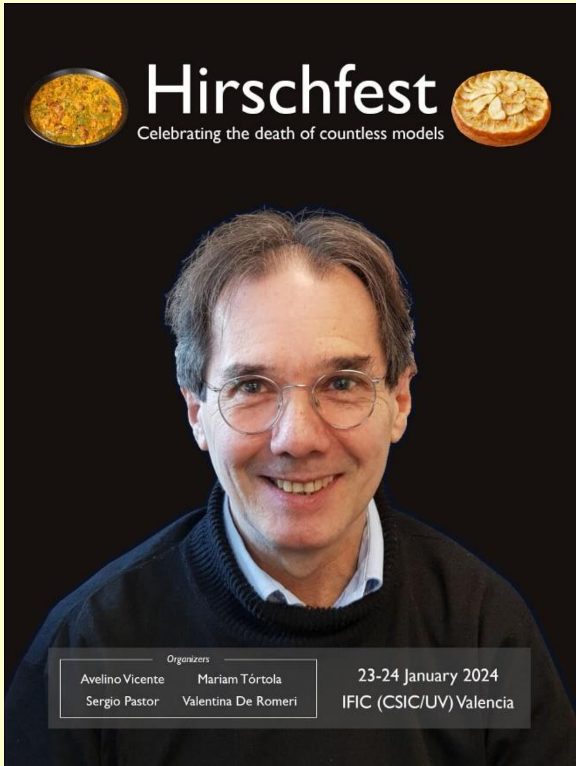
Congratulations !!

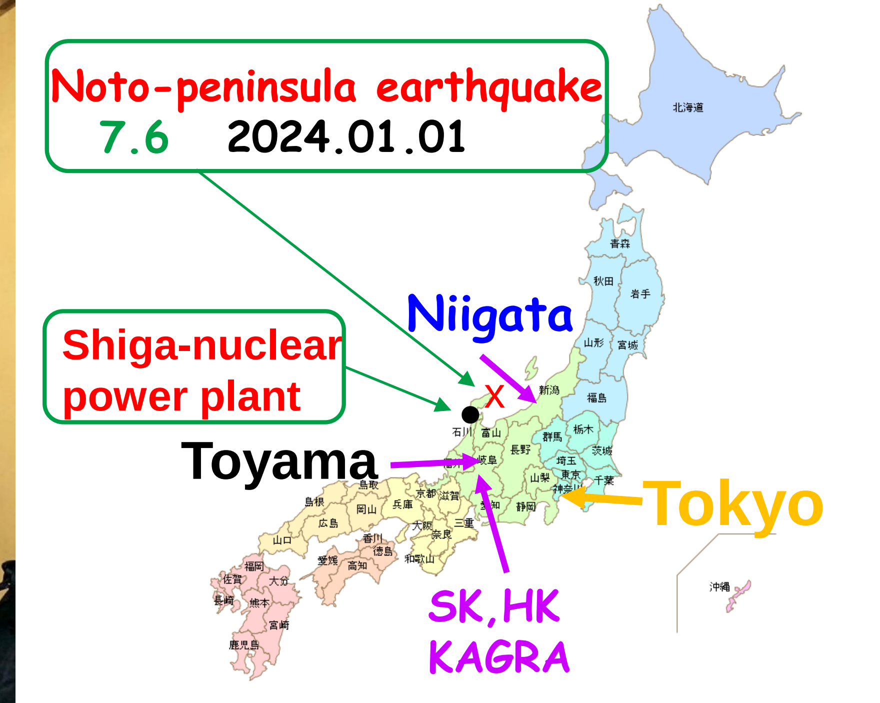
Martin's 60th birthday

Morimitsu Tanimoto

Niigata University

January 23, 2024





Sushi-Restaurant
at Niigata

Coffee
at Toyama

2019 September
TAUP at Toyama

Shiho was born
in hospital La Fe, Valencia

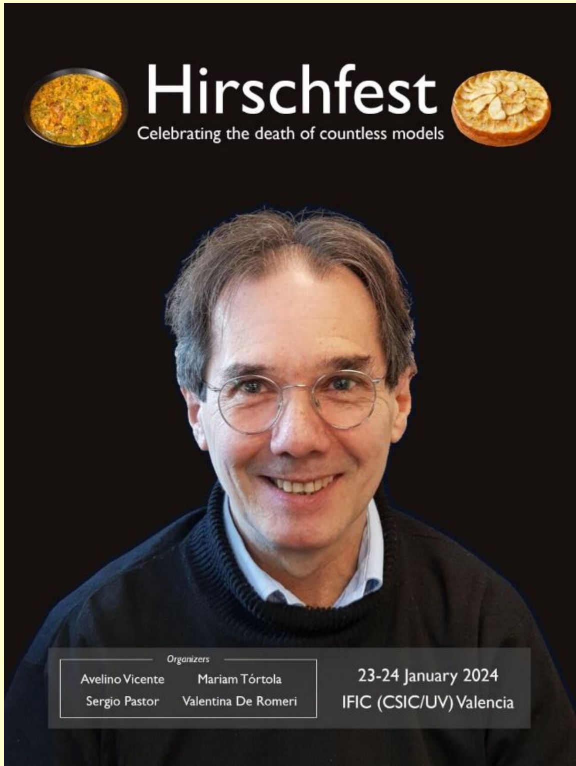
Hiroko
Osamu
新梅田
ドリンクメニュー
NYC

Modular symmetry of quark and lepton flavor

Morimitsu Tanimoto

Niigata University

January 23, 2024
IFIC, Valencia



1 Flavor meets modular forms

Flavor problem of quarks and leptons

Mass hierarchy

Flavor mixing

CP violation

What is the mechanism of fermion mass hierarchy ?

What is the origin of large flavor mixing ?

What is the origin of CP violation ?

Modular forms meet the flavor problem

What is Modular form ?

$$\boxed{f(x) = \sin 2\pi x\,, \qquad \mathrm{T}: x \to x+1 \Rightarrow f(x+1) = f(x)}$$ shift-symmetry

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (a, b, c, d) are integer and $ad - bc = 1$

$$\gamma : \boxed{z \to \frac{az+b}{cz+d}}$$

z is complex

(Modular transformation)

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \mathrm{T}: z \to z+1$$

Modular form f(z) is defined by imposing three conditions

① $f(z)$ is holomorphic @ Im Z > 0

② $f(z)$ is holomorphic @ $z \to i\infty$

③ $\cancel{f\left(\dfrac{az+b}{cz+d}\right) = f(z)}$

Modular function only constant

k: weight

$$\boxed{f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)}$$

Automorphy factor

Modular group

Three matrices construct γ (Modular transformation)

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \ f(z+1) = f(z)$$ z → z+1

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \ f\left(\frac{1}{-z}\right) = (-z)^k f(z)$$ z → -1/z

$$I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} : \ f\left(\frac{-z}{-1}\right) = (-1)^k f(z)$$ ⇒ k=even

$$S : \tau \longrightarrow -\frac{1}{\tau},$$
$$T : \tau \longrightarrow \tau + 1.$$

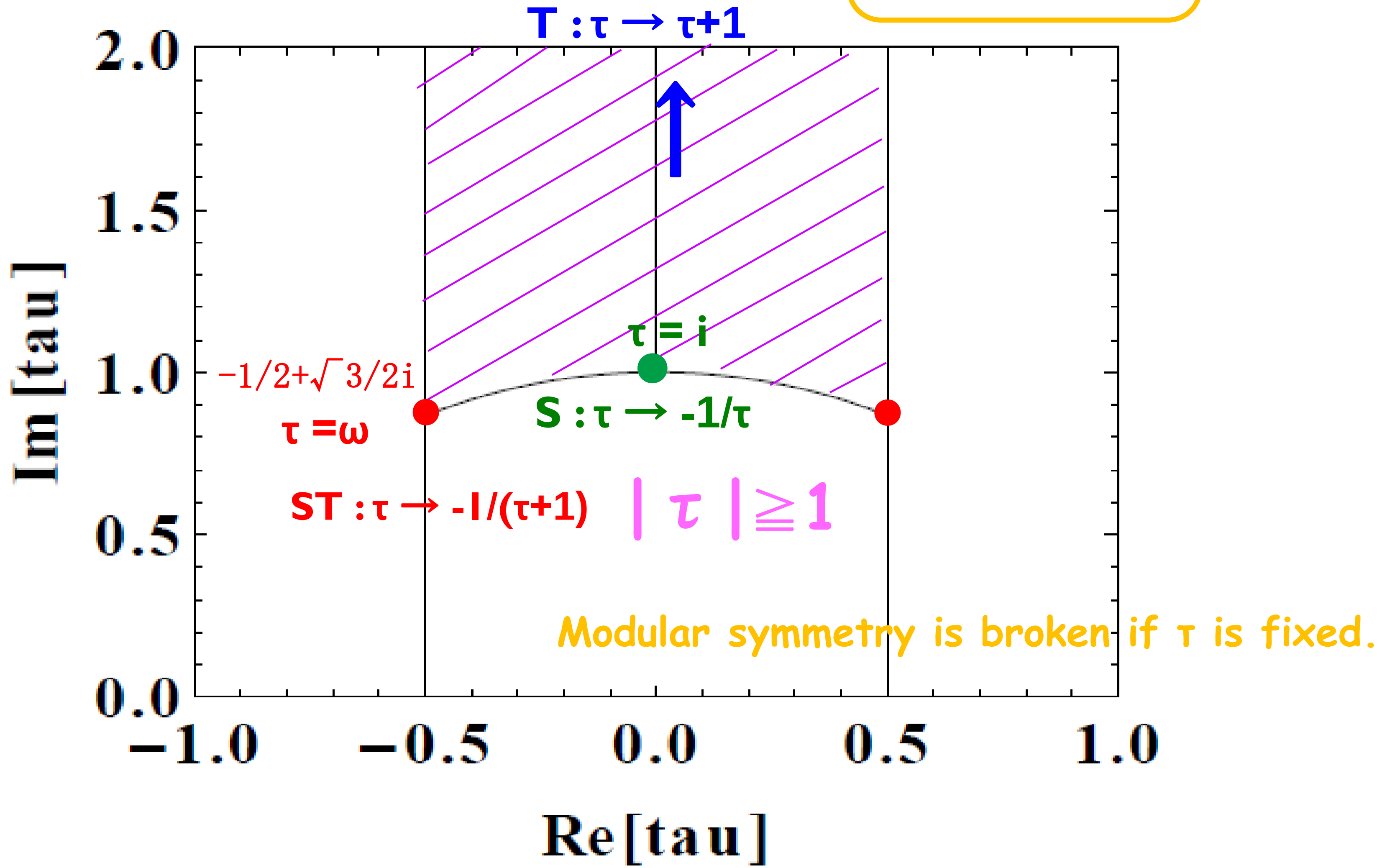
τ : modulus

$$S^2 = 1, \qquad (ST)^3 = 1.$$

generate infinite discrete group PSL(2,Z)

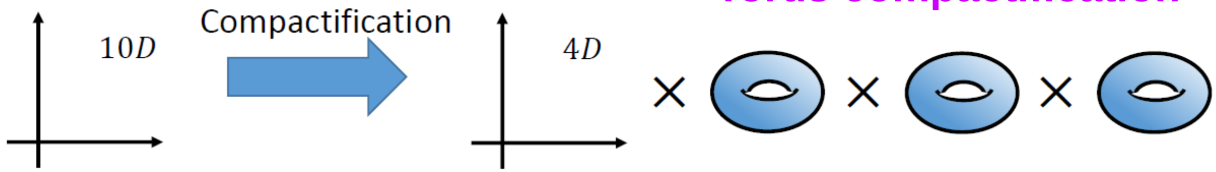
Fundamental Domain of τ

$$S : \tau \longrightarrow -\frac{1}{\tau},$$
$$T : \tau \longrightarrow \tau + 1.$$



● ● **Fixed point of τ (Residual symmetry)**

Torus compactification

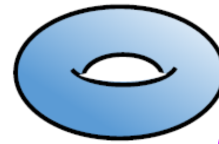


Modular forms appear naturally in top-down scenarios based on a class of string compactifications

We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \rightarrow \int d^4x \, \mathcal{L}_{\mathrm{eff}}$$

$\mathcal{L}_{\mathrm{eff}}$ depends on the structure of



Modular symmetry

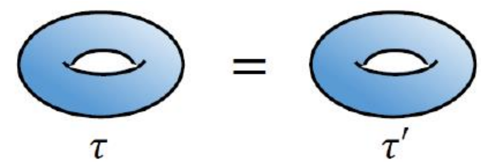
- $4D$ effective theory depends on internal space

Modular group

$\Gamma \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}\}$

infinite group

$\tau \rightarrow \tau'$



Modular group has subgroups

Γ_N **finite modular group of level N**

$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$

$\Gamma_2 \simeq S_3 \qquad \Gamma_3 \simeq A_4 \qquad \Gamma_4 \simeq S_4 \qquad \Gamma_5 \simeq A_5$

Consider effective theories with Γ_N symmetry.

$$\mathcal{L}_{\text{eff}} \ni f(\tau)\phi^{(1)}\cdots\phi^{(n)}$$

$f(\tau), \phi^{(I)}$: non-trivial rep. of Γ_N

Modular form of Level N

$$\tau \longrightarrow \tau' = \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

Modular transformation

Automorphy factor

$$f_i(\tau) \longrightarrow f_i(\gamma\tau) = \boxed{(c\tau + d)^k}\rho(\gamma)_{ij} f_j(\tau)$$

modular form of weight k (Level N)

K is modular weight

representation matrix of finite group

Chiral superfields

$$(\phi^{(I)})_i(x) \longrightarrow (c\tau + d)^{-k_I}\rho(\gamma)_{ij}(\phi^{(I)})_j(x)$$

$$\boxed{f_i(\tau)\ \phi^{(I)}\phi^{(J)}H}$$

$$(c\tau + d)^k(c\tau + d)^{-k_I}(c\tau + d)^{-k_J} = (c\tau + d)^{k - k_I - k_J}$$

Automorphy factor vanishes if $k = k_I + k_J$

Yukawa coupling is modular invariant !

2 Modular forms for Level 3 (N=3)

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$\Gamma_3 \simeq A_4$ group

Number of modular forms depend on weight k (even)

k+1 for A_4 (2k+1 for S_4)

For k=0, the modular form is constant (modular function)

For k=2, there are 3 linealy independent modular forms,

which form a A_4 triplet.

F. Feruglio, arXiv:1706.08749

A_4 triplet of modular forms with weight 2

$$\begin{aligned}
Y_1(\tau) &= \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}-\frac{27\eta'(3\tau)}{\eta(3\tau)}\right),\\
Y_2(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),\\
Y_3(\tau) &= \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right)
\end{aligned}$$

$$Y_2^2+2Y_1Y_3=0$$

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

Dedekind eta-function

$$Y=\begin{pmatrix}Y_1(\tau)\\Y_2(\tau)\\Y_3(\tau)\end{pmatrix}=\begin{pmatrix}1+12q+36q^2+12q^3+\dots\\-6q^{1/3}(1+7q+8q^2+\dots)\\-18q^{2/3}(1+2q+5q^2+\dots)\end{pmatrix}$$

$$\boxed{q=e^{2\pi i\tau}}$$

$$\rho(S)=\frac{1}{3}\begin{pmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{pmatrix},\quad \rho(T)=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\quad \omega=\exp(i\frac{2}{3}\pi)$$

Modular forms in favour

Prototype of mass matrix

F. Feruglio, arXiv:1706.08749

T.Kobayashi,N.Omoto,Y.Shimizu, K.Takagi,M.T,T.H.Tatsuishi, arXiv:1808.03012

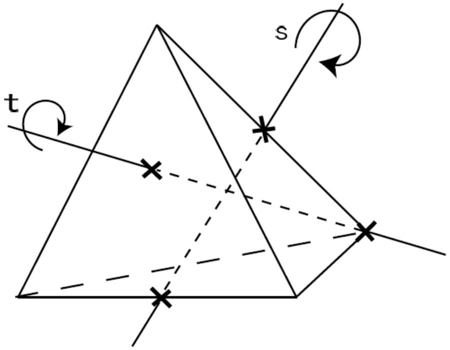
A_4 triplet 3 (Le, Lμ, Lτ) 3 (ν_{eR}, $\nu_{\mu R}$, $\nu_{\tau R}$)

A_4 singlets e_R 1 ; μ_R 1” ; τ_R 1’

Weight 2 modular forms $\boxed{\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}}$

Irreducible representations: 1 , 1’, 1”, 3

The minimum group containing triplet



A_4 symmetry

Symmetry of tetrahedron

E. Ma and G. Rajasekaran, PRD64(2001)113012

Lepton -- Seesaw model

	L	e^c, μ^c, τ^c	ν^c	H_u	H_d	$\mathbf{Y_3} = (Y_1, Y_2, Y_3)^T$
SU(2)	2	1	1	2	2	1
A_4	3	1, 1", 1'	3	1	1	3
k_I	1	1	1	0	0	$k=2$

1 0 1 2 　 1 0 1 2 　 1 0 1 2

$$w_e = \alpha\ E_1^c H_d (L\ Y)_1 + \beta\ E_2^c H_d (L\ Y)_{1'} + \gamma\ E_3^c H_d (L\ Y)_{1''}$$

$$w_\nu = g(N^c H_u L\ Y)_1 + \Lambda(N^c N^c Y)_1$$

1 0 1 2 　 1 1 2

w_e, w_ν are modular invariant if sum of weights satisfy $\sum k_I = k$.

$M_E \leftarrow \quad \alpha e_R H_d (LY) + \beta \mu_R H_d (LY) + \gamma \tau_R H_d (LY)$

A_4 　 1 1 3 3 　 1" 1 3 3 　 1' 1 3 3

$M_D \leftarrow \quad g(\nu_R H_u L Y)_1$ 　 $M_N \leftarrow \quad \Lambda(\nu_R \nu_R Y)_1$

A_4 　 3 1 3 3 　 A_4 　 3 3 3

ν_R: 3

Seesaw 　 $M_\nu = -M_D^T M_N^{-1} M_D$

Consider the case of Normal neutrino mass hierarchy

$$\mathcal{Y}_e = \begin{pmatrix} \alpha Y_1 & \alpha Y_3 & \alpha Y_2 \\ \beta Y_2 & \beta Y_1 & \beta Y_3 \\ \gamma Y_3 & \gamma Y_2 & \gamma Y_1 \end{pmatrix}$$

$$\mathcal{Y}_\nu = \begin{pmatrix} 2g_1 Y_1 & (-g_1+g_2)Y_3 & (-g_1-g_2)Y_2 \\ (-g_1-g_2)Y_3 & 2g_1 Y_2 & (-g_1+g_2)Y_1 \\ (-g_1+g_2)Y_2 & (-g_1-g_2)Y_1 & 2g_1 Y_3 \end{pmatrix}$$

$$\mathcal{M}_R = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \Lambda$$

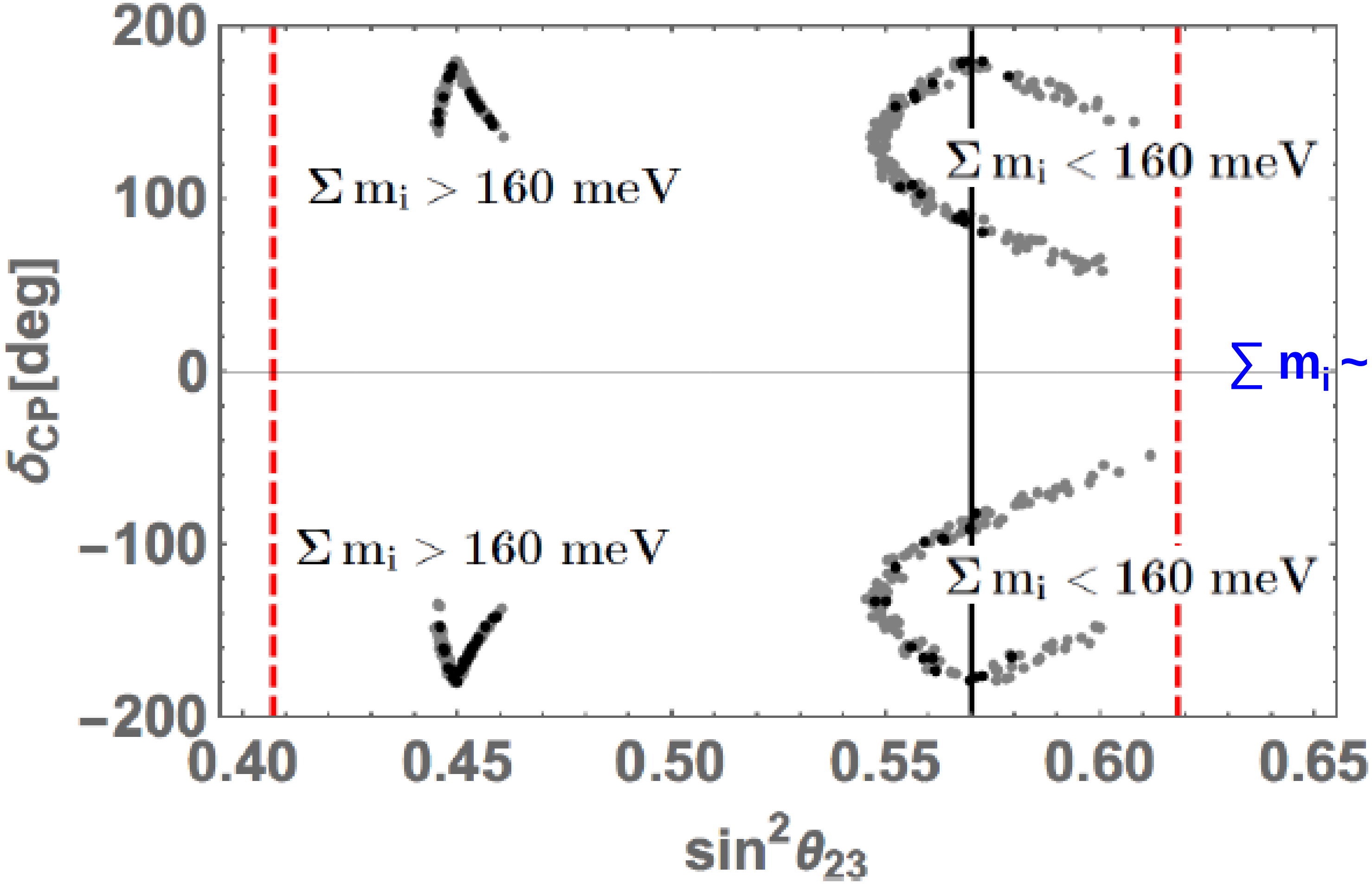
Parameters:

α, β, γ, $g_2/g_1 = g$, τ

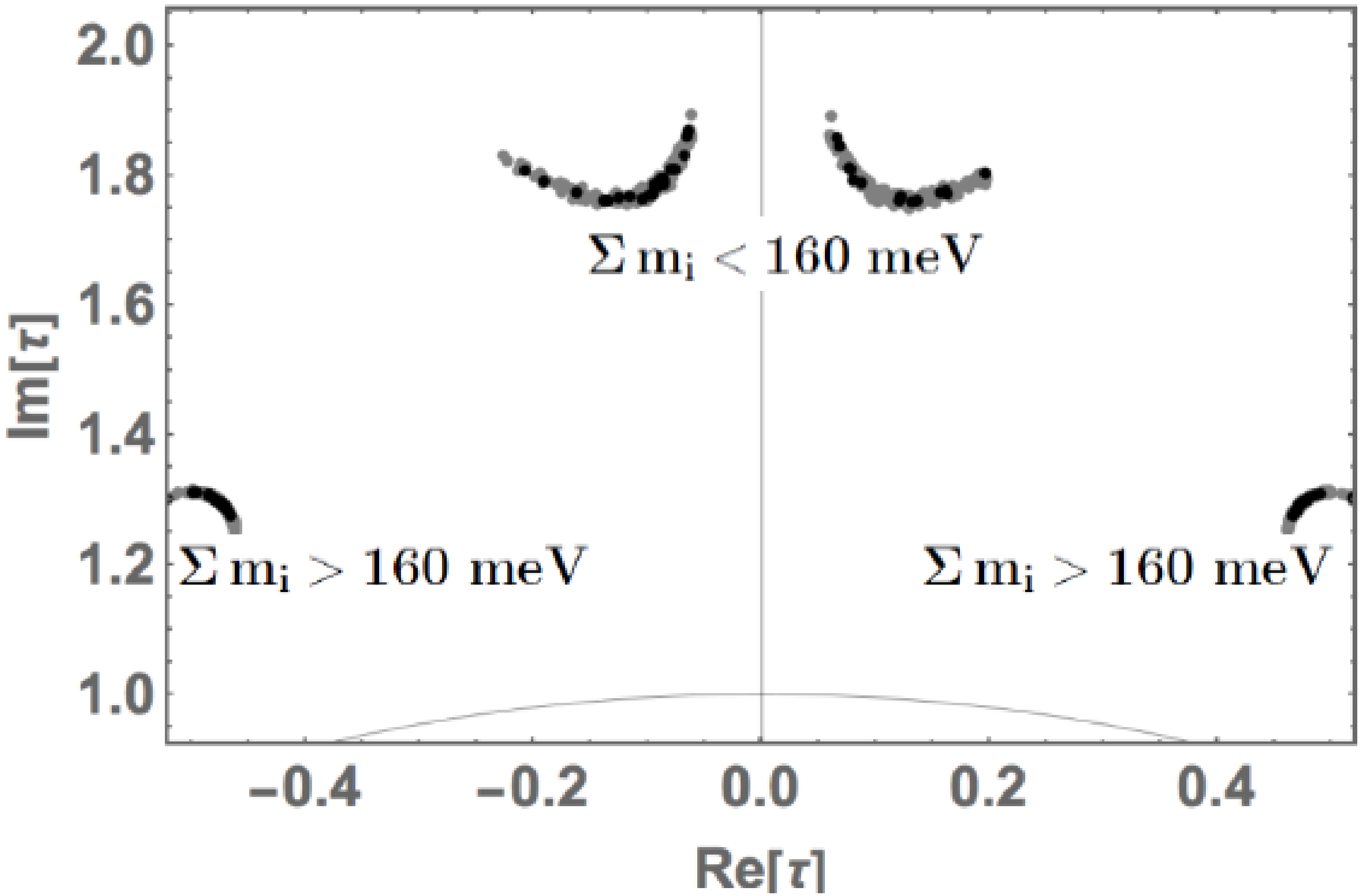
m_e, m_μ, m_τ fix α, β, γ .

$m_1 < m_2 < m_3$

$\Delta m^2_{sol} / \Delta m^2_{atm}$ and θ_{23}, θ_{12}, θ_{13} fix $g_2/g_1 = g$ and τ .



$\sum m_i \sim 140$ meV



Question: What is a principle to fix modulus τ ?

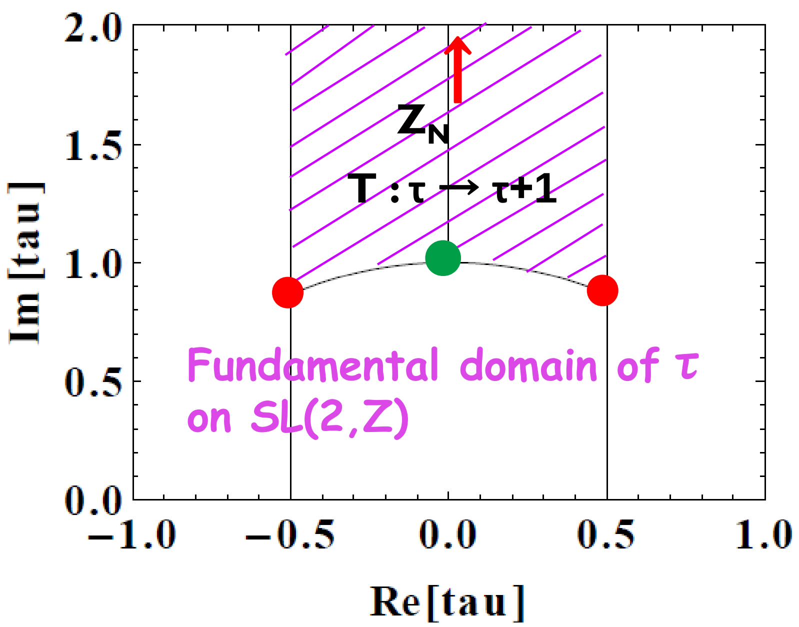
g is complex: What is the orign of CP violation ?

● **When τ is fixed, the modular symmetry is broken.**

What is a principle of fixing modulus τ ?

☆ **Modulus stabilization (non-perturbative effect, model dependent)**

Some models indicate the potential minimum at or near the boundary of fundamental domain.



Fixed point of τ

Residual symmetry

● Z_2: $\tau = i$ — S symmetry — $S: \tau \to -1/\tau$

● Z_3: $\tau = \omega$ ($\omega^3=1$) — ST symmetery — $ST: \tau \to -1/(\tau+1)$

3 CP violation and breaking of modular symmetry

CP transformation in A_4 modular invariant theory

$$\tau \xrightarrow{\mathrm{CP}} -\tau^* \,, \qquad \psi(x) \xrightarrow{\mathrm{CP}} \overline{\psi}(x_P) \,, \qquad \mathbf{Y}_{\mathbf{r}}^{(\mathrm{k})}(\tau) \xrightarrow{\mathrm{CP}} \mathbf{Y}_{\mathbf{r}}^{(\mathrm{k})}(-\tau^*) = \mathbf{Y}_{\mathbf{r}}^{(\mathrm{k})*}(\tau)$$

One can construct CP invariant mass matrices in modular invariant flavor theory.

$$M_E(-\tau^*) = M_E(\tau)^* \,, \qquad M_\nu(-\tau^*) = M_\nu(\tau)^*$$

P.P.Novichkov, J.T.Penedo, S.T.Petcov, A.V.Titov, JHEP 07(2019)165 [arXiv:1905.11970 [hep-ph]].

Model of CP violation in Lepton Sector

H.Okada, M.Tanimoto, JHEP 03(2021),010 [arXiv:2012.01688 [hep-ph]]

	L	(e^c,μ^c,τ^c)	H_u	H_d	$\mathbf{Y_r^{(2)}}, \quad \mathbf{Y_r^{(4)}}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$\boxed{(\mathbf{1},\ \mathbf{1''},\ \mathbf{1'})}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3},\ \{\mathbf{3},\mathbf{1},\mathbf{1'}\}$
$-k_I$	-2	$(0,\ 0,\ 0)$	0	0	$2, \quad 4$

Weight 4
Modular forms
3 + 1 + 1'

real matrix

$$M_E = v_d \begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_e & 0 \\ 0 & 0 & \gamma_e \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}_{RL}$$

$$\mathbf{Y_1^{(4)}} = Y_1^2 + 2Y_2Y_3 = E_4\,, \quad \mathbf{Y_{1'}^{(4)}} = Y_3^2 + 2Y_1Y_2$$

$$\mathbf{Y_3^{(4)}} = \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}$$

$$w_\nu = -\frac{1}{\Lambda}(H_u H_u L L \mathbf{Y_r^{(k)}})_\mathbf{1}$$

Weinberg operator

Weight 4 modular forms !

real   real

$$M_\nu = \frac{v_u^2}{\Lambda}\left[\begin{pmatrix} 2Y_1^{(4)} & -Y_3^{(4)} & -Y_2^{(4)} \\ -Y_3^{(4)} & 2Y_2^{(4)} & -Y_1^{(4)} \\ -Y_2^{(4)} & -Y_1^{(4)} & 2Y_3^{(4)} \end{pmatrix} + g_{\nu 1}\mathbf{Y_1^{(4)}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g_{\nu 2}\mathbf{Y_{1'}^{(4)}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]_{LL}$$

3 x 3 x 3
L L Y

$$\boxed{M_E(-\tau^*) = M_E(\tau)^*\,, \qquad M_\nu(-\tau^*) = M_\nu(\tau)^*}$$

CP invariance since Y(-τ*)=Y* (τ)

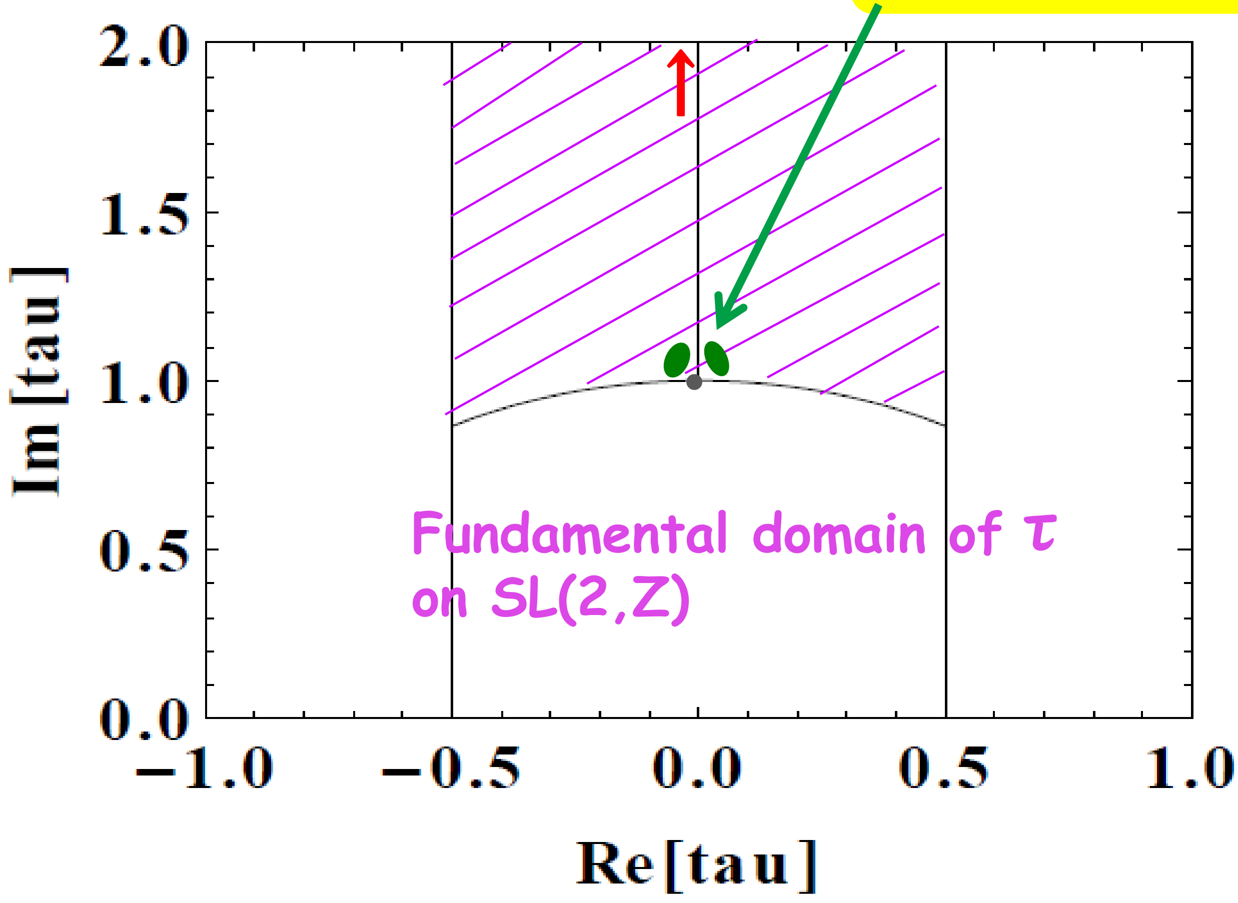
Input of 8 observed values:

m_e , m_μ , m_τ , θ_{12} , θ_{23} , θ_{13} ,

$\Delta m^2_{sol}=m_2^2-m_1^2$, $\Delta m^2_{atm}=m_3^2-m_1^2$

Output : δ_{CP} , $\langle m_{ee}\rangle$, $\sum m_i$

τ is close to i



Neutrino mass matrix at fixed point of τ

At $\tau = i$

tribimaximal-mixing

$$M_\nu = \frac{v_u^2}{\Lambda}(6-3\sqrt{3})Y_0^2 \left[\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + g_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + g_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right]$$

Putting g_2=0, tribimaximal-mixing is reproduced !

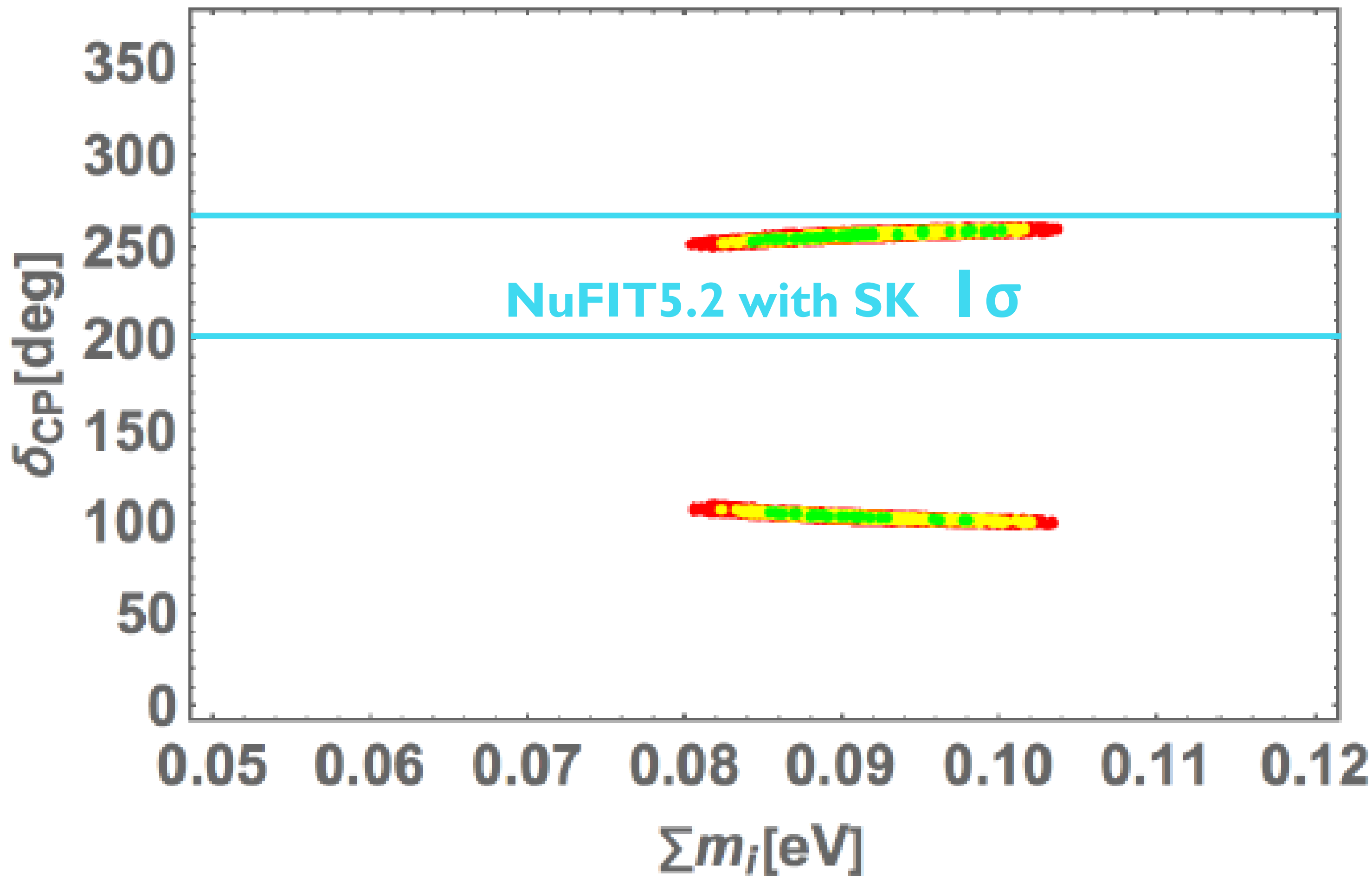
$$\sin^2\theta_{12} = 1/3, \ \sin^2\theta_{23} = 1/2, \ \sin^2\theta_{13} = 0,$$

$$U_{\text{tri-bimaximal}} = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & -\sqrt{1/2} \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}$$

Harrison, Perkins, Scott (2002)

$m_3 > m_2 > m_1$

$\langle m_{ee} \rangle$ = 12-20 meV



def: *Arg [Ve3/Vμ3] = $-\delta_{CP}$* → CP phase

4 Fermion mass hierarchy in modular invariance

P.P.Novichkov, J.T.Penedo, S.T.Petcov, JHEP 04(2021)206, arXiv:2102.07488

The hierarchical fermion masses are obtained
by using modular forms at nearby $\tau=\infty i$ and ω

$$\boxed{\mathcal{M}_q \sim v_q \begin{pmatrix} \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}_{RL}}$$

This hierarchical structure is not accidental.
Thanks to Residual symmetry Z_3 (N=3)

F. Feruglio, V. Gherardi, A. Romanino,A. Titov,
S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, H. Uchida
Y. Abe, T. Higaki, J. Kawamurab,T. Kobayashi,
S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, H. Uchida
Y. Abe, T. Higaki, J. Kawamura ,T. Kobayashi

Modular forms have hierarchy at nearby fixed points

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots). \end{aligned}$$

$$\boxed{q = e^{2\pi i\tau} = e^{2\pi i \mathrm{Re}\tau} e^{-2\pi \mathrm{Im}\tau}}$$

$$\boxed{\varepsilon = 6\,|q|^{1/3}}$$

$\tau \to \infty i$ $(Y_1, Y_2, Y_3)^T \longrightarrow (1, -\varepsilon, -1/2\,\varepsilon^2)^T$

A_4 triplet

$|\varepsilon| << 1$

Modular forms are hierarchical at $\tau = i\infty$ and ω !

$\tau = \omega$

$$ST = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix} \xrightarrow{\text{Unitary transformation}} \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$$\mathbf{Y_3^{(2)}} = Y_0 \begin{pmatrix} 1 \\ \omega \\ -\frac{1}{2}\omega^2 \end{pmatrix} \longrightarrow \mathbf{Y_3^{(2)}} = \frac{3}{2}\omega Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

An example in A_4 modular symmetry @ $\tau=\omega$

	Q	(u^c, c^c, t^c), (d^c, s^c, b^c)	H_q	$\mathbf{Y}_{\mathbf{3}}^{(6)}$, $\mathbf{Y}_{\mathbf{3'}}^{(6)}$	$\mathbf{Y}_{\mathbf{3}}^{(4)}$	$\mathbf{Y}_{\mathbf{3}}^{(2)}$
$SU(2)$	**2**	**1**	**2**	**1**	**1**	**1**
A_4	**3**	$(\mathbf{1},\ \mathbf{1''},\ \mathbf{1'})$	**1**	**3**	**3**	**3**
k_I	2	(4, 2, 0)	0	$k=6$	$k=4$	$k=2$

$$W_d = \left[\alpha_d(\mathbf{Y}_{\mathbf{3}}^{(6)}Q)_{\mathbf{1}}d^c_{\mathbf{1}} + \alpha'_d(\mathbf{Y}_{\mathbf{3'}}^{(6)}Q)_{\mathbf{1}}d^c_{\mathbf{1}} + \beta_d(\mathbf{Y}_{\mathbf{3}}^{(4)}Q)_{\mathbf{1'}}s^c_{\mathbf{1'}} + \gamma_d(\mathbf{Y}_{\mathbf{3}}^{(2)}Q)_{\mathbf{1''}}b^c_{\mathbf{1'}}\right] H_d$$

Suppose all coefficients are same order.

$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1^{(6)} + g_q Y_1'^{(6)} & Y_3^{(6)} + g_q Y_3'^{(6)} & Y_2^{(6)} + g_q Y_2'^{(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}_{RL}$$

$g_q = \alpha'_q / \alpha_q$

very small

$Y_2^2 + 2Y_1Y_3 = 0$

$$\tau = \omega + \epsilon$$

$$\frac{Y_2(\tau)}{Y_1(\tau)} \simeq \omega\,(1+\epsilon_1)\,, \quad \frac{Y_3(\tau)}{Y_1(\tau)} \simeq -\frac{1}{2}\omega^2\,(1+\epsilon_2)\,, \quad \boxed{\epsilon_1 = \frac{1}{2}\epsilon_2} \simeq 2.1\,i\,\epsilon\,.$$

$$\mathcal{M}_q = \frac{3}{2}\begin{pmatrix} \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & 2\tilde{\alpha}_q g_q \omega \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & \tilde{\beta}_q \omega^2 \\ \mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon) & \tilde{\gamma}_q \end{pmatrix} \quad \mathcal{M}_q^2 \sim \begin{pmatrix} |\epsilon^4| & |\epsilon|^2\epsilon^* & \epsilon^{2*} \\ |\epsilon|^2\epsilon & |\epsilon^2| & \epsilon^* \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}$$

$$m_{b\,(t)} : m_{s\,(c)} : m_{d\,(u)} \sim 1 : |\epsilon| : |\epsilon|^2$$

$$M(\tau) \sim \mathcal{O}(\epsilon^\ell) \quad \ell = 0, 1, 2$$

due to Z_3 at $\tau = \omega$

5 Summary

- CP violation is realized by fixing modulus τ, which causes the breaking of modular symmetry.

- Fermion mass hierarchy is realized at nearby fixed point of τ, which may be consistent with modulus stabilization.

- Is Modulus τ common in both quarks and leptons ?

F.~J.~de Anda, S.~F.~King and E.~Perdomo, Phys. Rev. D101 (2020) 015028
"SU(5) grand unified theory with A_4 modular symmetry"

Modular symmetry could be applied to other flavor phenomena. For example,

M.C.Chen, S.F.King, O.Medina, J.W.F.Valle, arXiv:2312.09255 [hep-ph]
``Quark-lepton mass relations from modular flavor symmetry''.

Flavor theory with modular symmetry is developing !

Move to diagonal base of ST for triplet @ τ=ω

Automorphy factor of Q_L
$(c\tau+d)^{-k}$

$$V_{\mathrm{ST-I}}\,\omega^{-4} ST\, V_{\mathrm{ST-I}}^{\dagger} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad V_{\mathrm{ST-I}} = \frac{1}{3}\begin{pmatrix} -2\omega^2 & -2\omega & 1 \\ 2\omega^2 & -\omega & 2 \\ -\omega^2 & 2\omega & 2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \frac{3}{2}\omega Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{Y}_{\mathbf{3}}^{(4)} = \frac{9}{4} Y_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{Y}_{\mathbf{3}}^{(6)} = 0\,, \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} = \frac{27}{8}\omega^2 Y_0^3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$M_q V_{\mathrm{ST-I}}^{\dagger} = \frac{3}{2}\begin{pmatrix} 0 & 0 & 2\tilde{\alpha}_q g_q^{\mathrm{I}}\omega \\ 0 & 0 & \tilde{\beta}_q\omega^2 \\ 0 & 0 & \tilde{\gamma}_q \end{pmatrix}$$

rank one matrix

$$\mathcal{M}_q^{2(0)} \equiv V_{\mathrm{ST-I}} M_q^{\dagger} M_q V_{\mathrm{ST-I}}^{\dagger} = \frac{9}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4|g_q^{\mathrm{I}}|^2\,\tilde{\alpha}_q^2 + \tilde{\beta}_q^2 + \tilde{\gamma}_q^2 \end{pmatrix}$$

Kinetic Term

Kinetic term of the modulus τ $$\frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$

Modular transformation $$\tau' = \frac{a\tau + b}{c\tau + d}, ad - bc = 1$$

- numerator

$$\partial_\mu \tau' = \frac{(a\partial_\mu \tau)(c\tau + d) - (a\tau + b)(c\partial_\mu \tau)}{(c\tau + d)^2} = \frac{(ad - bc)\partial_\mu \tau}{(c\tau + d)^2} = \frac{\partial_\mu \tau}{(c\tau + d)^2}$$

- denominator

$$\tau' - \bar{\tau}' = \frac{(a\tau + b)(c\bar{\tau} + d) - (a\bar{\tau} + b)(c\tau + d)}{|c\tau + d|^2} = \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} = \frac{\tau - \bar{\tau}}{|c\tau + d|^2}$$

$$\frac{|\partial_\mu \tau'|^2}{\langle -i\tau' + i\bar{\tau}'\rangle^2} = \frac{|\partial_\mu \tau|^2}{\langle -i\tau + i\bar{\tau}\rangle^2}$$ Modular invariant

Multiplication rule of A_4 group

Irreducible representations: 1, 1', 1'', 3

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad \omega = e^{2\pi i/3}$$ **for triplet**

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1 b_1 + a_2 b_3 + a_3 b_2)_{\mathbf{1}} \oplus (a_3 b_3 + a_1 b_2 + a_2 b_1)_{\mathbf{1'}}$$

$$\oplus\, (a_2 b_2 + a_1 b_3 + a_3 b_1)_{\mathbf{1''}}$$

$$\oplus\, \frac{1}{3}\begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_1 b_3 - a_3 b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix}_{\mathbf{3}}$$

A_4 invariant Majorana neutrino mass term

$$(LL)_1 = L_1 L_1 + L_2 L_3 + L_3 L_2$$

3 x 3

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$$

A_4 invariant

Modular transformation is the transformation of modulus τ

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

$$\boxed{\begin{aligned} S &: \tau \longrightarrow -\frac{1}{\tau}, \\ T &: \tau \longrightarrow \tau + 1. \end{aligned}}$$

weight 2; k=2

3 modular forms

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

S transformation

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

T transformation

$$\boxed{f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)}$$

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \qquad \begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$\boxed{(c\tau + d)^k}$ cτ+d = -τ

$\boxed{(c\tau + d)^k}$ cτ+d = 1

$$\rho(\mathrm{S}) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(\mathrm{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \omega = \exp(i\frac{2}{3}\pi)$$

Flavor symmetry acts non-linealy (Modular forms).

Superpotential

$$W_d = \left[\alpha_d(\mathbf{Y}_{\mathbf{3}}^{(6)}Q)_1 d_1^c + \alpha'_d(\mathbf{Y}_{\mathbf{3}'}^{(6)}Q)_1 d_1^c + \beta_d(\mathbf{Y}_{\mathbf{3}}^{(4)}Q)_{1'} s_{1'}^c + \gamma_d(\mathbf{Y}_{\mathbf{3}}^{(2)}Q)_{1''} b_{1'}^c\right] H_d$$

Kinetic terms

$$\sum_I \frac{|\partial_\mu \psi^{(I)}|^2}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}$$

We renormalize superfields to get canonical kinetic terms

$$\boxed{\psi^{(I)} \to \sqrt{(2\mathrm{Im}\tau_q)^{k_I}}\, \psi^{(I)}}$$

$$\alpha_u \to \hat{\alpha}_u = \alpha_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha_u\, (2\mathrm{Im}\tau)^4, \qquad \alpha'_u \to \hat{\alpha}'_u = \alpha'_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha'_u\, (2\mathrm{Im}\tau)^4\,,$$

$$\beta_u \to \hat{\beta}_u = \beta_u \sqrt{(2\mathrm{Im}\tau)^4} = \beta_u\, (2\mathrm{Im}\tau)^2\,, \qquad \gamma_u \to \hat{\gamma}_u = \gamma_u \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_u\, (2\mathrm{Im}\tau)\,,$$

$$\alpha_d \to \hat{\alpha}_d = \alpha_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha_d\, (2\mathrm{Im}\tau)^3, \qquad \alpha'_d \to \hat{\alpha}'_d = \alpha'_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha'_d\, (2\mathrm{Im}\tau)^3\,,$$

$$\beta_d \to \hat{\beta}_d = \beta_d \sqrt{(2\mathrm{Im}\tau)^4} = \beta_d\, (2\mathrm{Im}\tau)^2\,, \qquad \gamma_d \to \hat{\gamma}_d = \gamma_d \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_d\, (2\mathrm{Im}\tau)\,.$$

$2\mathrm{Im}\,\tau$ **is large**

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = (Y_1^2 + 2Y_2Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = (Y_3^2 + 2Y_1Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(8)} \equiv \begin{pmatrix} Y_1^{(8)} \\ Y_2^{(8)} \\ Y_3^{(8)} \end{pmatrix} = (Y_1^2 + 2Y_2Y_3) \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix}, \quad \mathbf{Y}_{\mathbf{3'}}^{(8)} \equiv \begin{pmatrix} Y_1'^{(8)} \\ Y_2'^{(8)} \\ Y_3'^{(8)} \end{pmatrix} = (Y_3^2 + 2Y_1Y_2) \begin{pmatrix} Y_2^2 - Y_1 Y_3 \\ Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(8)} = (Y_1^2 + 2Y_2Y_3)\mathbf{Y}_{\mathbf{3}}^{(4)}$$

Observed Yukawa ratios at GUT scale with tanβ=10

S. Antusch, V. Maurer, JHEP 1311 (2013) 115 [arXiv:1306.6879].

$$\frac{y_d}{y_b} = 9.21 \times 10^{-4}\ (1 \pm 0.111)\,, \qquad \frac{y_s}{y_b} = 1.82 \times 10^{-2}\ (1 \pm 0.055)$$

$$\frac{y_u}{y_t} = 5.39 \times 10^{-6}\ (1 \pm 0.311)\,, \qquad \frac{y_c}{y_t} = 2.80 \times 10^{-3}\ (1 \pm 0.043)$$

$$m_{b\,(t)} : m_{s\,(c)} : m_{d\,(u)} \sim 1 : |\epsilon| : |\epsilon|^2$$

For down quark sector $\varepsilon_d = 0.02 \sim 0.03$

For up quark sector $\varepsilon_u = 0.002 \sim 0.003$

We have only one parameter $|q| = \varepsilon$

$g_q \sim 1$

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : |\epsilon_1| : |\epsilon_1|^2 \simeq 1 : |\epsilon| : |\epsilon|^2$$

$g_q >> 1$

$$m_{q3} : m_{q2} : m_{q1} \simeq \quad 1 : \frac{|\epsilon_1|}{|g_q|} : \left(\frac{|\epsilon_1|}{|g_q|}\right)^2$$