The Cosmological Tree Theorem

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17th January 2024

Future = History × Past

The night sky encodes information about the history of the universe:

$$\underbrace{\langle \delta T \delta T \delta T \rangle}_{\text{Observations}} \sim \underbrace{\int f(k) f(k)}_{\text{Thermal history}} \underbrace{\langle \zeta_k \zeta_k \zeta_k \rangle}_{\text{Very early universe}} \tag{1}$$

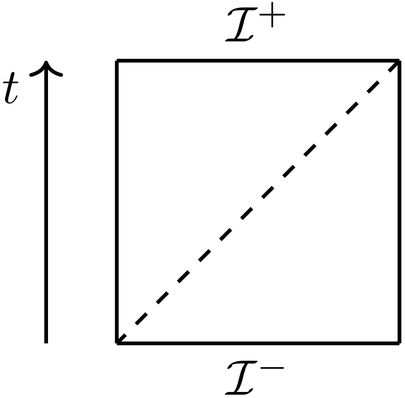
Studying the past is beneficial in three ways:

- It constrains templates for observations.
- It tells us something about the thermal history of the universe.
- It provides a window into high energy physics.

Theoretical knowledge $\leftrightarrow$ Primordial perturbations

The very early universe

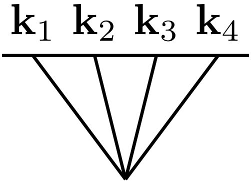
We model the very early universe by a (quasi) de Sitter universe.



- Primordial perturbations live on $\mathcal{I}^+$:

$$\langle \hat{\zeta}_{k_1} \hat{\zeta}_{k_2} \hat{\zeta}_{k_3} \hat{\zeta}_{k_4} \rangle$$

- An expanding spacetime leads to particle production:



The tradition in Inflation

The usual predicting process in Theoretical Physics is:

Fundamental principles $\rightarrow$ Models $\rightarrow$ Observables

For the very early universe we have:

- CMB power spectrum.
- Matter power spectrum.

Number of models $>>$ Measurements

What is the impact of fundamental principles in predictions?

The analytic S-matrix I

The microscopic description of inflation is still at large.

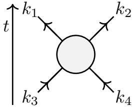
- Approximate scale invariance.
- Low tensor-scalar ratio.

Particle physicists developed two approaches to face this situation:

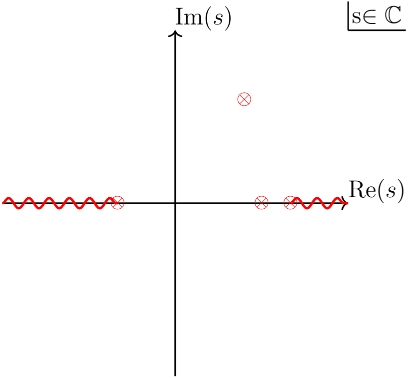
Effective Field Theories	**S-matrix programme**
What kind of observations are compatible with IR symmetries?	What kind of observations are compatible with fundamental principles?

What can we, as cosmologists, learn from amplitudes?

The analytic structure of amplitudes encodes the details of the theory.

$$A(s,t) = \sum_{\text{diag}}$$ 

$$, \quad s \in \mathbb{C} \qquad (2)$$



- Poles $\text{Im}(s) = 0$: Exchange of stable particles.
- Poles $\text{Im}(s) > 0$: Exchange of unstable particles.
- Branch points: Loop diagrams.
- Causality: Analyticity $\text{Im}(s) < 0$.
- Analyticity: Sum rules.

The Analytic S-matrix III

Analytic $A(s,t)$ is reflected on $\sigma(s) \to \sigma(s) \sim |A(s,t)|^2$

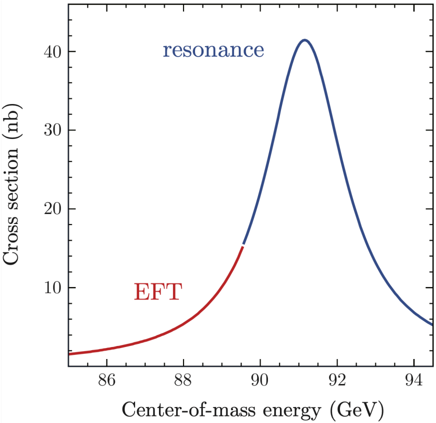


Figure from 1811.00024v3

- Low energies: local EFT for light sector.
- width: Set by the lifetime.
- Height: Strength of the interaction.

There is so much to learn from the analytic structure!

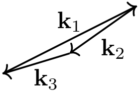
Analytic Cosmological Correlators I

The analytic structure of Cosmological Correlators encodes a lot of information about the theory.

$$\langle \hat{\zeta}_{\mathbf{k}_1} \hat{\zeta}_{\mathbf{k}_2} \hat{\zeta}_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \tag{3}$$

About the initial state of primordial perturbations:

Non standard vacuum $\rightarrow$ Folded singularities



$$, \quad \lim_{k_{12} \to k_3} B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \infty$$

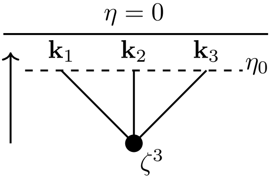
We can also establish a link to flat space amplitudes:

Total energy residue: $k_T = k_1 + k_2 + k_3 \rightarrow 0$

$$\lim_{k_T \to 0} B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \sim \frac{A(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{k_T^{\alpha}}$$

The analytic structure of Cosmological Correlators can tell us something about the inflationary potential. An example is the presence of a total energy branch point.

Late time divergences $\rightarrow$ Total energy branch point



$$, \quad \lim_{\eta_0 \to 0^-} B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \eta_0) \sim \text{Log}(-k_T \eta_0) \qquad (4)$$

Resonant non-Gaussianities $\rightarrow$ Oscillatory total energy branch point

$$V(\phi) = V_{\text{SR}}(\phi) + \Lambda^4 \cos(\frac{\phi}{f}) \qquad (5)$$

$$B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \sim \frac{A(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{k_T^{\alpha}} \cos\left(\frac{\omega}{H} \text{Log}\left(\frac{k_T}{k_*}\right)\right) \qquad (6)$$

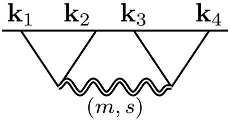
Analytic Cosmological Correlators III

The analytic structure of Cosmological Correlators can also tell us something about the field content of Inflation.

$$\langle \hat{\zeta}_{k_1} \hat{\zeta}_{k_2} \hat{\zeta}_{k_3} \hat{\zeta}_{k_4} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \tag{7}$$

One probe is the Cosmological Collider signal:

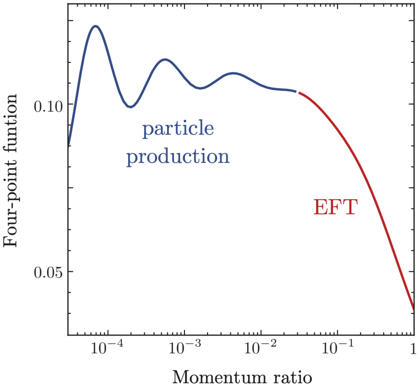
Exchange of Heavy Particles $\rightarrow$ Oscillations



$$, \quad T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \sim \cos\left(\mathrm{Log}\left(\frac{|\mathbf{k}_1 + \mathbf{k}_2|}{k_1 + k_2}\right)\right) P_s(\cos\theta)$$

Analytic Cosmological Correlators IV

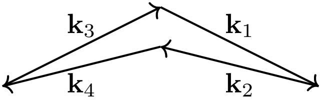
The Cosmological Collider signal appears in the squeezed limit.



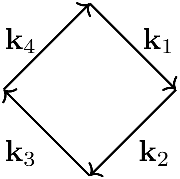
$$u = \frac{|\mathbf{k}_1 + \mathbf{k}_2|}{k_1 + k_2}$$

Figure from 1811.00024v3

Squeezed limit $\sim$ Particle production



Equilateral shapes $\sim$ Local EFTs



Summary

- The present is a window into the past and very high energy physics

$$\underbrace{\langle \delta T \delta T \delta T \rangle}_{\text{Observations}} \sim \underbrace{\int f(k) f(k)}_{\text{Thermal history}} \underbrace{\langle \zeta_k \zeta_k \zeta_k \rangle}_{\text{Very early universe}} \tag{8}$$

- We study the analytic structure of Cosmological Correlators following the spirit of the Analytic S-matrix programme.
- Total energy singularities have multiple sources.

Can loop effects generate total energy branch points?

Outline

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The Bootstrap philosophy

Direct calculations in cosmology (even at tree level!) are very difficult and fundamental principles are obscured.

In the Bootstrap we jump over the models:

Fundamental principles $\rightarrow$ Observables

Fundamental Principles

Locality, Causality,
Unitarity, BD vacuum

Shift symmetry,
de Sitter Isometries

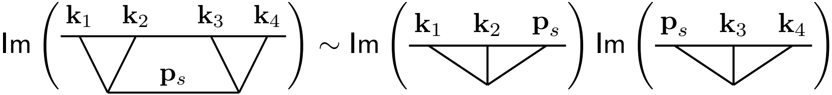
A success story I

This programme has a rich tool-box:

- Manifest locality is understood at any loop order [Jazayeri, Pajer, Stefanyszyn 2021]

$$\partial_{k_a} \langle \zeta_{\mathbf{k}_1} \ldots \zeta_{\mathbf{k}_n} \rangle \Big|_{k_a=0} = 0$$

- Unitary yields a set of cutting rules similar to those in Amplitudes [Goodhew, Jazayeri, Pajer, 2020] [Melville, Pajer 2021]



- Causality explains the analytic structure of Cosmological Correlators [Agui, Lee, Melville, Pajer 2022] [Agui, Melville 2023]

A success story II

This programme has been very successful at the observational level:

- All three-point contact functions have been classified [Pajer, 2020] [Jazayeri, Pajer, Stefanyszyn 2021]

$$\langle \hat{\zeta}_{\mathbf{k}_1} \hat{\zeta}_{\mathbf{k}_2} \hat{\zeta}_{\mathbf{k}_3} \rangle^{\text{con}} = (2\pi)^3 B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$$

- Parity-odd four-point functions are forbidden at tree level [Cabass, Jazayeri, Pajer, Stefanyszyn 2022]

$$\langle \prod_{a=1}^{4} \hat{\zeta}_{\mathbf{k}_a} \rangle^{\text{tree}} = (2\pi)^3 T_{\text{PE}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$$

- Four-point function of tensor perturbations [Bonifacio, Goodhew, Joyce, Pajer, Stefanyszyn 2023]

Born rules

We go back to Quantum Mechanics 101. The expectation value of observables obeys the Born rule:

$$\langle \hat{\mathcal{O}} \rangle = \frac{\int dx \psi^*(x) \hat{\mathcal{O}} \psi(x)}{\int dx |\psi(x)|^2} \tag{9}$$

In cosmology, we promote the Born rule from QM:

$$\text{Position integral: } \int dx \ \rightarrow \ \text{Configuration integral: } \int d\zeta$$

$$\text{Wavefunction: } \psi(x) \ \rightarrow \ \text{Wavefunctional: } \Psi[\zeta]$$

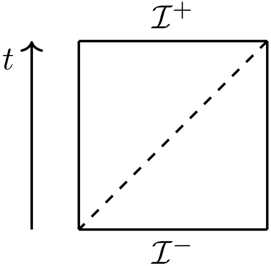
At the level of the Hilbert space they have a similar definition:

$$\psi(x) = \langle x | \Omega \rangle \ , \ \Psi[\zeta] = \langle \zeta | \text{BD} \rangle \tag{10}$$

The wavefunction of the universe

$\Psi[\zeta]$ is an object of QFT in curved spacetime:

$$\Psi[\zeta] = \int_{\text{BD}}^{\zeta} \mathcal{D}\Phi \; e^{iS_{\text{cl}}[\Phi]} = e^{i\Gamma[\zeta]} \tag{11}$$



Bunch Davies vacuum condition:

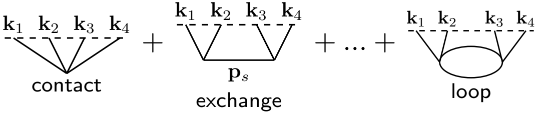
- It is analogous to the asymptotically free states condition.
- It shuts off interactions at the beginning of inflation.

It can be expanded in a power series:

$$\Gamma[\zeta] = \frac{1}{2!}\int_{\mathbf{k}_1\mathbf{k}_2} \psi_2(\mathbf{k}_1,\mathbf{k}_2)\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2} + \frac{1}{3!}\int_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \psi_3(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3} + \dots \tag{12}$$

Perturbation Theory I

Feynman diagrams for $\psi_n(\mathbf{k}_a)$ involve a boundary at $t = t_0$:

$\psi_4(\mathbf{k}_a) =$  $+ \ldots$ (13)

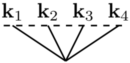
External lines are bulk-to-boundary propagators. They obey the free theory equations of motion and Dirichlet boundary conditions:

$$\mathcal{D}_\Phi K_k(t) = 0\,,\;\; K_k\big|_{-\infty} = 0\,,\;\; K_k\big|_{t_0} = 1 \tag{14}$$

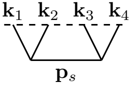
Internal lines are bulk-to-bulk propagators:

$$\mathcal{D}_\Phi G_k(t,t') = -\,\delta(t-t')\,,\;\; G_k\big|_{-\infty} = 0\,,\;\; G_k\big|_{t_0} = 0 \tag{15}$$

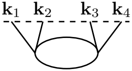
The analytic structure of wavefunction coefficients is constrained by the Bunch-Davies vacuum:

Contact 

$$= \psi_4^{\text{con}}(k_a) \to k_T = 0 \tag{16}$$

Exchange 

$$= \psi_4^{\text{exc}}(k_a, p_s) \to \begin{cases} k_T = 0 \\ E_L = 0 \\ E_R = 0 \end{cases} \tag{17}$$

Loops 

$$= \psi_4^{1-\text{loop}} \to \begin{cases} k_T = 0 \\ k_1 + k_2 + \min(q_1 + q_2) = 0 \\ k_3 + k_4 + \min(q_1 + q_2) = 0 \end{cases} \tag{18}$$

Long lived (zero energy) vertices lead to singularities.

Tree-level results

The Born rule for correlators reads:

$$\langle \prod_a \hat{\zeta}_{\mathbf{k}_a} \rangle = \frac{1}{\int d\zeta |\Psi[\zeta]|^2} \int d\zeta \left(\prod_a \zeta_{\mathbf{k}_a} \right) |\Psi[\zeta]|^2 \tag{19}$$

At leading order:

$$\langle \hat{\zeta}_{\mathbf{k}_1} \hat{\zeta}_{\mathbf{k}_2} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2) P_k = -\frac{1}{2\mathrm{Re}(\psi_2(\mathbf{k}_1, \mathbf{k}_2))} \tag{20}$$

$$\frac{\langle \hat{\zeta}_{\mathbf{k}_1} \hat{\zeta}_{\mathbf{k}_2} \hat{\zeta}_{\mathbf{k}_3} \rangle}{P_{k_1} P_{k_2} P_{k_3}} = \psi_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) + \psi_3^*(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3) \tag{21}$$

Cosmological Correlators $\subset$ Wavefunction coefficients

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Causality in the classical theory

In Classical field theory the general solution is of the form:

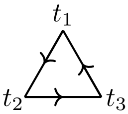
$$\mathcal{D}_\Phi \Phi_k(t) = -J_k(t)\ , \Phi_k(t) = K_k(t)\phi_k + \int dt' G_k^R(t,t')J_k(t') \tag{22}$$

The retarded propagator is of the form:

$$G_k^R(t,t') = 2P_k \mathrm{Im}(K_k(t)K_k^*(t'))\Theta(t-t') \tag{23}$$

We can push causality to give a constrain for time ordered loops.

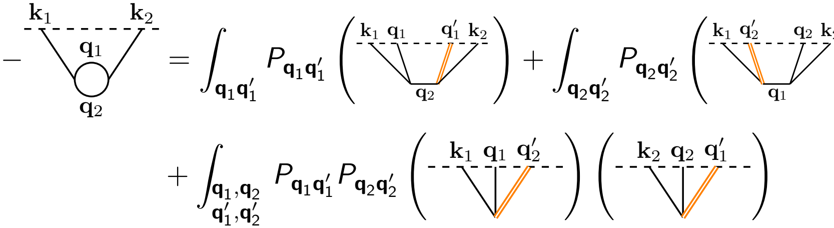
$$\Theta(t_1-t_2)\Theta(t_2-t_3)\Theta(t_3-t_1) = 0\ ,$$



$$= 0 \tag{24}$$

A loop of retarded propagators vanishes. This yields an expansion for products of bulk-to-bulk propagators and cutting rules for loop diagrams. We call these the **Cosmological Tree Theorem**. An example is the sunset diagram:

$$G_{q_1}(t_1,t_2)G_{q_2}(t_2,t_1) = 2P_{q_1}G_{q_2}(t_2,t_1)K_{q_1}(t_1)\mathrm{Im}(K_{q_1}(t_2)) + \begin{pmatrix} q_1 \leftrightarrow q_2 \\ t_1 \leftrightarrow t_2 \end{pmatrix}$$
$$- 4P_{q_1}P_{q_2}K_{q_1}(t_1)\mathrm{Im}(K_{q_1}(t_2))K_{q_2}(t_1)\mathrm{Im}(K_{q_2}(t_2))$$



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Causality Cutting rules II

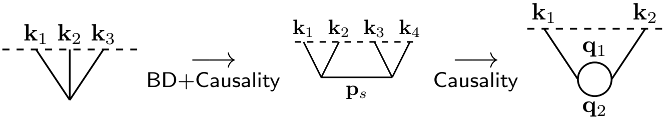
For the Cosmological Tree Theorem we need only need Causality.

- CTT fixes the whole wavefunction coefficient.
- It applies to derivative and IR divergent unitary interactions.
- Extension to spacetimes beyond Minkowski or pure dS.

These have two important consequences:

- Remove nested time integrals.
- Explain the analytic structure of loop corrections.

With the further assumption of the Bunch-Davies vacuum it removes the need for time integrals:



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The (appropriate) questions

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Where is Gravity in all this?

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Is there an amplitudes version?

The (appropriate) questions

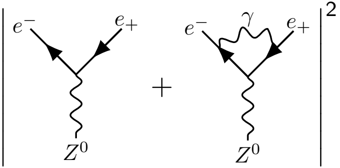
Where is Gravity in all this?

Is there an amplitudes version?

Are we there yet?

In Amplitudes, loops of massless particles generate IR divergences:

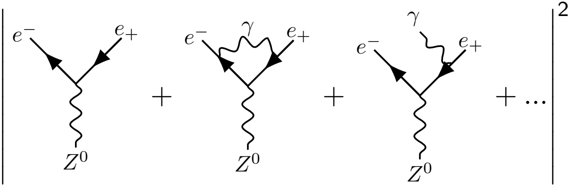
$\sigma_{Z\to ee} \sim$



$\to \infty$ (27)

Inclusive observables cancel these divergences:

$\sigma_{Z\to ee(\gamma)} \sim$

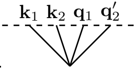


$< \infty$ (28)

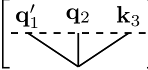
KLN Theorem: Inclusive observables are IR finite.

Loop effects in Cosmological Correlators are present in the NLO of the Born rule:

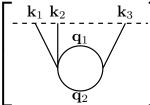
$$\frac{\langle\hat{\zeta}_{\mathbf{k}_1}\hat{\zeta}_{\mathbf{k}_2}\hat{\zeta}_{\mathbf{k}_3}\rangle^{\text{NLO}}}{P_{k_1}P_{k_2}P_{k_3}} = 2\int_{\substack{\mathbf{q}_1\mathbf{q}_1' \\ \mathbf{q}_2\mathbf{q}_1'}} P_{\mathbf{q}_1\mathbf{q}_1'}P_{\mathbf{q}_2\mathbf{q}_2'}\,\mathrm{Re}\,[$$



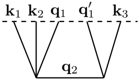
$$]\,\mathrm{Re}\,[$$



$$] + \mathrm{Re}\,[$$



$$] + \int_{\mathbf{q}_1\mathbf{q}_1'} P_{\mathbf{q}_1\mathbf{q}_1'}\,\mathrm{Re}\,[$$

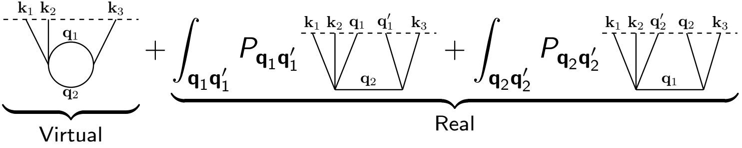


$$] + (q_1 \leftrightarrow q_2)\ldots \quad (29)$$

Does the total energy branch point survive the Born rule?

Real-virtual cancellation

There is a cancellation between virtual and real emissions to remove the total energy branch point:

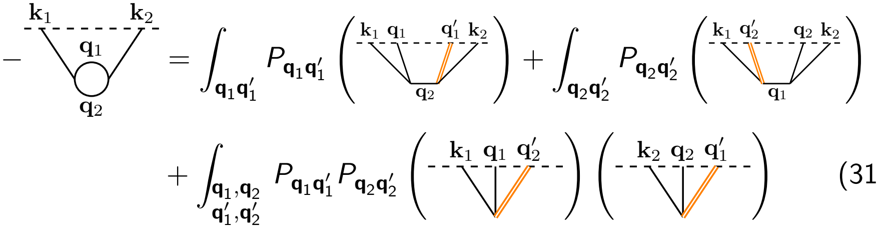


(30)

This cancellation extends to higher loops!

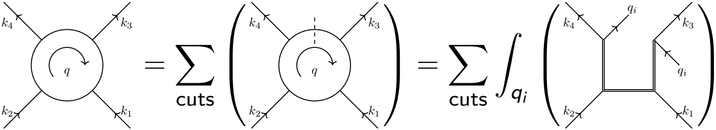
Cosmological KLN theorem:
Any total-energy branch point produced in the wavefunction by **loop integration** *will not appear in the corresponding equal-time correlation functions.*

Causality cutting rules yield a sum over partial and total cuts, with $2^E - 1$ terms:



(31)

The Tree-Loop duality in amplitudes expresses loop integrals in terms of single cut diagrams, with only E terms in the sum:



(32)

The Cosmological Tree-Loop duality II

Upon cutting, the internal lines' propagators are modified:

$$G_{q_i}(t_i, t_{i+1}) \to \tilde{G}_{q_i}(t_i, t_j) = G_{q_i}(t_i, t_j) + \sum_{j \neq i} \omega^i{}_j(q_i, q_j) H_j(q_i, t_i, t_{i+1}) \quad (33)$$

The case for amplitudes and for cosmology is different:

Amplitudes: $H_j \sim K_{q_i}(t_i) K^*_{q_i}(t_{i+1})$, Cosmology: $H_j \sim K_{q_i}(t_i) \mathrm{Im} K^*_{q_i}(t_{i+1})$

The coefficients have to verify:

$$\sum_i^N \prod_{j \neq i} \omega^i{}_j = 1 \to \begin{cases} \text{Amplitudes: } \omega^i{}_j = \Theta(q_i^0 - q_j^0) \\ \text{Cosmology: } \omega^i{}_j = \Theta(\hat{\mathbf{n}}(\mathbf{q}_i - \mathbf{q}_j)) \end{cases} \quad (34)$$

In Cosmology energy is not conserved and hence we need to change the coefficients $\omega^i{}_j(q_i, q_j)$.

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Summary

Studying the very early universe is a window of opportunity:

$$\underbrace{\langle \delta T \delta T \delta T \rangle}_{\text{Observations}} \sim \underbrace{\int f(k) f(k)}_{\text{Thermal history}} \underbrace{\langle \hat{\zeta}_k \hat{\zeta}_k \hat{\zeta}_k \rangle}_{\text{Very early universe}} \tag{35}$$

The Cosmological Bootstrap tries to overcome two problems:

$$\left.\begin{array}{l}\text{A plethora of models} \\ \text{Few measurements}\end{array}\right\} \to \text{Need for model independent predictions.}$$

Causality is a very valuable tool:

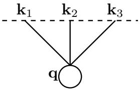
- Removes the need for nested time integrals.
- Explains the analytic structure of wavefunction coefficients and Cosmological Correlators.

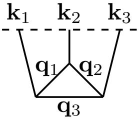
Causality→Analyticy→Observations

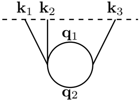
Outlook

Near future open questions:

- How to further constrain observables?
- Characterisation of one-loop three-point functions:

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle =$$ 

$+$ 

$+$ 

$+ \ldots$ (36)

Long term questions:

- Study of dS S-matrix?
- Role of entanglement in cosmology?

Thank you very much!