FLOW-ORIENTED PERTURBATION THEORY

Collaborators: M. Borinsky, Z. Capatti and E. Laenen.

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Alexandre Salas-Bernárdez

Outline

1. Introduction and motivation.

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2. Derivation of FOPT.

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3. Hybrid S-Matrix representation and the Flow polytope.

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2. Derivation of FOPT. Examples.
3. Hybrid S-Matrix representation and the Flow polytope.
4. Unitarity and cut integrals in FOPT.

Based on *"Flow-oriented perturbation theory"*,
JHEP 01 (2023), 172 `https://arxiv.org/abs/2210.05532`**.**

Introduction

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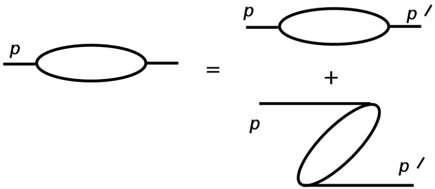
3D representations of Feynman integrals

Famous non-(manifestly)-covariant approaches:

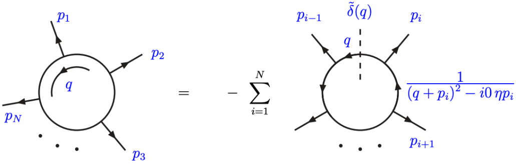
- Time Ordered Perturbation Theory (TOPT)



3D representations of Feynman integrals

Famous non-(manifestly)-covariant approaches:

- Time Ordered Perturbation Theory (TOPT)
- Loop-tree duality.



Coordinate space formulation of QFTs

Coordinate space treatments:

- Unitarity and the Largest Time equation.
- Multi-loop renormalization group invariants.
- Factorization results.
- Axiomatic QFT.
- PDFs.
- ...

Coordinate space scalar QFT

$$\Delta_F(z) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot z} \frac{i}{p^2 + i\epsilon} = \frac{1}{(2\pi)^2} \frac{1}{-z^2 + i\epsilon} .$$

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- Scalar n-point Green's function

$$\Gamma(x_1, ..., x_{|V^{\text{ext}}|}) = \left\langle 0 \right| T(\varphi(x_1) \cdots \varphi(x_{|V^{\text{ext}}|})) |0\rangle$$
$$= \sum_G \frac{1}{\text{Sym } G} A_G(x_1, \ldots, x_{|V^{\text{ext}}|}),$$

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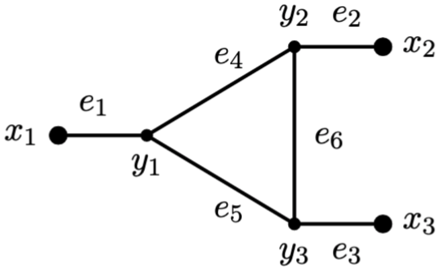
- A graph G contributing to the Green's function

$$A_G(x_1, \ldots, x_{|V^{\mathrm{ext}}|}) = \frac{(-ig)^{|V^{\mathrm{int}}|}}{(2\pi)^{2|E|}} \left[\prod_{v \in V^{\mathrm{int}}} \int \mathrm{d}^4 y_v \right] \prod_{e \in E} \frac{1}{-z_e^2 + i\varepsilon}$$

Coordinate space triangle diagram



$$A_G(x_1, x_2, x_3) = \frac{(-ig)^3}{(2\pi)^{12}} \int \left[\prod_{v \in V^{\mathrm{int}}} \mathrm{d}^4 y_v \right] \times$$

$$\times \frac{1}{(x_1 - y_1)^2 (x_2 - y_2)^2 (x_3 - y_3)^2 (y_1 - y_2)^2 (y_2 - y_3)^2 (y_1 - y_3)^2},$$

Flow-oriented perturbation theory

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Performing time integrations

In the spirit of TOPT we perform $\left[\int \mathrm{d}y_v^0\right]$ integrations to obtain a 3D representation of coordinate space diagrams:

- In doing so we introduce auxiliary energy variables.
- Perform Cauchy integrations.

The result is a sum over the different energy flows (orientations $\boldsymbol{\sigma}$) in the diagram, with energies being conserved at each vertex.

$$\frac{A_G(x_1,\ldots,x_{|V^{\mathrm{ext}}|})}{\mathrm{Sym}\, G} = \sum_{\langle\boldsymbol{\sigma}\rangle} \frac{A_{G,\boldsymbol{\sigma}}(x_1,\ldots,x_{|V^{\mathrm{ext}}|})}{\mathrm{Sym}(G,\boldsymbol{\sigma})},$$

Energy cycles

It is possible to resolve the energy integrations and conservation conditions for each orientation $\boldsymbol{\sigma}$ on a graph G in terms of "cycle" energy variables

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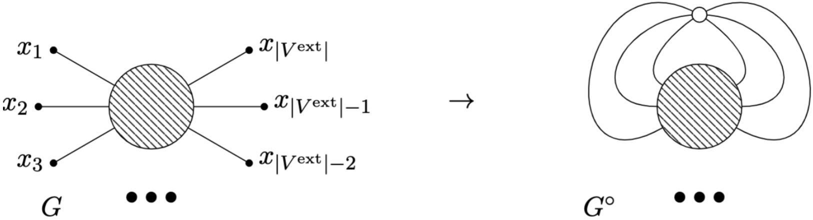

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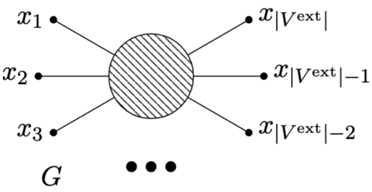

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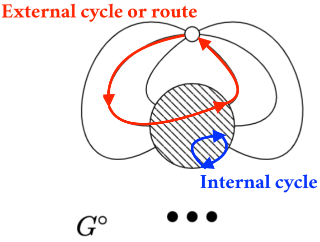


Energy conservation in $(G, \boldsymbol{\sigma}) \Rightarrow$ Strongly connected closed graph $(G^\circ, \boldsymbol{\sigma})$

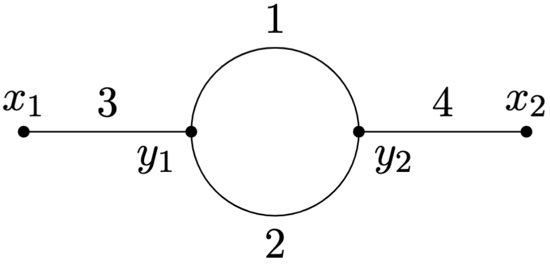
Energy cycles

It is possible to resolve the energy integrations and conservation conditions for each orientation $\boldsymbol{\sigma}$ on a graph G in terms of "cycle" energy variables:

$$A_{G,\boldsymbol{\sigma}}(x_1,\ldots,x_{|V^{\mathrm{ext}}|}) \propto \left(\prod_{v\in V^{\mathrm{int}}}\int \mathrm{d}^3\vec{y}_v\right)\times \\ \times\left(\prod_{e\in E}\frac{1}{2|\vec{z}_e|}\right)\prod_{p\in\text{cycles}}\frac{1}{\gamma_p+\tau_p+i\varepsilon}.$$

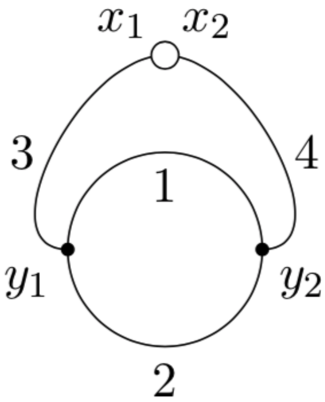
τ_p is the time difference and γ_p the sum of the lengths of the edges passed in the cycle.

One loop self energy graph



$$A_G(x_1, x_2) = \frac{(-ig)^2}{(4\pi^2)^4} \int \mathrm{d}^4 y_1 \mathrm{d}^4 y_2 \frac{1}{-z_1^2 + i\varepsilon} \frac{1}{-z_2^2 + i\varepsilon} \frac{1}{-z_3^2 + i\varepsilon} \frac{1}{-z_4^2 + i\varepsilon}$$

One loop self energy closed graph



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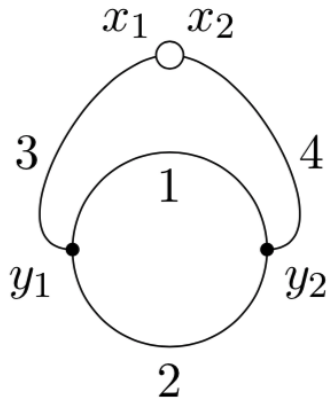
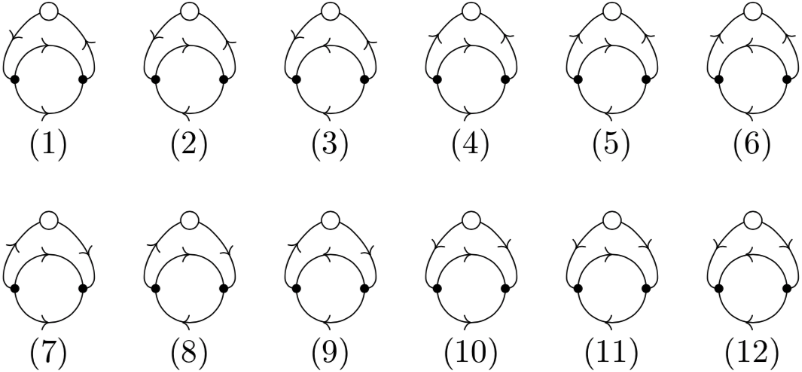


Next, draw all possible energy flows.

Energy flows through the closed bubble



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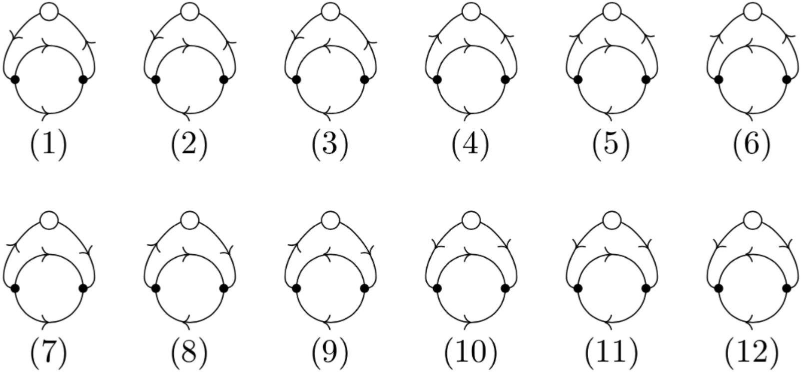
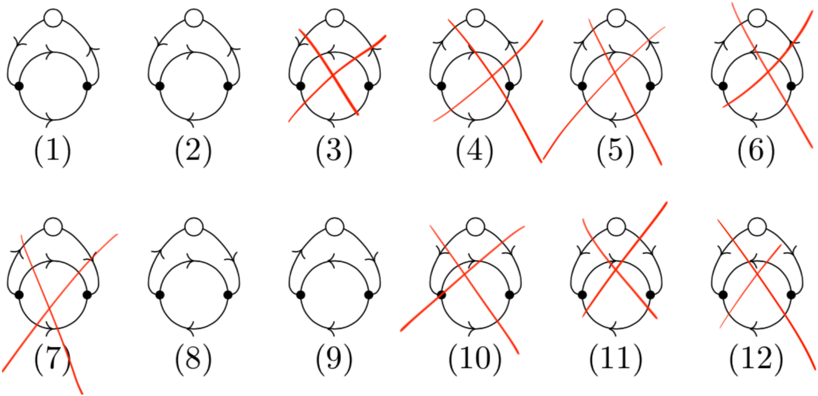
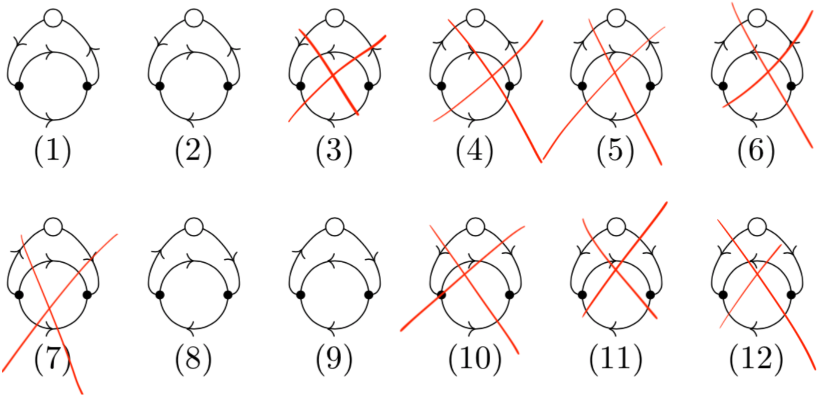


Energy must be conserved at each vertex.

Energy flows (= orientations) through the closed bubble



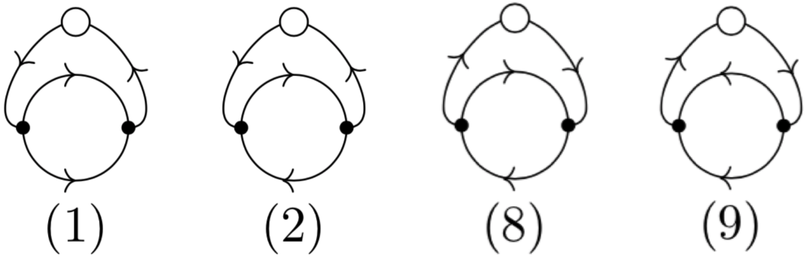
Energy flows (= orientations) through the closed bubble



(1) is equal to (8) and (2) to (9)

(under $\tau \equiv x_2^0 - x_1^0 \to -\tau$)

3D representation of the bubble



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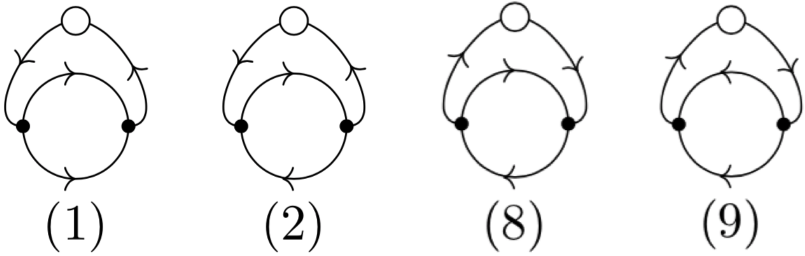
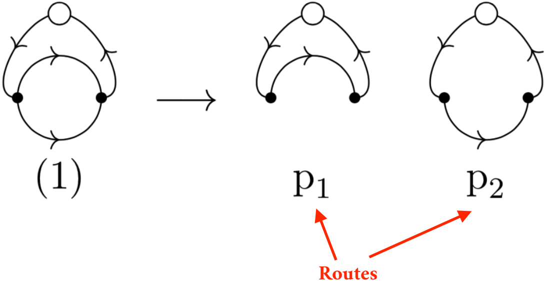


By collecting the overall and individual symmetry factors, we have that

$$\frac{1}{2}A(x_1, x_2) = \frac{1}{2}A_{G,\boldsymbol{\sigma}_{(1)}} + A_{G,\boldsymbol{\sigma}_{(2)}} + A_{G,\boldsymbol{\sigma}_{(8)}} + \frac{1}{2}A_{G,\boldsymbol{\sigma}_{(9)}} \ .$$

Decomposition of an orientation into cycles



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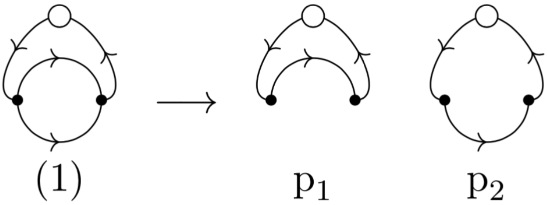

$$A_{G,\boldsymbol{\sigma}_{(1)}}(x_1,x_2) = \frac{(2\pi g)^2}{(8\pi^2)^4}\int \frac{\mathrm{d}^3\vec{y}_1\mathrm{d}^3\vec{y}_2}{|\vec{z}_1||\vec{z}_2||\vec{z}_3||\vec{z}_4|}\times$$
$$\times \frac{1}{|\vec{z}_3|+|\vec{z}_1|+|\vec{z}_4|+\tau+i\varepsilon}\frac{1}{|\vec{z}_3|+|\vec{z}_2|+|\vec{z}_4|+\tau+i\varepsilon},$$

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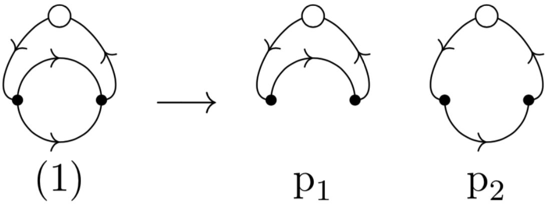

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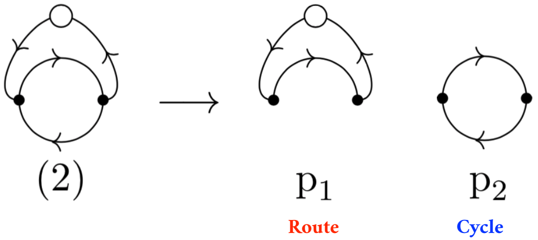
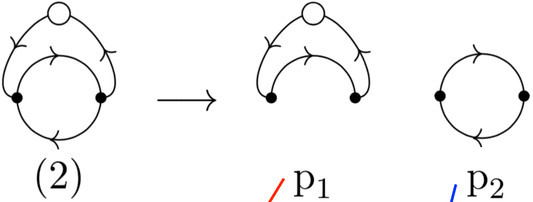


Decomposition of an orientation into cycles



$$A_{G,\boldsymbol{\sigma}_{(2)}}(x_1,x_2) = \frac{(2\pi g)^2}{(8\pi^2)^4}\int \frac{\mathrm{d}^3\vec{y}_1\mathrm{d}^3\vec{y}_2}{|\vec{z}_1||\vec{z}_2||\vec{z}_3||\vec{z}_4|}\times$$

$$\times\frac{1}{|\vec{z}_3|+|\vec{z}_1|+|\vec{z}_4|+\tau+i\varepsilon}\frac{1}{|\vec{z}_1|+|\vec{z}_2|}$$

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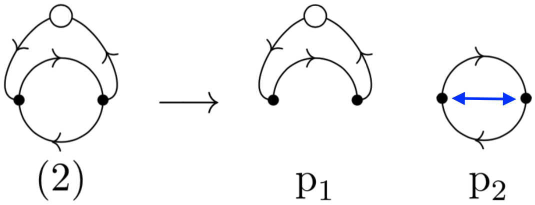

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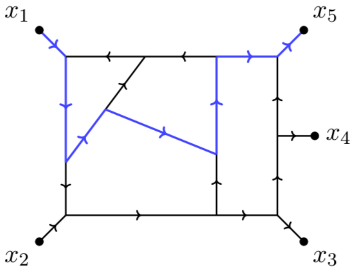
$$\times \frac{1}{|\vec{z}_3| + |\vec{z}_1| + |\vec{z}_4| + \tau + i\varepsilon} \frac{1}{|\vec{z}_1| + |\vec{z}_2|}$$

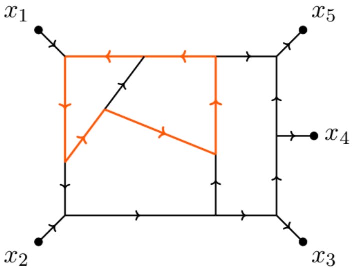
UV divergent if y1->y2

Routes and cycles: UV singularities in FOPT

Two types of paths:



(a) Route, $\mathrm{r} \in \Gamma^{\mathrm{ext}}$.



(b) Cycle, $\mathrm{c} \in \Gamma^{\mathrm{int}}$.

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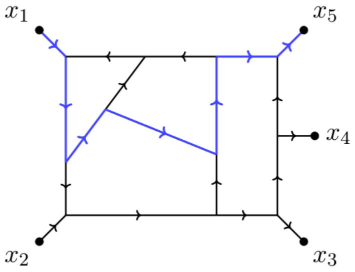

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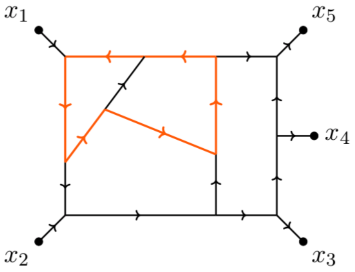

(b) Cycle, $\mathrm{c} \in \Gamma^{\mathrm{int}}$.

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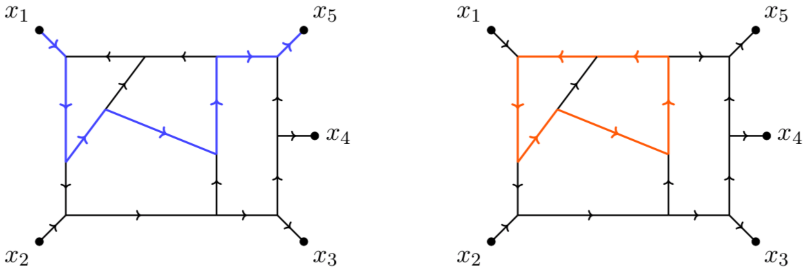

(a) Route, $\mathrm{r} \in \Gamma^{\mathrm{ext}}$. **(b)** Cycle, $\mathrm{c} \in \Gamma^{\mathrm{int}}$.

UV singularities of cycles match those of the covariant Feynman diagrams.

$\Rightarrow$ Amplitudes can be regularised as "usual"

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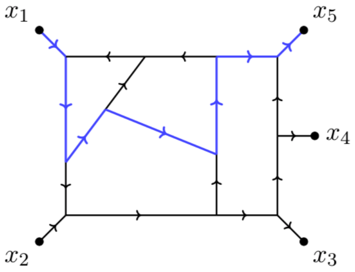
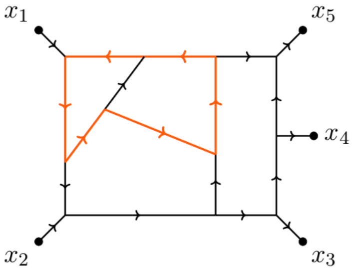


(a) Route, r $\in \Gamma^{\text{ext}}$.



(b) Cycle, c $\in \Gamma^{\text{int}}$.

UV singularities of cycles match those of the covariant Feynman diagrams.
$\Rightarrow$ Amplitudes can be regularised as "usual"(it is coordinate space).

Long and finite distance singularities in FOPT

Diagrams in FOPT fail to reproduce the finite distance (collinear) and long distance (soft) divergent behaviour expected from momentum space results.

Long and finite distance singularities in FOPT

Diagrams in FOPT fail to reproduce the finite distance (collinear) and long distance (soft) divergent behaviour expected from momentum space results.

$\Rightarrow$ We shift our attention to the S-matrix and construct an FOPT representation of it.

The p-x representation of the S-matrix

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Hybrid representation of the S-matrix

We construct a representation of the S-matrix where external data is given in momentum space whereas the internal integrals are in coordinate space (FOPT).

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$$S(\{p_i\}_{i\in V^{\mathrm{ext}}_{\mathrm{in}}}, \{p_f\}_{f\in V^{\mathrm{ext}}_{\mathrm{out}}}) = Z^{|V_{\mathsf{ext}}|/2}\widetilde{\Gamma}_T(\{p_a\}_{a\in V^{\mathrm{ext}}})\,,$$

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$$\widetilde{\Gamma}_T(\{p_a\}_{a\in V^{\mathrm{ext}}}) = \left[\prod_{i\in V_{\mathrm{in}}^{\mathrm{ext}}} \widetilde{\Delta}_R(p_i)\right]^{-1}\left[\prod_{f\in V_{\mathrm{out}}^{\mathrm{ext}}} \widetilde{\Delta}_A(p_f)\right]^{-1}\widetilde{\Gamma}(\{p_a\}_{a\in V^{\mathrm{ext}}}).$$

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$$\widetilde{\Gamma}(\{p_a\}_{a\in V^{\mathrm{ext}}}) = \int\left[\prod_{a\in V^{\mathsf{ext}}} \mathrm{d}^4 x_a\, e^{ix_a\cdot p_a}\right]\Gamma(\{x_a\}_{a\in V^{\mathrm{ext}}}),$$

Hybrid representation of the S-matrix

Next we use FOPT representation of the Green's function

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where the Fourier transform of a FOPT orientation is given by

$$\widetilde{A}_{G,\boldsymbol{\sigma}}(\{p_a\}_{a\in V^{\text{ext}}}) = \int \left[\prod_{a\in V^{\text{ext}}} \mathrm{d}^4 x_a \, e^{ix_a\cdot p_a}\right] A_{G,\boldsymbol{\sigma}}(\{x_a\}_{a\in V^{\text{ext}}}).$$

p-x representation of the S-matrix

It is possible to perform the Fourier transform explicitly and the final result equals

$$S(\{p_i\}_{i\in V_{\text{in}}^{\text{ext}}},\{p_f\}_{f\in V_{\text{out}}^{\text{ext}}}) = \sum_{(G,\boldsymbol{\sigma})} \frac{S_{G,\boldsymbol{\sigma}}(\{p_i\}_{i\in V_{\text{in}}^{\text{ext}}},\{p_f\}_{f\in V_{\text{out}}^{\text{ext}}})}{\text{Sym}(G,\boldsymbol{\sigma})},$$

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$$S_{G,\boldsymbol{\sigma}} \propto \delta\left(\sum_{a\in V^{\mathrm{ext}}} p_a^0\right) \times \\ \times \int \frac{\left[\prod_{v\in V^{\mathsf{int}}} \mathrm{d}^3\vec{y}_v\right]\left[\prod_{a\in V^{\mathrm{ext}}} e^{-i\vec{y}_{\bar{a}}\cdot\vec{p}_a}\right]}{\left[\prod_{e\in E^{\mathrm{int}}} 2|\vec{z}_e|\right]\left[\prod_{\mathrm{c}\in\Gamma^{\mathrm{int}}}\gamma_{\mathrm{c}}\right]} \widehat{\mathcal{F}}_{G,\boldsymbol{\sigma}}^{\{p_a^0\}}(\gamma^t + i\varepsilon 1).$$

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Fourier Transform of the Flow Polytope

The flow polytope

$$\widehat{\mathcal{F}}^{\{p_a^0\}}_{G,\boldsymbol{\sigma}}(\gamma + i\varepsilon 1) = \int_{\mathcal{F}^{\{p_a^0\}}_{G,\boldsymbol{\sigma}}} \mathrm{d}\boldsymbol{E}\, e^{i\boldsymbol{E}\cdot(\gamma + i\varepsilon 1)}.$$

The flow polytope

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$\mathcal{F}^{\{p_a^0\}}_{G,\boldsymbol{\sigma}}$ is swept out by all tuples $(E_{\mathrm{r}})_{\mathrm{r}\in\Gamma^{\mathrm{ext}}}$ which fulfill

$$\begin{aligned} E_{\mathrm{r}} &\geq 0 \text{ for all } \mathrm{r} \in \Gamma^{\mathrm{ext}}\,, \\ \sum_{\mathrm{r}\ni i} E_{\mathrm{r}} &= \quad p_i^0 \text{ for all } i \in V^{\mathrm{ext}}_{\mathrm{in}}, \\ \sum_{\mathrm{r}\ni f} E_{\mathrm{r}} &= -p_f^0 \text{ for all } f \in V^{\mathrm{ext}}_{\mathrm{out}}. \end{aligned}$$

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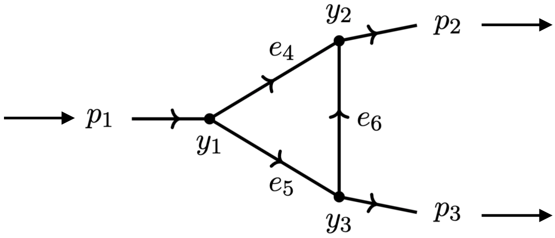
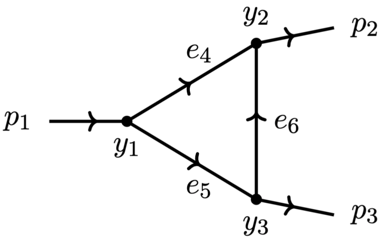
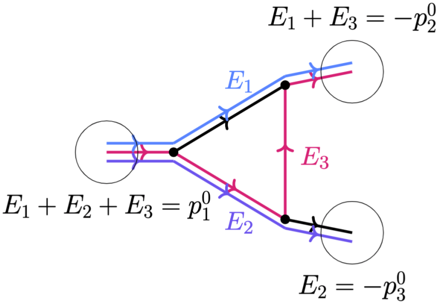
Nice features regarding the cancellation of spurious singularities.

Example: The *p*-*x* representation of the triangle diagram



Example: The p-x representation of the triangle diagram



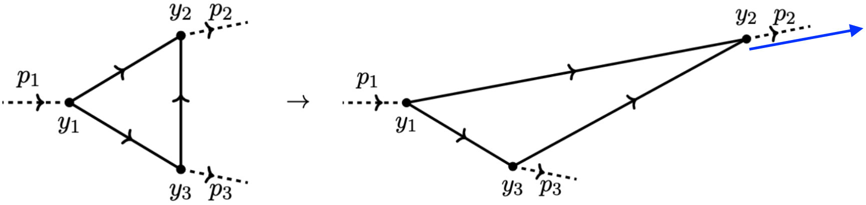


The flow polytope is cut out by the conditions:

$$E_1, E_2, E_3 \geq 0\,,$$
$$E_1 + E_2 + E_3 = p_1^0\,,$$
$$E_1 + E_3 = -p_2^0\,,$$
$$E_2 = -p_3^0\,,$$

IR singularities in the p-x representation

Collinear singularities are studied taking limits as ($\lambda \to \infty$):



$$\vec{y}_2 = \lambda \vec{p}_2/|\vec{p}_2|^2 + \sqrt{|\lambda|}\vec{y}_2^{\perp}, \quad \vec{y}_3 = \vec{y}_3\,,$$

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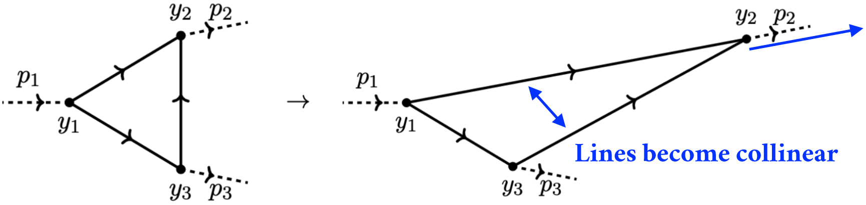

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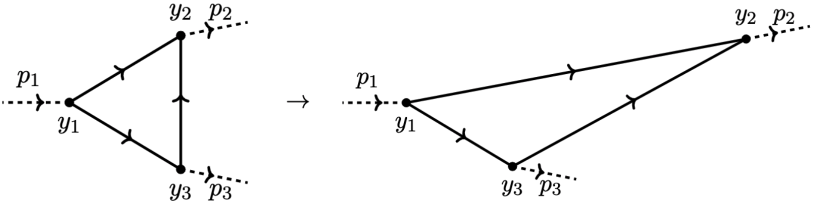

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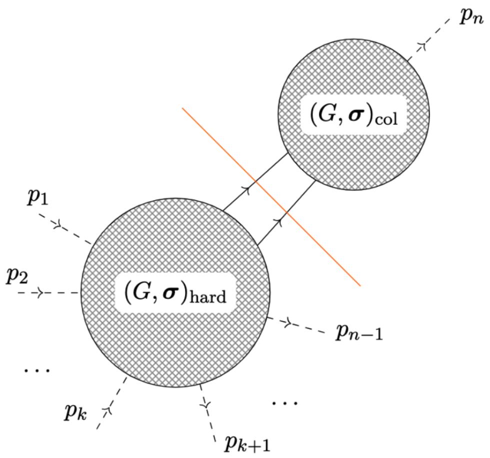
We find a per-diagram factorization of collinear and hard singularities!

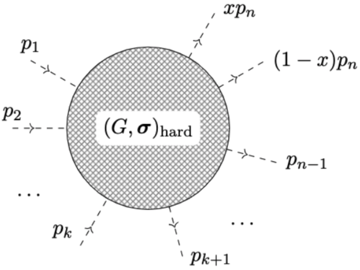
Per-diagram factorization



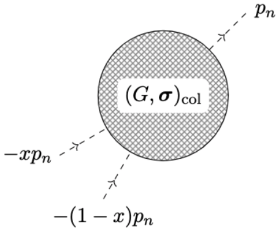
only two-cut yield collinear singularities

Per-diagram factorization of the S-matrix IR singularities

$$s_{G,\boldsymbol{\sigma}}(\{p_1,\ldots,p_k\},\{p_{k+1},\ldots,p_n\}) =$$
$$= -\frac{2\pi i}{4}\log\frac{p_n^2}{Q^2}\int_0^1 \mathrm{d}x\, s_{(G,\boldsymbol{\sigma})_{\text{hard}}} s_{(G,\boldsymbol{\sigma})_{\text{col}}} + \mathcal{O}_{p_n^2\to 0}(1)\,,$$

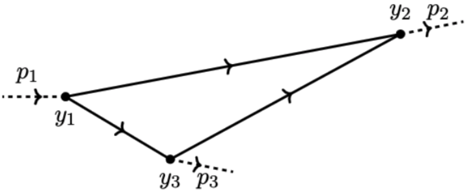


and



Soft-collinear singularity in the triangle diagram

We can study in the triangle the overlap of the collinear singularity with the soft singularity



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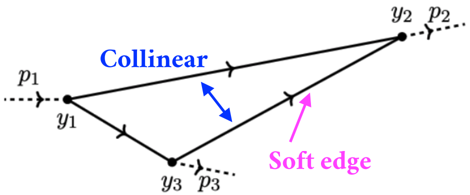

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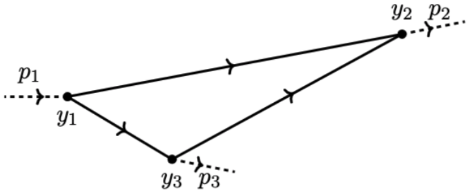


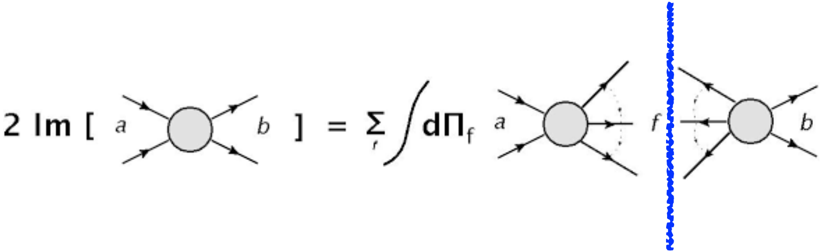
$\Rightarrow$ appearance of double-log Sudakov logs:

$$s_{G,\boldsymbol{\sigma}}(\{p_1\},\{p_2,p_3\}) = -i\frac{(2\pi)^2}{8}\frac{\log\frac{p_2^2}{p_1^2}\log\frac{p_3^2}{p_1^2}}{p_2\cdot p_3} + \mathcal{O}_{\substack{p_2^2\to 0\\ p_3^2\to 0}}(1).$$

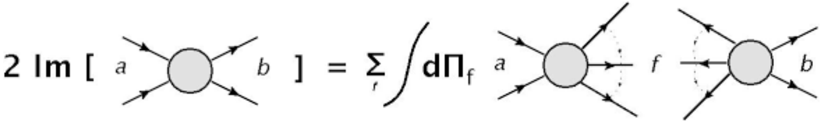
Unitarity and cut integrals

Alexandre Salas-Bernárdez

Cuts relating virtual and real processes



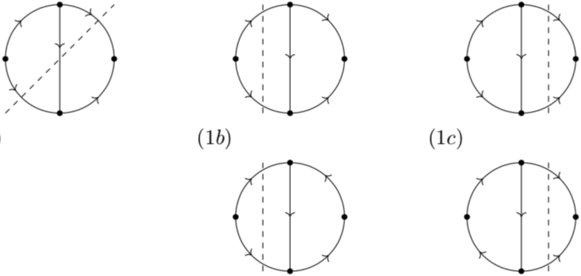
Cuts relating virtual and real processes



Loop Tree Duality puts all virtual and real corrections to a cross section under the same integral sign.

FOPT cut integrals

It is possible to extend FOPT to cut integrals.



(1a) (1b) (1c) (2b) (2c)

A remarkable property arises: different sized cuts have the **same integral measure**.

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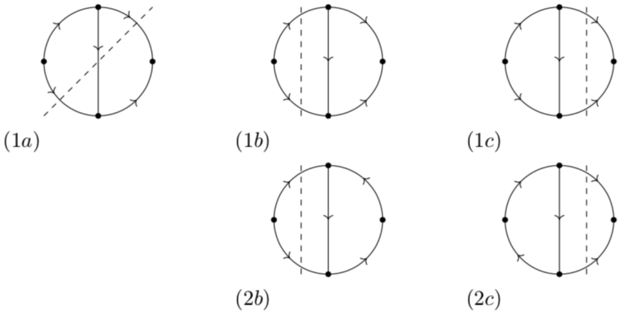

A remarkable property arises: different sized cuts have the **same integral measure**.

⇒ Advantage: IR singularities in numerical evaluations will cancel locally (no need for Loop Tree Duality).

Outlook

- FOPT offers promising features:
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  - Factorization.

Extensions

See the short handbook for using FOPT: 2310.09708

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The extension of FOPT to arbitrary dimensions is performed by using the dispersive representation of a scalar propagator of mass, m, in $D = 4 - 2\varepsilon$ dimensions,

$$\Delta(z^2, m) \; = \; \int_0^\infty \frac{dz'^2}{\pi} \, \frac{\mathrm{Im}\, \Delta\left(z'^2 + i\eta, m\right)}{-z^2 + z'^2 + i\eta} \; .$$

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$$\mathrm{Im}\,\Delta(z^2, m) \;= \theta\left(z^2\right) \, \frac{m^{1-\varepsilon}}{2^{3-\varepsilon}\pi^{1-\varepsilon}\left(\sqrt{z^2}\right)^{1-\varepsilon}} \, H^{(2)}_{1-\varepsilon}\left(m\sqrt{z^2}\right)$$

$$\mathrm{Im}\,\Delta(z^2, 0) \;=\; \frac{1}{(z^2)^{1-\varepsilon}} \, \frac{\pi}{\Gamma(\varepsilon)} \,,$$

with $H_{1-\varepsilon}$ a Bessel function.

Extensions: D dimensions and massive

$$A_{G,\boldsymbol{\sigma}}(x_1,\ldots,x_{|V^{\mathrm{ext}}|}) = \frac{(2\pi g)^{|V^{\mathrm{int}}|}}{(-4\pi^2)^{|E|}} \left(\prod_{v \in V^{\mathrm{int}}} \int \mathrm{d}^3 \vec{y}_v \right)$$
$$\times \left(\prod_{e \in E} \int_0^\infty \frac{dz_e'^{\,2}}{\pi} \frac{\mathrm{Im}\,\Delta\left(z_e'^{\,2} + i\eta, m\right)}{2\sqrt{|\vec{z}_e|^2 + z_e'^{\,2}}} \right) \prod_{\mathrm{p} \in \Gamma} \frac{1}{\gamma_\mathrm{p} + \tau_\mathrm{p} + i\eta} .$$

Where now each path length, γ_p, is modified as

$$\gamma_\mathrm{p} = \sum_{e \in \mathrm{p}} \left(\sqrt{|\vec{z}_e|^2 + z_e'^{\,2}} \right) .$$

Extensions: fermion lines

Each fermion line, e, contributes with an extra factor

$$\boxed{\gamma_\mu \frac{\tilde{z}^\mu_{e,\sigma_e}}{(2|\vec{z}_e|)^2}\left(2\sum_{i=1}^{3}\delta^{\mu i} - 2|\vec{z}_e|\frac{\partial}{\partial|\vec{z}_e|}\right)},$$

with the lightlike vector $\tilde{z}^\mu_{e,\sigma_e=\pm 1} = (\pm|\vec{z}_e|, \vec{z}_e)$