell}(r) \to \frac{i}{2r}\left[e^{-i\left(kr-\frac{1}{2}\ell\pi\right)} - S_\ell(k)e^{i\left(kr-\frac{1}{2}\ell\pi\right)}\right] \qquad r \to \infty$$

$$\mathscr{A}_\ell(k)\mathscr{J}_\ell(k) + i\mathscr{B}_\ell(k)\mathscr{K}_\ell(k) = 1 \qquad S_\ell(k) = \frac{\mathscr{J}_\ell(-k)}{\mathscr{J}_\ell(k)}\left[1 - i\mathscr{B}_\ell(k)\mathscr{K}_\ell(k)\right] - i\mathscr{B}_\ell(k)\mathscr{K}_\ell(-k)$$

Full scattering state

The other combination is fixed by renormalization condition

$$\mathcal{B}_\ell(k) = \frac{k^{2\ell+1}}{p_\ell(k)\mathcal{J}_\ell(k)} \qquad \tilde{R}_{k,\ell}(r) = \mathcal{B}_\ell(k)\left(\left[p_\ell(k) - k^{2\ell+1}\frac{i\mathcal{K}_\ell(k)}{\mathcal{J}_\ell(k)}\right]\frac{\mathcal{R}_{k,\ell}(r)}{k^{2\ell+1}} + \mathcal{S}_{k,\ell}(r)\right)$$

- potential has delta-function (contact) term

$$V \supset u\delta^3(\vec{x})$$

- kinetic term of singular solution has contact term at the origin

$$\left(-\frac{\nabla^2}{2\mu}\right)\frac{1}{r} = \frac{4\pi}{2\mu}\delta^3(\vec{x})$$

- cancellation between them leads to renormalization condition

$$p_\ell(k) - k^{2\ell+1}\frac{i\mathcal{K}_\ell(k)}{\mathcal{J}_\ell(k)} + \frac{d^{2\ell+1}}{dr^{2\ell+1}}\left[\frac{(kr)^\ell}{(2\ell)!!}r\mathcal{S}_{k,\ell}\right](0)$$

$$= p_\ell(k_0) - k_0^{2\ell+1}\frac{i\mathcal{K}_\ell(k_0)}{\mathcal{J}_\ell(k_0)} + \frac{d^{2\ell+1}}{dr^{2\ell+1}}\left[\frac{(k_0 r)^\ell}{(2\ell)!!}r\mathcal{S}_{k_0,\ell}\right](0)$$

- renormalization scale

Full scattering state

Bound state with decay width

AK, Matsumoto and Watanabe, in progress

- bound state is a pole of S-matrix
- non-Unitarized S-matrix has a pole at pure imaginary momentum

$$\mathcal{J}_\ell(k = i\kappa) = 0$$

- one can find the correction to the pole from Unitarized S-matrix by using properties of Jost function

$$\mathrm{Im} E_B = -\frac{1}{2(4\pi)} \frac{\sigma^\ell_{\mathrm{ann},0} v}{(2\ell+1)p^{2\ell}} \left| \frac{(2\ell+1)!!}{\ell!} \frac{d^\ell R^\ell_B}{dr^\ell}(0) \right|^2$$

$$\mathrm{Re} E_B = -\frac{\kappa^2}{2\mu} + \frac{1}{2(4\pi)\mu} \eta \sqrt{\frac{4\pi\sigma^\ell_{\mathrm{sc},0}}{(2\ell+1)p^{4\ell}} - \left(\frac{\sigma^\ell_{\mathrm{ann},0}}{(2\ell+1)p^{2\ell-1}} \right)^2} \left| \frac{(2\ell+1)!!}{\ell!} \frac{d^\ell R^\ell_B}{dr^\ell}(0) \right|^2$$