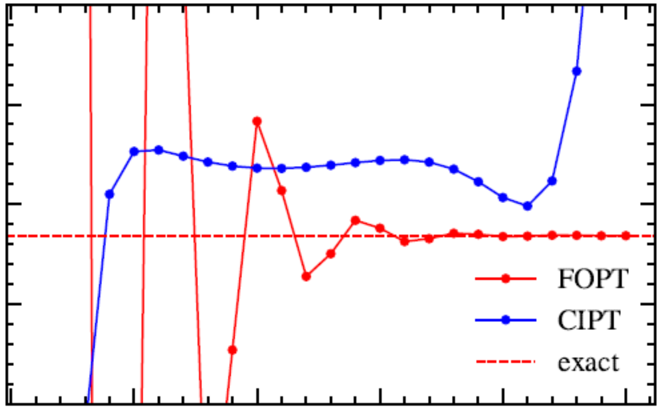
XVI CPAN days (2024)

Mathematical aspects of FOPT/CIPT discrepancy in Tau decays



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What is the FOPT/CIPT discrepancy?

Context

- **FOPT/CIPT discrepancy:** long-standing problem

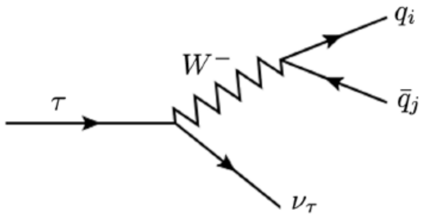
Fixed **O**rder **P**erturbation **T**heory and **C**ontour **I**mproved **P**erturbation **T**heory

handle differenly the resummation of QCD series, systematically yielding different results

- Prominent effect in **Tau decays:** affects the determination of α_s

Where does the discrepancy arise?

- $\tau^- \longrightarrow \text{hadrons}\, \nu_\tau$: clean frame to study QCD



- The extraction of α_s is competitive with the world average

Dominated by FOPT/CIPT discrepancy

$$\alpha_s(m_\tau) = 0.332 \pm 0005_{\text{exp}} \pm \boxed{0.011_{\text{theo}}}$$

M. Davier, A. Höcker, B. Malaescu, C. Z. Yuan, and Z. Zhang. Update of the ALEPH non-strange spectral functions from hadronic τ decays. *The European Physical Journal C*, 74(3), mar 2014.

- Extracted from matching invariant mass spectral functions (experiment) onto the integral of the vacuum polarization function (theory):

Perturbative QCD corrections to $\tau \to \text{hadrons}\, \nu_\tau$

$$\boxed{\delta^{(0)}_{W(x)} = \frac{1}{2i\pi} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W(x) \hat{D}(m_\tau^2 x)}$$

$$D(s) = \frac{N_c}{12\pi^2}\left(1 + \hat{D}(s)\right) = -s\frac{\mathrm{d}}{\mathrm{d}s}\Pi(s)$$

$$W_\tau(x) = 1 - 2x + 2x^3 - x^4$$

$$\delta^{(0)}_{W(x)} = \frac{1}{2i\pi}\oint_{|x|=1}\frac{\mathrm{d}x}{x}W(x)\hat{D}(m_\tau^2 x)$$

- $\hat{D}(\mu)$ is μ-independent

$$\begin{aligned}
\hat{D}(s) &= \sum_{n=1}^{\infty}\left[\frac{\alpha_s(\mu^2)}{\pi}\right]^n \sum_{k=1}^{n} k\, c_{n,k} \ln^{k-1}\left(-\frac{s}{\mu^2}\right) \\
&= \boxed{\sum_{n=1}^{\infty}\left[\frac{\alpha_s(m_\tau^2)}{\pi}\right]^n \sum_{k=1}^{n} k\, c_{n,k} \ln^{k-1}\left(-\frac{s}{m_\tau^2}\right)} \qquad \text{FOPT: } \mu^2 = m_\tau^2 \\
&= \boxed{\sum_{n=1}^{\infty}\left[\frac{\alpha_s(-s)}{\pi}\right]^n c_{n,1}} \qquad \text{CIPT: } \mu^2 = -s
\end{aligned}$$

$$a(\mu^2) = \frac{\beta_0 \alpha_s(\mu^2)}{4\pi}$$
$$a = a(m_\tau^2)$$
$$\bar{c}_{n,k} = (4/\beta_0)^n c_{n,k}$$

- Build the $\delta^{(0)}_{W(x)}$ from contributions from moments $(-x)^\ell$

$$\delta^{(0)}_{\mathrm{FOPT},\ell} = \frac{1}{2i\pi}\sum_{n=1}^{\infty}\left[\frac{\alpha_s(m_\tau^2)}{\pi}\right]^n \sum_{k=1}^{n} k\, c_{n,k} \oint_{|x|=1}\frac{\mathrm{d}x}{x}(-x)^\ell \ln^{k-1}(-x)$$

$$\delta^{(0)}_{\mathrm{CIPT},\ell} = \frac{1}{2i\pi}\sum_{n=1}^{\infty}\frac{c_{n,1}}{\pi} \oint_{|x|=1}\frac{\mathrm{d}x}{x}(-x)^\ell\, \alpha_s^n(-x\, m_\tau^2)$$

Expansion in powers a^n

$$\boxed{\delta^{(0)}_{\mathrm{FOPT},\ell} = \sum_{n=1}^{\infty} a^n \sum_{k=1}^{n} k\, \bar{c}_{n,k} I_{k-1,\ell}}$$

$$\boxed{\delta^{(0)}_{\mathrm{CIPT},\ell} = \sum_{n=1}^{\infty} \bar{c}_{n,1} H_{n,\ell}(a)}$$

Expansion in functions $H_{n,\ell}(a)$

Our analysis

$$\delta^{(0)}_{\mathrm{FOPT},\ell} = \sum_{n=1}^{\infty} a^n \sum_{k=1}^{n} k\, \bar{c}_{n,k}\, I_{k-1,\ell}$$

$$I_k = \frac{1}{2i\pi} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^\ell \ln^k(-x)$$

$$\delta^{(0)}_{\mathrm{CIPT},\ell} = \sum_{n=1}^{\infty} \bar{c}_{n,1}\, H_{n,\ell}(a)$$

$$H_{n,\ell}(a) = \frac{1}{2i\pi} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^\ell a^n(-x m_\tau^2)$$

- **Large-β_0 limit:** full analytic control
- **Full QCD:** numerical confirmation

Our analysis

$$\delta^{(0)}_{\mathrm{FOPT},\ell} = \sum_{n=1}^{\infty} a^n \sum_{k=1}^{n} k\, \bar{c}_{n,k}\, I_{k-1,\ell}$$

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$$H_{n,\ell}(a) = \frac{1}{2i\pi} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^\ell a^n(-x m_\tau^2)$$

Large-β_0 limit: full analytic control

❶ Running of α_s

$$a(\mu^2) = \frac{a(\mu_0^2)}{1 + a(\mu_0^2)\ln\left(\frac{\mu^2}{\mu_0^2}\right)}$$

❷ Adler function in closed form (Borel sum)

$$\hat{D}(s) = \int_0^\infty \mathrm{d}u \left[B(u)\right]_{\mathrm{Taylor}} e^{-\frac{u}{a(-s)}}$$

$$B(u) = \frac{128}{3\beta_0} e^{\frac{5u}{3}} \left\{ \frac{3}{16(2-u)} + \sum_{p=3}^{\infty} \left[\frac{d_2(p)}{(p-u)^2} - \frac{d_1(p)}{p-u} \right] - \sum_{p=-1}^{-\infty} \left[\frac{d_2(p)}{(u-p)^2} + \frac{d_1(p)}{u-p} \right] \right\}$$

Our model: series arising from a single-pole contribution

$$B_p(u) = \frac{1}{p-u} \quad \Longrightarrow \quad \bar{c}_{n,k,p} = \frac{(-1)^{k+1}\Gamma(n)}{p^{n-k+1}\Gamma(k+1)}$$

Our analysis

$$\delta^{(0)}_{\mathrm{FOPT},\ell} = \sum_{n=1}^{\infty} a^n \sum_{k=1}^{n} k\, \bar{c}_{n,k}\, I_{k-1,\ell}$$

$$I_k = \frac{1}{2i\pi} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^{\ell} \ln^k(-x)$$

$$\delta^{(0)}_{\mathrm{CIPT},\ell} = \sum_{n=1}^{\infty} \bar{c}_{n,1}\, H_{n,\ell}(a)$$

$$H_{n,\ell}(a) = \frac{1}{2i\pi} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^{\ell} a^n(-x m_\tau^2)$$

Large-β_0 limit: full analytic control

$$d^{\mathrm{FOPT}}_{n,\ell} = \sum_{k=1}^{n} k\, \bar{c}_{n,k} I_{k-1,\ell}$$

$$d^{\mathrm{FOPT}}_{n,0,p} = \frac{(-1)^p\, \Gamma(n+1, i\pi p, -i\pi p)}{2i\pi n p^{n+1}}$$

$$d^{\mathrm{FOPT}}_{n,\ell>0,p\neq\ell} = \frac{\ell^{-n}\Gamma(n, -i\pi\ell, i\pi\ell) - (-1)^{\ell+p} p^{-n}\Gamma(n, -i\pi p, i\pi p)}{2i\pi(p-\ell)}$$

$$d^{\mathrm{FOPT}}_{n,\ell>0,p=\ell} = \frac{\Gamma(n, i\pi p) + \Gamma(n, -i\pi p)}{2p^n} + \frac{\Gamma(n+1, -i\pi p, i\pi p)}{2\pi i p^{n+1}}$$

$$H_{n,\ell}(a) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \mathrm{d}\phi\, e^{i\ell\phi} \left[\frac{a}{1+ia\phi}\right]^n$$

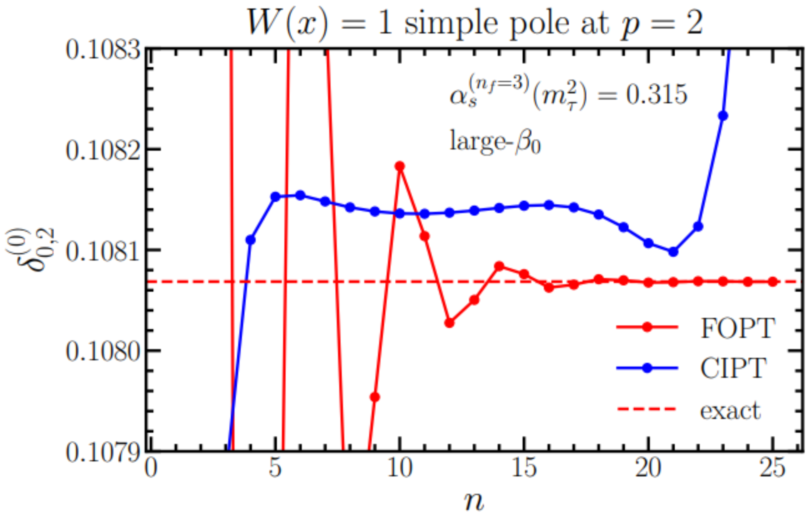
$$H_{1,0}(a) = \frac{1}{\pi} \arctan(a\pi)$$

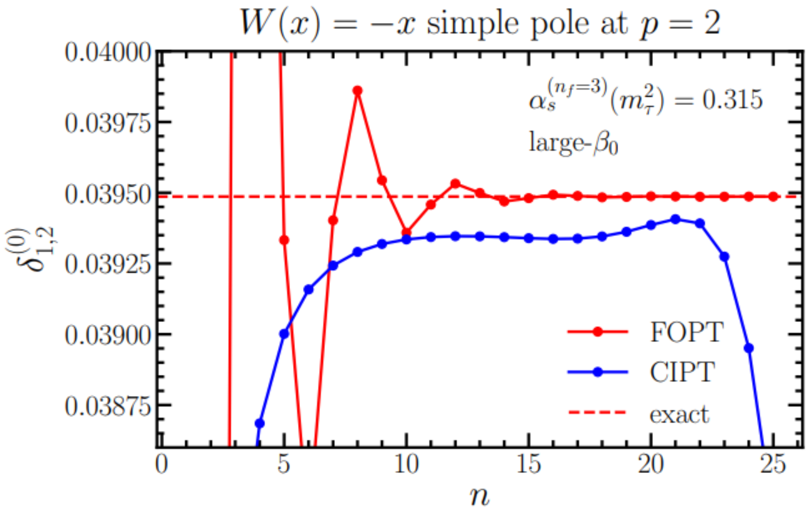
$$H_{n\geq 2,0} = \left(\frac{a}{\sqrt{1+a^2\pi^2}}\right)^n \frac{\sin[(n-1)\arctan(a\pi)]}{\pi(n-1)}$$

$$H_{n,\ell\geq 1} = \frac{(-1)^n}{2i\pi} e^{-\ell/a} \ell^{n-1} \Gamma\left(1-n, -\frac{\ell}{a} - i\pi\ell, -\frac{\ell}{a} + i\pi\ell\right)$$

1) Convergence and large order behavior

Our analysis





- FOPT is an **absolute convergent series** within the circle $|a| < 1/\pi$ in the complex plane of a (large n and root test)

- CIPT is an has **zero convergence radius**

$$\limsup_{n\to\infty} |H_{n,\ell}(a)|^{1/n} = \frac{|a|}{\sqrt{1+|a|^2\pi^2 - 2\pi|\mathrm{Im}(a)|}}$$

$$\delta^{(0)}_{\mathrm{CIPT},\ell} = \sum_{n=1}^{\infty} \underbrace{\bar{c}_{n,1}}_{\text{divergent}} \overbrace{H_{n,\ell}(a)}^{\text{bounded}} \qquad \bar{c}_{n,1} = \Gamma(n)/p^n$$

Bonus observation: for $a \in \mathbb{R}$, $H_{n,\ell}(a)$ scales in powers of $\frac{|a|}{\sqrt{1+a^2\pi^2}} < \min\left(|a|, \frac{1}{\pi}\right)$

$\implies$ a reason for CIP's better convergence at low orders

2) A deeper look: asymptotic sequences

Our analysis

Definition (Asymptotic Sequence). *A sequence of functions $\{\phi_n(x)\}$ is an asymptotic sequence as $x \to x_0$ in R if $\phi_{n+1} = o(\phi_n)$ for all n.*

Definition (o-relation). *It is said that $f = o(g)$ as $x \to x_0$, if for any $\epsilon > 0$ there exists a neighborhood U_ϵ of x_0 such that $|f(x)| < \epsilon |g(x)|$ for all $x \in R \cap U_\epsilon$. When $g(x) \neq 0$ in some neighborhood of x_0, the condition is equivalent to $\lim_{x\to x_0} f(x)/g(x) = 0$.*

FOPT: $\{a^n\}_{n=1}^{\infty}$ is an **asymptotic sequence** as $a \to 0$:

$$\lim_{n\to\infty} \frac{a^{n+1}}{a^n} = 0$$

It is also **uniform in n**: U_ϵ is the same for all n

CIPT: $H_{n,\ell}(a)$ are **analytic** in the cirlce $|a| < 1/\pi$ and vanish as a^{n+1} as $a \to 0$

$\{H_{n,\ell}(a)\}_{n=1}^{\infty}$ is an **asymptotic sequence** as $a \to 0$

but it is **not uniform in n**: $\lim_{n\to\infty} \frac{H_{n+1,\ell}(a)}{H_{n,\ell(a)}} = 0$ only in regions U_ϵ that srink to zero with n

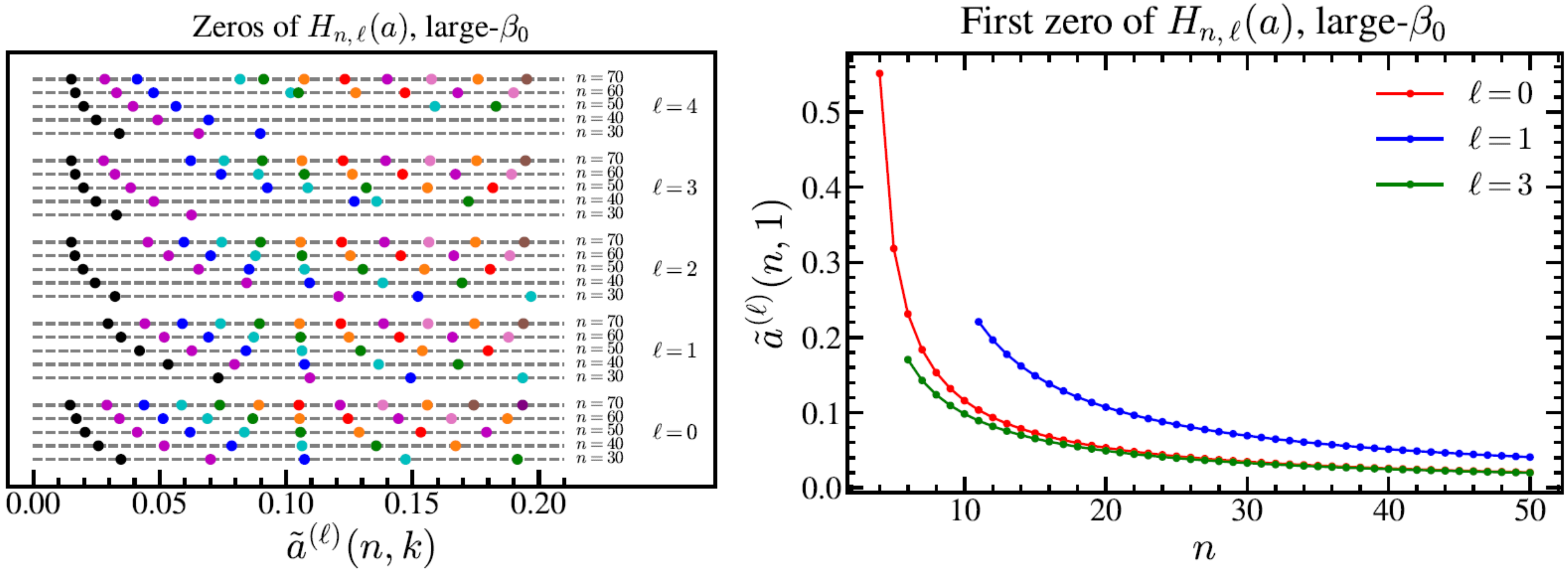
$$H_{n\geq 2,0} = \left(\frac{a}{\sqrt{1+a^2\pi^2}}\right)^n \frac{\sin[(n-1)\arctan(a\pi)]}{\pi(n-1)}$$

zeros that approach $a = 0$ with n

$$\tilde{a}(n,k) = \frac{1}{\pi} \tan\left(\frac{k\pi}{n-1}\right), \qquad -\frac{n-1}{2} < k < \frac{n-1}{2}$$

2) A deeper look: asymptotic sequences

Our analysis



- The number of zeros grows with n
- As n increases, each zero moves closer to $a = 0$
- The zeros of $H_{n,\ell}(a)$ may provide another reason for the better convergence of CIPT at intermediate orders

3) A deeper look: double series transformation

Our analysis

- **CIPT** $\longrightarrow$ **FOPT** by analiticity, $H_{n,\ell}(a)$ can be Taylor-expanded around $a=0$ $\quad H_{n,\ell}=\sum_{k=n}^{\infty} s_{n,k}^{\ell}\, a^k$

$$s_{n,k}^{0}=\frac{\Gamma(k)\cos[\frac{\pi}{2}(k-n)]}{\Gamma(n)\Gamma(k-n+2)} \qquad s_{n,k}^{\ell}=\frac{\Gamma(k)\Gamma(k-n+1,-i\pi\ell,i\pi\ell)}{2\pi i\,\ell^{k+n-1}\,\Gamma(n)\Gamma(k-n+1)}$$

The expansion is **absolutely convergent** in the circle $|a|<1/\pi$

A convergent CIPT will turn into a convergent FOPT

Even if the original CIPT is not convergent, it may turn into a convergent FOPT

- **FOPT** $\longrightarrow$ **CIPT**: the coefficients of the inverse transformation can be determined $\quad a^n=\sum_{k=n}^{\infty} t_{n,k}^{\ell} H_{n,\ell}(a)$

$$t_{n,k}^{0}=\frac{(i\pi)^{k-n}(2-2^{k-n})\Gamma(k)}{\Gamma(k-n+1)\Gamma(n)}B_{k-n}, \qquad t_{n>1,k}^{\ell>0}=\begin{cases}0, & k<n-1\\ (-1)^{\ell}\frac{k-1}{n-1}t_{n-1,k-1}^{0}, & n+k\quad \text{even}\\ (-1)^{\ell+1}\frac{\ell}{n-1}t_{n-1,k}^{0}, & n+k\quad \text{odd}\end{cases}$$

Bernoulli numbers

The expansion has **zero convergence radius**

An absolute convergent FOPT will turn into a divergent CIPT

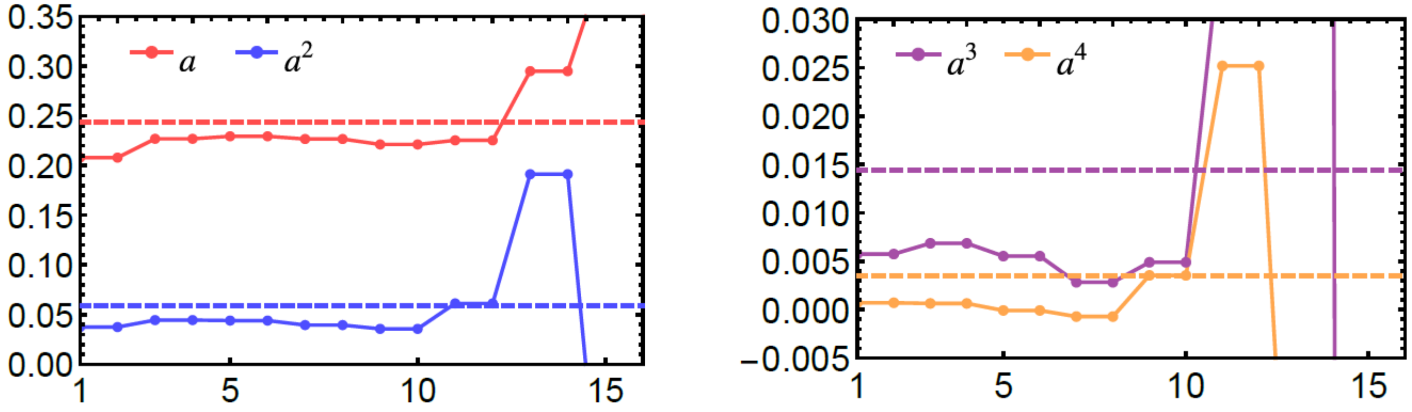
- **FOPT** $\longrightarrow$ **CIPT**: the coefficients of the inverse transformation can be determined $a^n = \sum_{k=n}^{\infty} t^{\ell}_{n,k} H_{n,\ell}(a)$

$$t^0_{n,k} = \frac{(i\pi)^{k-n}(2-2^{k-n})\Gamma(k)}{\Gamma(k-n+1)\Gamma(n)} \underbrace{B_{k-n}}_{\text{Bernoulli numbers}}, \qquad t^{\ell>0}_{n>1,k} = \begin{cases} 0, & k < n-1 \\ (-1)^{\ell}\frac{k-1}{n-1} t^0_{n-1,k-1}, & n+k \quad \text{even} \\ (-1)^{\ell+1}\frac{\ell}{n-1} t^0_{n-1,k}, & n+k \quad \text{odd} \end{cases}$$

The expansion has **zero convergence radius**

An absolute convergent FOPT will turn into a divergent CIPT

Example: power series with a single power a^n



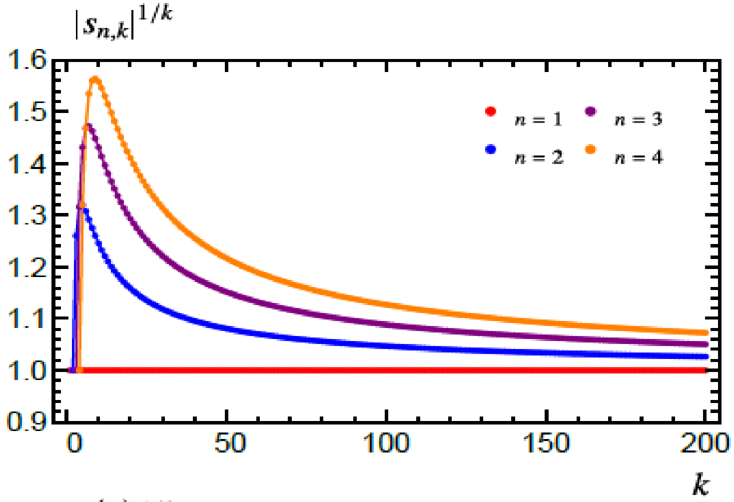
4) A deeper look: remaining question

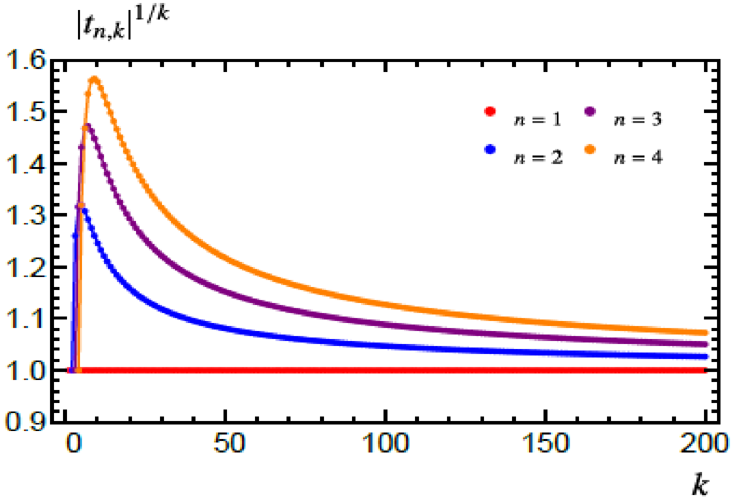
Our analysis

Connection between the non-uniformity of $\{H_{n,\ell}(a)\}_{n=1}^{\infty}$ induced by the zeros and the divergence of $a^n = \sum_{n=1}^{\infty} t_{n,k}^{\ell} H_{n,\ell}(a)$

Model of $H_{n,\ell}(a)$ with n-independent zeros

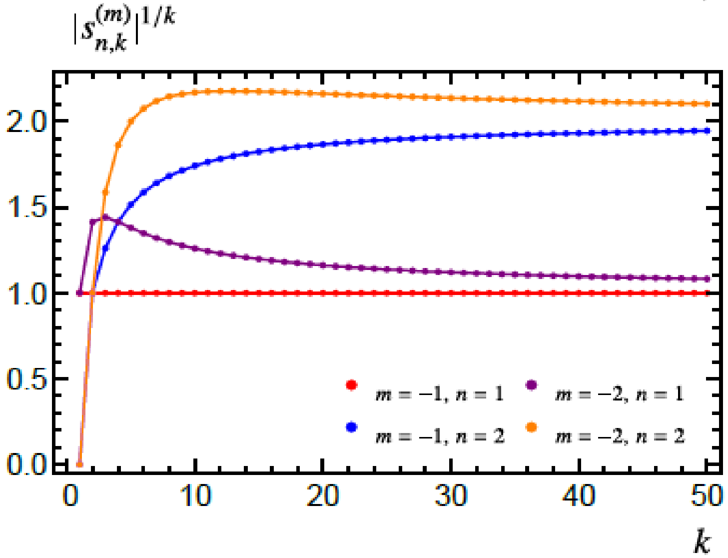
$$h_n(a) = \frac{a^n}{(1+aL)^n}$$

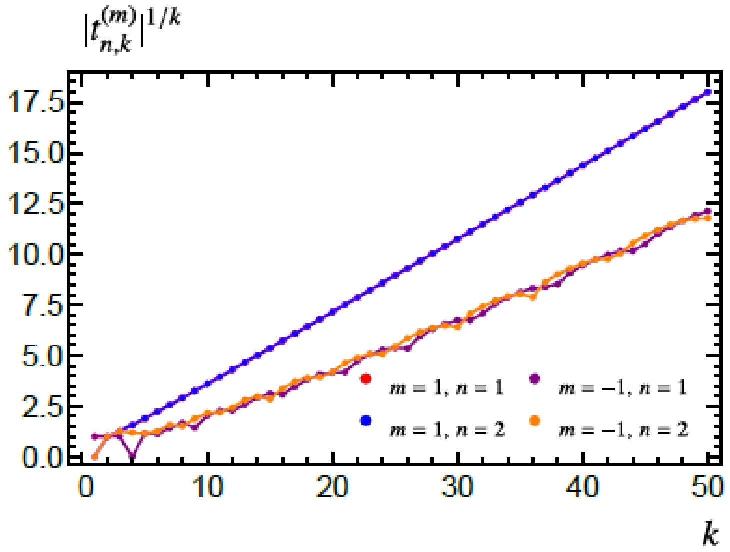




Model of $H_{n,\ell}(a)$ with zeros that grow closer to $a=0$ with n

$$\hat{h}_n^{(m)}(a) = a^n(1-\xi n a)^m$$





Conclussions

- **FOPT** is an expansion in powers of the strong coupling, a^n, while **CIPT** is an expansion in non-trivial functions of the strong coupling $H_{n,\ell}(a)$
- While both $\{a^n\}$ and $\{H_{n,\ell}(a)\}$ are asymptotic sequences, **FOPT** is uniform in n but **CIPT** is not, due to the zeros of $H_{n,\ell}(a)$ that increase in number and approach $a = 0$ with n.
- The **CIPT** expansion of a single power a^n is divergent, a fact that arises for every H-model with zeros approaching the origin.
- We believe the convergence properties of $a^n = \sum_{n=1}^{\infty} t^{\ell}_{n,k} H_{n,\ell}(a)$ provide a diagnostic tool to test expansions in functions of the running coupling.
- CIPT can still be used as a viable phenomenological method once the dominant IR renormalons contained in the Adler function are removed by subtractions (M. A. Benitez-Rathgeb, D. Boito, A. H. Hoang and M. Jamin, 2022)
- Our claims are valid for linear combinations of moments $(-x)^{\ell}$, in particular for $W_{\tau}(x) = 1 - 2x + 2x^3 - x^4$, which recovers the τ-decay width...
- $\ldots$ and also in full-QCD, where we confirm them numerically.

Thank you for your attention

Questions?

Backup slides

leading term. The expansion of $\Gamma(n,z)$ as $|n|\to\infty$ for any finite z and $n\neq 0,-1,-2\ldots$ reads

$$\Gamma(n,z)-\Gamma(n)\overset{-n\notin\mathbb{N}_0}{=}-z^n e^{-z}\sum_{k=0}^{\infty}\frac{z^k}{(n)_{k+1}}\,. \tag{A4}$$

This expansion is even absolutely convergent for any finite z. A particular case frequent in this work occurs when $z=\pm i\ell\pi$, with ℓ a positive integer. The following identify holds:

$$\Gamma(n,-i\ell\pi,i\ell\pi)=\Gamma(n,-i\ell\pi)-\Gamma(n,i\ell\pi)\overset{-n\notin\mathbb{N}_0}{=}2i(-1)^{\ell}\sum_{k=0}^{\infty}\frac{(\pi\ell)^{n+k}}{(n)_{k+1}}\sin\Big[(n+k)\frac{\pi}{2}\Big]. \tag{A5}$$

FOPT convergence radius (large-β_0)

the leading large-n expression (for $k=1$)

$$d_{n,\ell,p}^{\rm FOPT}-\frac{\Gamma(n)}{p^n}\delta_{\ell,p}=(-1)^{\ell+1}\pi^n\sum_{k=1}^{\infty}\frac{(p\pi)^{k-1}}{(n)_{k+1}}\sin\Big[\frac{\pi}{2}(k+n)\Big]\sum_{j=0}^{k-1}\Big(\frac{\ell}{p}\Big)^j$$

CIPT->FOPT convergence radius (large-β_0)

$$s_{n,k}^{\ell}=\frac{(-1)^{\ell}\pi^{k-n}}{\Gamma(n)}\sum_{j=0}^{\infty}\frac{(\ell\pi)^j}{(k)_{j-n+2}}\cos\Big[\frac{\pi}{2}(j+k-n)\Big]$$

rely on numerical studies and evidences rather than strict proofs. Still some closed formulas can be presented in the C-scheme for the strong coupling α_s [20]. The C-scheme uses that only the one- and two-loop coefficients of the QCD β-function are renormalization-scheme invariant. It is therefore possible to adopt a scheme for α_s where the QCD β-function takes the all-order form

$$\frac{\mathrm{d}a(\mu^2)}{\mathrm{d}\ln\mu^2} = -\frac{[a(\mu^2)]^2}{1-2\,\hat{b}_1 a(\mu^2)}\,, \tag{4.1}$$

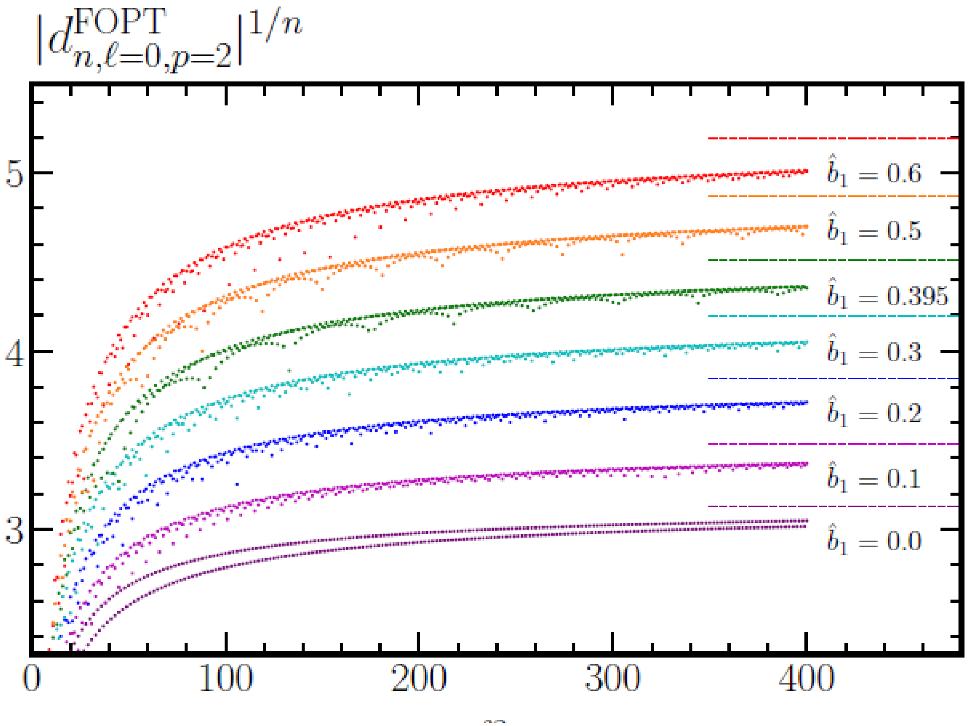
where $\hat{b}_1 = \beta_1/(2\beta_0^2)$ and $\beta_1 = 102 - 38\,n_f/3$.

We start by considering the full QCD generalization of the spectral function moment analysis of Sec. II. The full QCD generalization (in the C-scheme) of the single pole Borel function in the large-β_0 approximation corresponding to an OPE matrix-element corrections of the form const. $\times\,\langle\mathcal{O}_p\rangle/s_0^p$ in the Adler function reads $B_{p,\hat{b}_1}(u) = 1/(p-u)^{1+2p\hat{b}_1}$ (with $p=2,3,\ldots$) and consistency with the OPE again demands that the resulting perturbative series of the spectral function moment $\delta^{(0)}_{\ell\neq p}$ have a finite region of convergence. The resulting coefficients in the Adler function series of Eq. (2.1) read

$$\bar{c}_{n,1,p} = \frac{\Gamma(n+2p\hat{b}_1)}{p^{n+2p\hat{b}_1}\,\Gamma(1+2p\hat{b}_1)}\,, \tag{4.2}$$

$$\bar{c}_{n,k\geq 2,p} = -\frac{1}{k}\sum_{j=k-1}^{n-1}(2\hat{b}_1)^{n-j-1}\,j\,\bar{c}_{j,k-1,p}\,.$$

FOPT: limit superior is computed numerically



For $n_f = 3$, which is relevant for hadronic τ decays, we have $\hat{b}_1 = 32/81 \simeq 0.395$ which yields $\limsup_{n\to\infty} |d^{\mathrm{FOPT}}_{n,\ell=0,p=2}|^{1/n} = 4.528 \pm 0.020$ giving $\alpha_s(s_0) = 0.3083 \pm 0.0013$ as the convergence radius in the C-scheme. This corresponds to $\alpha_s(s_0) = 0.3151 \pm 0.0012$ in the $\overline{\mathrm{MS}}$ scheme which for $s_0 = m_\tau^2$ is right within the world average of the $\overline{\mathrm{MS}}$ strong coupling at the τ lepton scale $\alpha_s(m_\tau^2) = 0.312 \pm 0.015$. This interesting fact was already pointed out

The CIPT expansion functions $H_{n,\ell}(a)$ in full QCD have been determined analytically in

A. H. Hoang and C. Regner, *Borel representation of τ hadronic spectral function moments in contour-improved perturbation theory*, *Phys. Rev. D* **105** (2022) 096023, [2008.00578].

$$H_{n,\ell}(a) = \frac{i}{\pi} e^{-\frac{\ell}{a}} a^{-2\ell\hat{b}_1} \int_{t_-}^{t_+} \mathrm{d}t \left(1 + \frac{\hat{b}_1}{t}\right) e^{-2\ell t} (-2t)^{-2\ell\hat{b}_1 - n} . \tag{4.4}$$

The integral reduces to Eq. (2.19) in the large-β_0 approximation, when $\hat{b}_1 = 0$. The results can be given in closed form in terms of $a_\pm = a(-s_0 \pm i0)$ and read

$$H_{1,0}(a) = \frac{1}{2\pi i}\left[2\hat{b}_1(a_+ - a_-) + \log\left(\frac{a_-}{a_+}\right)\right], \tag{4.5}$$

$$H_{n\geq 2,0}(a) = \frac{1}{2\pi i}\left[\frac{2\hat{b}_1(a_+^n - a_-^n)}{n} - \frac{a_+^{n-1} - a_-^{n-1}}{n-1}\right], \tag{4.6}$$

$$H_{n,\ell>0}(a) = \frac{\ell^{n+2\ell\hat{b}_1-1}}{a^{2\ell\hat{b}_1}} e^{-\frac{\ell}{a}}\left[\frac{n}{\Gamma(n+2\ell\hat{b}_1-1)} + e^{-i\pi\hat{b}_1\ell} h_{n,\ell}(a_+, \hat{b}_1) - e^{i\pi\hat{b}_1\ell} h_{n,\ell}(a_-, \hat{b}_1)\right], \tag{4.7}$$

where

$$h_{n,\ell}(a, \hat{b}_1) \equiv \frac{(-1)^{n+1}}{2\pi i}\left[n\,\Gamma\left(-n - 2\hat{b}_1\ell, -\frac{\ell}{a}\right) - e^{\frac{\ell}{a}}\left(-\frac{a}{\ell}\right)^{n+2\hat{b}_1\ell}\right]. \tag{4.8}$$

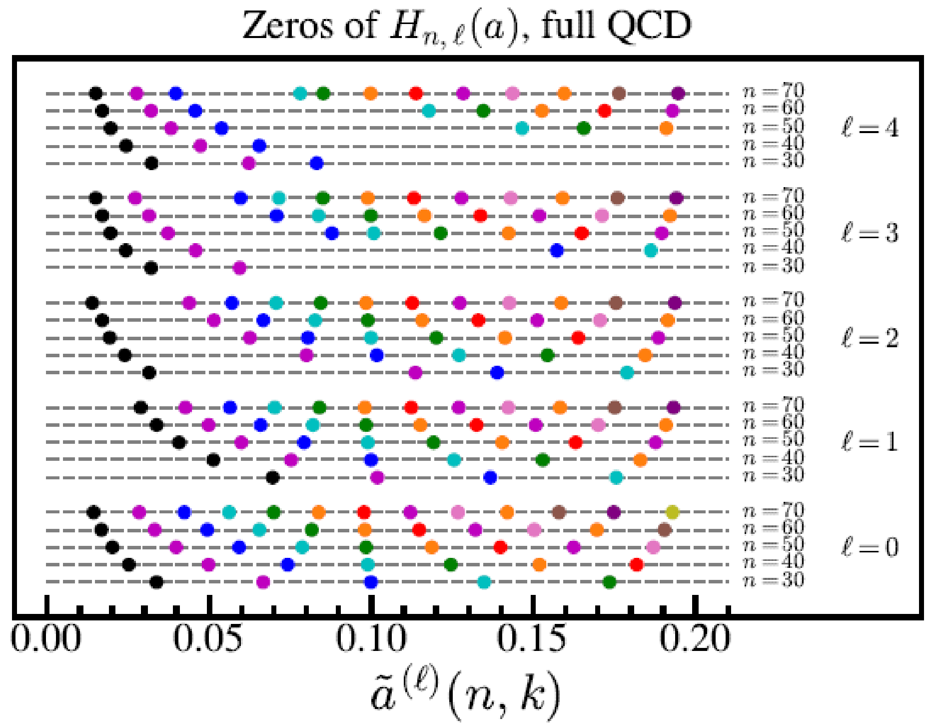
$$H_{n,\ell}(a) \overset{n\to\infty}{\asymp} e^{\ell\left(\frac{\mathrm{Re}(a_+)}{|a_+|}-\frac{1}{a}\right)} \frac{(|a_+|/a)^{2\ell\hat{b}_1}|a_+|^n}{n\pi}\left[2\hat{b}_1\sin(\theta_n) - \frac{\sin(\theta_{n-1})}{|a_+|}\right], \tag{4.9}$$

with $\theta_n \equiv (2\ell\hat{b}_1 + n)\arctan[\mathrm{Im}(a_+)/\mathrm{Re}(a_+)] - \ell\,\mathrm{Im}(a_+)/|a_+|^2$. This allows us to also determine the limit superior of $|H_{n,\ell}(a)|^{1/n}$ as $n\to\infty$ in full QCD, which is surprisingly simple,

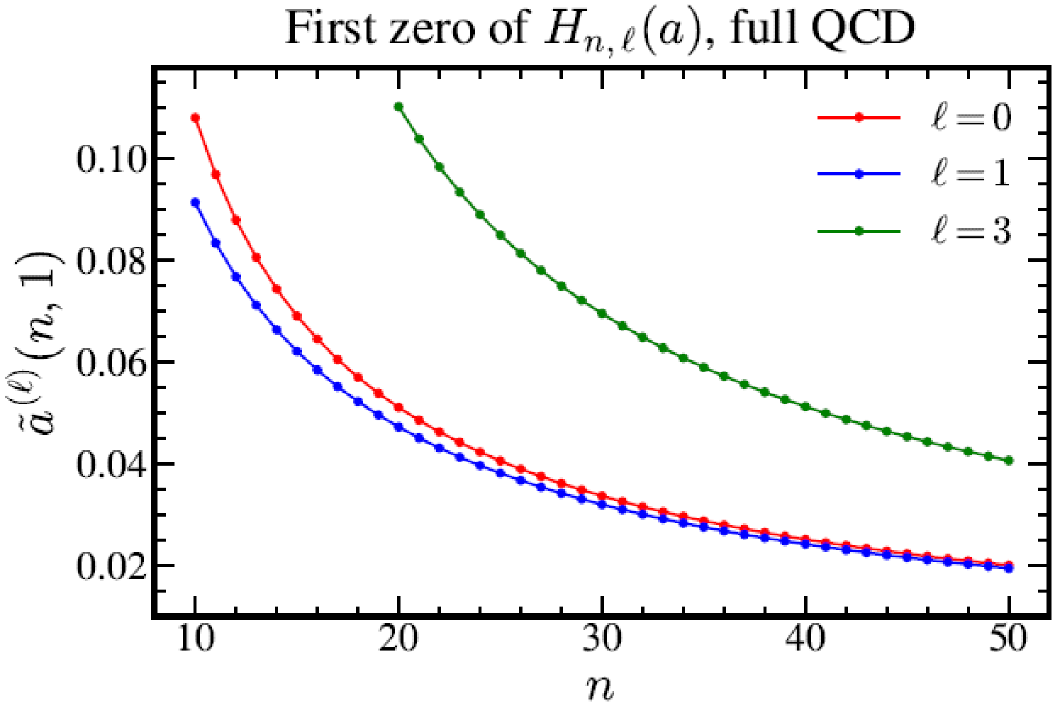
$$\limsup_{n\to\infty}|H_{n,\ell}(a)|^{1/n} = |a_+|\,. \tag{4.10}$$

As for the large-β_0 approximation, we can now see analytically that the CIPT series $\delta^{(0)}_{\mathrm{CIPT},\ell}$ of Eq. (2.4) with the coefficients in Eq. (4.2) diverge for any value of a,

$$\limsup_{n\to\infty}|\bar{c}_{n,1}H_{n,\ell}(a)|^{1/n} = \infty\,. \tag{4.11}$$

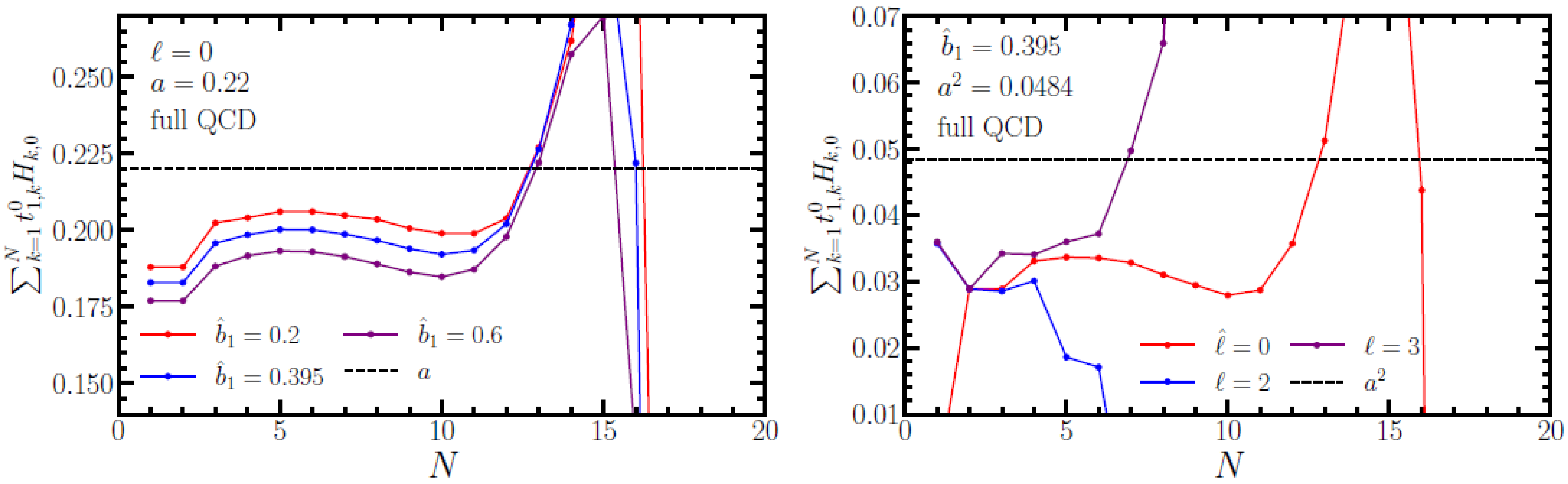


Zeros of $H_{n,\ell}(a)$, full QCD
$n=70$
$n=60$
$n=50$
$n=40$
$n=30$
$\ell=4$
$\ell=3$
$\ell=2$
$\ell=1$
$\ell=0$
0.00
0.05
0.10
0.15
0.20
$\tilde{a}^{(\ell)}(n,k)$



First zero of $H_{n,\ell}(a)$, full QCD
$\ell=0$
$\ell=1$
$\ell=3$
0.10
0.08
0.06
0.04
0.02
$\tilde{a}^{(\ell)}(n,1)$
10
20
30
40
50
n

FOPT->CIPT transformation of a singe power a^n



Uniformity for convergent series

Definition A.4 (Uniform Convergence). *A series of functions $\sum_{i=1}^{\infty} f_i(x)$ that converges to the function $F(x)$ in the interval I is said to be uniformly convergent in an interval $I' \subseteq I$, if for any $\epsilon > 0$, a number $N(\epsilon)$ independent of x exists such that $|\sum_{i=1}^{n} f_n(x) - F(x)| < \epsilon$ for all $n \geq N(\epsilon)$ and all $x \in I'$.*

Uniformity for asymptotic sequences

Definition A.10 (Uniform Asymptotic Sequence). *A sequence of functions $\{\phi_n(x)\}$ is an uniform asymptotic sequence as $x \to x_0$ in R if $\phi_{n+1} = o(\phi_n)$ uniformly in n. This means that for any $\epsilon > 0$ there exists a neighborhood U_ϵ of x_0 such that $|\phi_{n+1}(x)| < \epsilon|\phi_n(x)|$ for all $x \in R \cap U_\epsilon$ and all n.*

The Bernoulli numbers B_n arise in the summation of powers of integers and can be obtained from the Taylor coefficients of the generating function $x/(e^x - 1)$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \,. \tag{A7}$$

For odd $n \geq 3$ the Bernoulli numbers are zero. The leading term in the asymptotic expansion of the Bernoulli numbers as $n \to \infty$ reads

$$B_n \overset{n\to\infty}{\asymp} = -\frac{2n!}{(2\pi)^n} \cos\left(\frac{n\pi}{2}\right), \tag{A8}$$

$$\Pi_{\mu\nu,ij}^{V/A}(p) \equiv i \int dx\, e^{ipx} \langle\Omega|\, T\{J_{\mu,ij}^{V/A}(x)\, J_{\nu,ij}^{V/A}(0)^\dagger\}|\Omega\rangle \qquad J_{\mu,ij}^{V/A}(x) = [\bar{q}_j \gamma_\mu (\gamma_5) q_i](x)$$

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \mathrm{Im}\,\Pi^{(1)}(s) + \mathrm{Im}\,\Pi^{(0)}(s)\right]$$

$$\Pi^{(J)}(s) \equiv |V_{ud}|^2 \left[\Pi_{ud}^{V,(J)}(s) + \Pi_{ud}^{A,(J)}(s)\right] + |V_{us}|^2 \left[\Pi_{us}^{V,(J)}(s) + \Pi_{us}^{A,(J)}(s)\right]$$

$$R_\tau = 6\pi i \oint_{|s|=M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) \Pi^{(1)}(s) + \Pi^{(0)}(s)\right]$$

$$R_\tau = -i\pi \oint_{|x|=1} \frac{dx}{x} (1-x)^3 \left[3\,(1+x)\, D^{(1+0)}(M_\tau^2 x) + 4\, D^{(0)}(M_\tau^2 x)\right] \qquad x \equiv s/M_\tau^2$$

$$B(u) = \frac{128}{3\beta_0}\, e^{\frac{5u}{3}} \left\{ \frac{3}{16(2-u)} + \sum_{p=3}^{\infty} \left[\frac{d_2(p)}{(p-u)^2} - \frac{d_1(p)}{p-u} \right] - \sum_{p=-1}^{-\infty} \left[\frac{d_2(p)}{(u-p)^2} + \frac{d_1(p)}{u-p} \right] \right\}$$

tions) are called "IR renormalons". Each such IR renormalon contribution has a one-to-one association with a higher dimensional OPE correction term $\sim \langle \mathcal{O}_p \rangle / s_0^p$, which effectively compensates for the resulting ambiguity of the perturbation series. Here, $\langle \mathcal{O}_p \rangle$ stands for a dimension $2p$ nonperturbative low-energy QCD matrix element that cannot be determined with perturbative methods [15–17].

For a moment with weight function $W(x) = (-x)^\ell$ all OPE matrix-element corrections of the form const. $\times \langle \mathcal{O}_p \rangle / s_0^p$ for $p \neq \ell$ are eliminated by the contour integration as can be seen from the residue theorem. As a consequence, perturbative expansion methods consistent with the OPE must yield a convergent series, at least within some region of a, for those terms in the Borel function $B(u)$ with a single pole renormalon of the form $1/(p-u)$ [15–17]. In