On the CIPT-FOPT Discrepancy for Hadronic τ Decays

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arXiv:2008.00578 , arXiv:2105.11222

(with **Christoph Regner**)

arXiv:2202.10957, arXiv:2207.01116

(with **Miguel Benitez-Rathgeb, Diogo Boito** and **Matthias Jamin**)

arXiv:2305.10288

(with **Néstor Gracia** and **Vicent Mateu**)

Doktoratskolleg Particles and Interactions

universität wien

FWF Der Wissenschaftsfonds.

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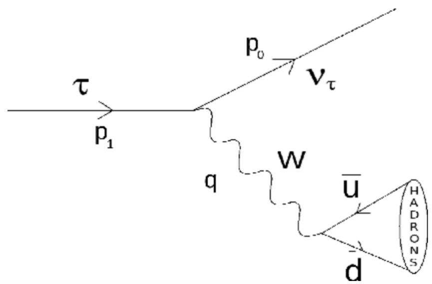
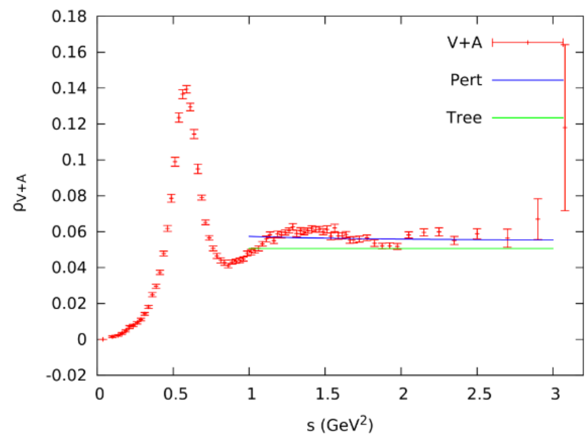
Hadronic τ Spectral Function Moments

ALEPH: τ hadronic width (HFLAV 2019)

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \text{hadrons}\, \nu_\tau(\gamma)]}{\Gamma[\tau^- \to e^- \overline{\nu}_e \nu_\tau(\gamma)]} = 3.6355 \pm 0.0081$$



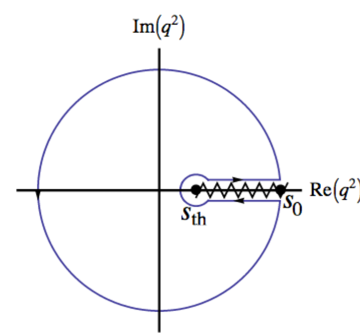
Inclusive hadronic mass spectrum



$$\left(p^\mu p^\nu - g^{\mu\nu} p^2\right) \Pi(p^2) \equiv i \int dx\, e^{ipx} \langle \Omega |\, T\{ j^\mu_{v/av,jk}(x)\, j^\nu_{v/av,jk}(0)^\dagger \} \Omega \rangle$$

Braaten, Narison, Pich, Le Diberder, ... 90's

$$A^\omega_{V/A}(s_0) \equiv \int_{s_{\rm th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\, {\rm Im}\,\Pi_{V/A}(s) = \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\, \Pi_{V/A}(s)$$



Hadronic τ Spectral Function Moments

Theory: Operator product expansion

Adler function: $\frac{1}{4\pi^2}\Big(1+D(s)\Big) \equiv -s\frac{d\Pi(s)}{ds}$

Braaten, Narison, Pich, Le Diberder, ... 90's

$$A_{W_i}(s_0) = \frac{N_c}{2}|V_{ud}|^2\left[\delta^{\rm tree}_{W_i} + \delta^{(0)}_{W_i}(s_0) + \sum_{d\geq 2}\delta^{(d)}_{W_i}(s_0) + \delta^{\rm DV}_{W_i}(s_0)\right]$$

$$W_i(x) = \sum_{n=0}^{m} a_n\, x^n$$

pQCD — OPE — Duality violation

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1}\left(\frac{\alpha_s(-s)}{\pi}\right)^n$$

← Perturbative

Shifman, Vainshtein, Sacharow 1978

OPE non-pert. corrections

$$\hat{D}^{\rm OPE}(s) = \frac{C(\alpha_s(-s))}{(-s)^2}\langle\alpha_s G^2\rangle + \sum_{p=3}^{\infty}\frac{1}{(-s)^p}\Big[C_0(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_1}\rangle + C_1(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_2}\rangle + \dots\Big]$$

$$\delta^{(0)}_{W_i}(s_0) = \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s}\,W_i\!\left(\tfrac{s}{s_0}\right)\hat{D}(s) \qquad \delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s}\,W_i\!\left(\frac{s}{s_0}\right)\frac{\Lambda^d_{\rm QCD}}{s^{d/2}}$$

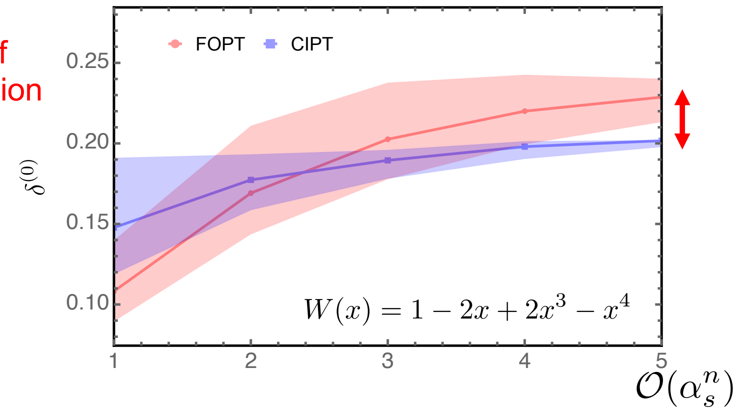
FOPT-CIPT Discrepancy Problem

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n ,$$

$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \ln^{k-1}\left(\frac{-s}{s_0}\right)$$

Change of renormalization scale

FOPT CIPT



$$c_{0,1} = c_{1,1} = 1 , \quad c_{2,1} = 1.640$$
$$c_{3,1} = 6.371$$
$$c_{4,1} = 49.076$$
$$c_{5,1} = 280 \pm 140$$

4-loop: Gorishni etal., Surguladze etal. 1991

5-loop: Baikov etal. 2008

6-loop estimate Beneke, Boito, Jamin; Caprini

- CIPT resums powers of π with respect to FOPT
- CIPT leads in general to smaller moments than FOPT
- OPE and DV corrections assumed to be universal
- Strong coupling from CIPT larger than from FOPT
- CIPT disfavored from plausibility studies of Borel models for the Adler function

Beneke, Boito, Jamin 2008, 2012

- **Situation inconclusive until 2020**

Contour-improved perturbation theory (CIPT):

$$\delta^{(0),\mathrm{CIPT}}_{W_i}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \boxed{\left(\frac{\alpha_s(-x s_0)}{\pi}\right)^n}$$

Fixed-order perturbation theory (FOPT):

$$x = \frac{s}{s_0}$$

$$\delta^{(0),\mathrm{FOPT}}_{W_i}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \boxed{\left(\frac{\alpha_s(s_0)}{\pi}\right)^n} \sum_{k=1}^{n+1} k\, c_{n,k} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \ln^{k-1}(-x)$$

FOPT-CIPT Discrepancy

$$\mathcal{L}_{\rm QCD} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\,\delta_{ij}\right)\psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

$$G^a_{\mu\nu} = \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu$$

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - ig (T_a)_{ij} \mathcal{A}^a_\mu$$

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n ,$$

$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \ln^{k-1}\left(\frac{-s}{s_0}\right)$$

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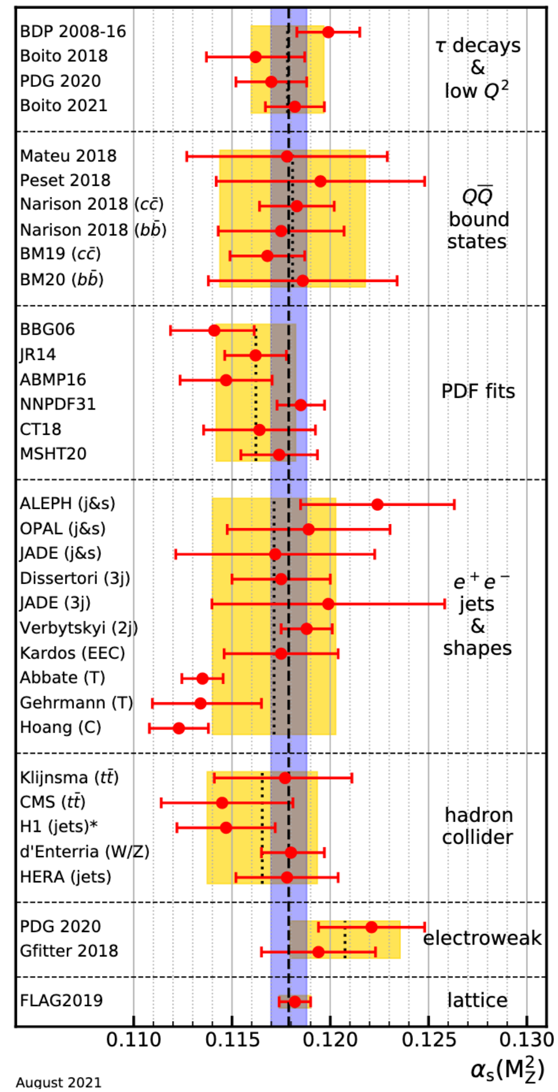


Outline

- Introduction
- **Renormalon view:**
  Asymptotic Separation Δ
  → **FOPT and CIPT expansions describe different quantities**
  → **CIPT inconsistent with the operator product expansion**
- **Mathematical view:**
  → **CIPT is a non-uniform asymptotic expansion**
- **Reconciling CIPT and FOPT:**
  renormalon-free gluon condensate scheme
- Impact on determinations of $\alpha_s(m_\tau^2)$

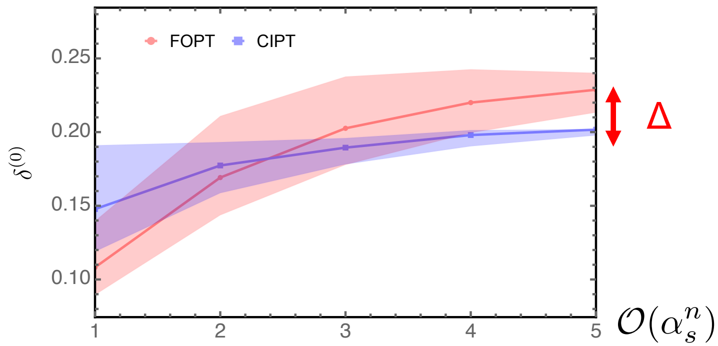
Interesting Observations: Total Decay Rate

$$W_\tau(x) = 1 - 2x + 2x^3 - x^4 \qquad \delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0} \frac{ds}{s}\left(\frac{s}{s_0}\right)^m \frac{\Lambda^4_{\rm QCD}}{s^2} = \frac{\Lambda^4_{\rm QCD}}{s_0^2}\,\delta_{m2}$$

→ Sensitivity to leading O($\Lambda^4_{\rm QCD}$) gluon condensate strongly suppressed

Δ

Moment's perturbation series:



- Discrepancy between CIPT and FOPT scales as $\sim \frac{\Lambda^4_{\rm QCD}}{s_0^2}$

  → Accidental or indication of a quartic IR sensitivity?

  → Contradiction to standard OPE

  → How can there be O($\Lambda^4_{\rm QCD}$) sensitivity left ?

- CIPT is not an expansion in powers of α_s at a definite renormalization scale. It is impossible to switch between the CIPT and FOPT moment series terms through a change of scheme of the strong coupling and a reexpansion of the series due to the **contour integration** !!

→ Worth to reconsider CIPT and FOPT from scratch: OPE ↔ IR renormalons

Interesting Observations: Total Decay Rate

$$W_\tau(x) = 1 - 2x + 2x^3 - x^4 \qquad \delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0} \frac{ds}{s}\left(\frac{s}{s_0}\right)^m \frac{\Lambda^4_{\rm QCD}}{s^2} = \frac{\Lambda^4_{\rm QCD}}{s_0^2}\,\delta_{m2}$$

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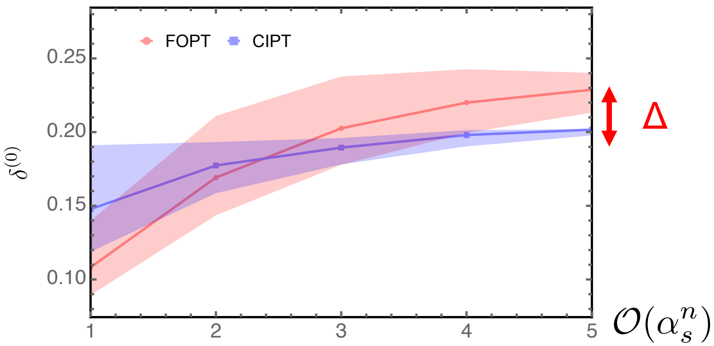

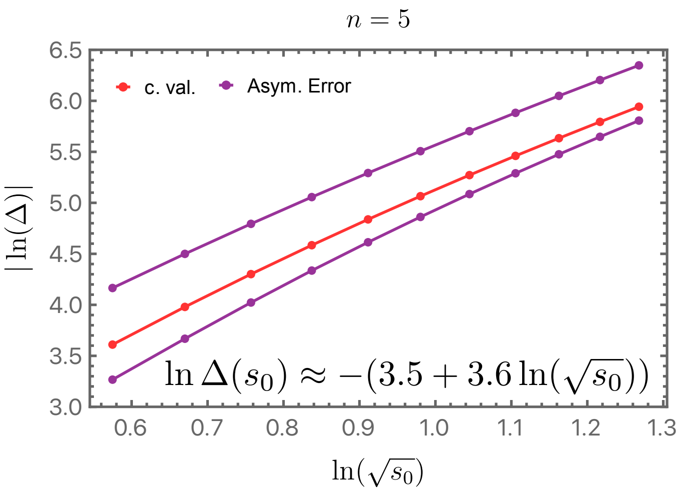

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Renormalon Calculus: Euclidean Adler Function

Perturbative series in QCD are not convergent, but asymptotic in expansion variable $\alpha_s(s_0)$. $\rightarrow$ $\hat{D}(-s_0) \sim \sum_{n=1}^{\infty} n! \left(\frac{\alpha_s(s_0)}{\pi}\right)^n$

\`t Hooft; David; Müller; ... Beneke; ...

Borel calculus:

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \quad \Longrightarrow \quad B[\hat{D}](u) = \sum_{n=1}^{\infty} \frac{c_{n,1}}{\Gamma(n)} u^{n-1}$$

Borel sum:

$$\hat{D}_{\text{Borel}}(-s_0) = \text{PV} \int_0^{\infty} \mathrm{d}u \; B[\hat{D}](u) \, e^{-\frac{4\pi u}{\beta_0 \alpha_s(s_0)}}$$

Association: IR renormalon poles/cuts $\Leftrightarrow$ (standard) OPE Corrections

$$\hat{D}^{\text{OPE}}(s) = \frac{C(\alpha_s(-s))}{(-s)^2} \langle \alpha_s G^2 \rangle + \sum_{p=3}^{\infty} \frac{1}{(-s)^p} \left[C_0(\alpha_s(-s)) \langle \mathcal{O}_{2p,\gamma_1} \rangle + C_1(\alpha_s(-s)) \langle \mathcal{O}_{2p,\gamma_2} \rangle + \ldots \right]$$

Leading Gluon Condensate:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

FOPT vs. CIPT Borel Representation (large-β_0)

FOPT expansion: $\rightarrow$ Expansion parameter: $\alpha_s(s_0)$

$$\hat{D}(s) = \underbrace{\sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n}_{\text{expansion variable}} \underbrace{\sum_{k=1}^{n} k\, c_{n,k}\, \ln^{k-1}(-x)}_{\text{coefficient}}$$

Borel sum:

$$\mathrm{PV}\int_0^\infty du \left[B[\hat{D}](u)\, e^{-u\ln(-x)}\right] e^{-\frac{4\pi u}{\beta_0 \alpha_s(s_0)}}$$

$$e^{-u\ln(-x)}\, e^{-\frac{4\pi u}{\beta_0\alpha_s(s_0)}} = e^{-\frac{4\pi u}{\beta_0\alpha_s(-s)}}$$

➡ FOPT Borel representation = “true“ Borel representation

$$\delta^{(0),\mathrm{FOPT}}_{W_i,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\, \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x}\, W_i(x)\, B[\hat{D}](u)\, e^{-\frac{4\pi u}{\beta_0\alpha_s(-x s_0)}}$$

FOPT vs. CIPT Borel Representation

CIPT expansion: $\rightarrow$ No obvious expansion parameter !

$$\delta^{(0),\mathrm{CIPT}}_{W_i}(s_0) = \frac{1}{2\pi i}\sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\pi}\right)^n = \frac{1}{2\pi i}\sum_{n=1}^{\infty} \underbrace{\left(\frac{\alpha_s(s_0)}{\pi}\right)^n}_{\text{expansion variable}} \underbrace{c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)^n}_{\text{coefficient}},$$

➡ CIPT Borel representation: NEW !

Regner, Hoang arXiv:2008.00578

$$\delta^{(0),\mathrm{CIPT}}_{W_i,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\, \frac{1}{2\pi i} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x} W_i(x) \left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right) B[\hat{D}]\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}\right) e^{-\frac{4\pi\bar{u}}{\beta_0\alpha_s(s_0)}}$$

Contour needs to be deformed from |x|=1 due to the cut in $\alpha_S(-x\, s_0)$

„Asymptotic Separation“

$$\Delta_W(s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{W,\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{W,\mathrm{Borel}}(s_0)$$

$$\Delta_W(s_0) \sim \frac{\Lambda^d_{\mathrm{QCD}}}{s_0^{d/2}} \quad \text{for } \mathcal{O}(\Lambda^d_{\mathrm{QCD}}) \text{ IR renormalon contained in } \hat{D}$$

Character of the Asymptotic Separation

FOPT Borel representation

$$\delta^{(0),\mathrm{FOPT}}_{W_i,\mathrm{Borel}}(s_0) = \mathrm{PV}\int_0^\infty \mathrm{d}u\,\frac{1}{2\pi i}\oint_{|x|=1}\frac{\mathrm{d}x}{x}\,W_i(x)\,B[\hat{D}](u)\,e^{-\frac{4\pi u}{\beta_0\alpha_s(-xs_0)}}$$

CIPT Borel representation

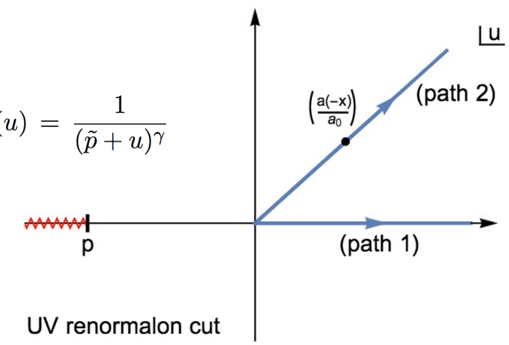
$$\delta^{(0),\mathrm{CIPT}}_{W_i,\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\,W_i(x)\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\right)B[\hat{D}]\left(\frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}\right)e^{-\frac{4\pi\bar{u}}{\beta_0\alpha_s(s_0)}}$$

- Related through complex-valued change of variables

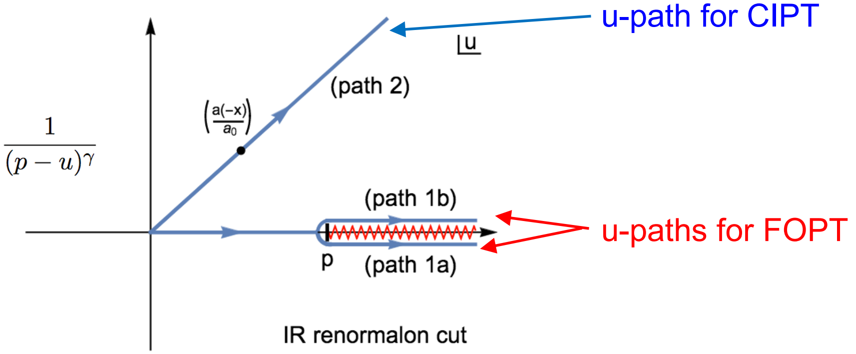
$$u = \frac{\alpha_s(-xs_0)}{\alpha_s(s_0)}\bar{u}$$

- Equivalent in perturbation theory (u-Taylor series)
- Agree at Euclidean point x = -1
- Difference in presence of IR renormalon cuts

$$B^{\mathrm{UV}}_{\hat{D},-\tilde{p},\gamma}(u) = \frac{1}{(\tilde{p}+u)^\gamma}$$



$$B^{\mathrm{IR}}_{\hat{D},p,\gamma}(u) = \frac{1}{(p-u)^\gamma}$$



UV renormalons:

FOPT and CIPT Borel representations equivalent because closing up paths 1 and 2 does not contain cuts

IR renormalons: finite difference !

FOPT and CIPT Borel representations inequivalent

- FOPT: PV prescription needs to be imposed
- CIPT: automatically well-defined by complex-valued α_s
- Difference because closing paths 1a/1b and 2 always contains cuts

CIPT Borel Sum Contour Integration

The contour integration for the CIPT Borel representation must be deformed away from |x| = 1.

(Leaves FOPT Borel sum unchanged!)

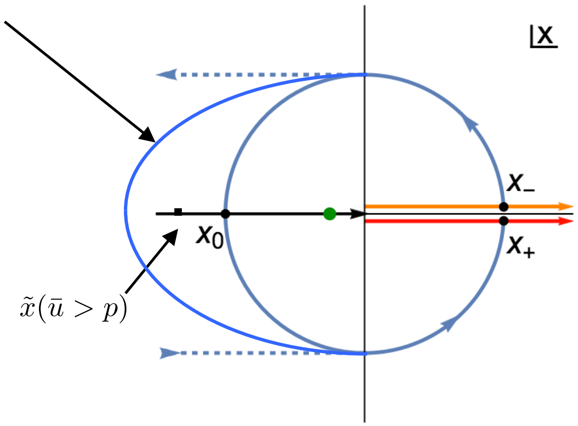
Do the contour-integral first:

$$\delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) = \int_0^\infty \mathrm{d}\bar{u}\,\frac{1}{2\pi i}\oint_{\mathcal{C}_x}\frac{\mathrm{d}x}{x}\,(-x)^m\left(\frac{a(-x)}{a_0}\right)\frac{e^{-\frac{\bar{u}}{a_0}}}{\left(p-\frac{a(-x)}{a_0}\bar{u}\right)^\gamma}$$

$$= \int_0^\infty \mathrm{d}\bar{u}\, e^{-\frac{\bar{u}}{a_0}}\,\tilde{C}(p,\gamma,m,s_0;\bar{u})\,.$$

pole in x-plane at (large-β_0)

Contour must always cross real axis for $x < \tilde{x}(\bar{u})$



$$\tilde{x}(\bar{u}) = -e^{(\bar{u}-p)/pa_0} = -\left(\frac{\Lambda^2_{\mathrm{QCD}}}{s_0}\right)^{\frac{p-\bar{u}}{p}}$$

$$< -1 \quad \text{for} \quad \bar{u} > p$$

$$\tilde{x}(0) = -\left(\frac{\Lambda^2_{\mathrm{QCD}}}{s_0}\right) \quad \text{(Landau pole)}$$

$$\tilde{x}(\bar{u}\to\infty) \to -\infty$$

Character of the Asymptotic Separation

Alternative view [for the large-β_0 approximation]

Golterman, Maltman, Peris arXiv:2305.10386

FOPT Borel representation

$$\delta^{(0),\mathrm{FOPT}}_{(-x)^\ell,\mathrm{Borel}}(s_0) \;=\; \mathrm{PV}\int_0^\infty \mathrm{d}u\,\frac{1}{2\pi i}\oint\limits_{|x|=1}\frac{\mathrm{d}x}{x}\,(-x)^\ell\,\frac{1}{(p-u)^\gamma}\,e^{-\frac{4\pi u}{\beta_0\alpha_s(-xs_0)}}$$

- α_S-dependent change of variables and contour deformation can be avoided due to the identities

$$e^{-\frac{4\pi u}{\beta_0\alpha_s(-xs_0)}} = (-x)^{-u}e^{-\frac{4\pi u}{\beta_0\alpha_s(s_0)}}$$

$$\frac{1}{2\pi i}\oint\limits_{|x|=1}\frac{\mathrm{d}x}{x}\,x^\ell\,(-x)^u = \frac{\sin(\pi u)}{u-\ell} = \frac{1}{2\pi i}\,\frac{e^{i\pi u}-e^{-i\pi u}}{u-\ell}$$

valid in the large-β_0 approximation

- Leads to the same result.

CIPT Borel representation

$$\delta^{(0),\mathrm{CIPT}}_{(-x)^\ell,\mathrm{Borel}}(s_0) \;=\; \frac{1}{2\pi i}\left[\int_0^{(1+i)\infty}\mathrm{d}u\,e^{i\pi u} - \int_0^{(1-i)\infty}\mathrm{d}u\,e^{-i\pi u}\right]e^{-\frac{4\pi u}{\beta_0\alpha_s(s_0)}}$$
$$\times\,\frac{1}{2}\left[\frac{1}{(\ell+i\epsilon-u)(p+i\epsilon-u)^\gamma}+\frac{1}{(\ell-i\epsilon-u)(p-i\epsilon-u)^\gamma}\right]$$
$$-\;[\text{residues at } u=\ell\pm i\epsilon\;]$$

Brief Numerical Analysis

Single renormalon model:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

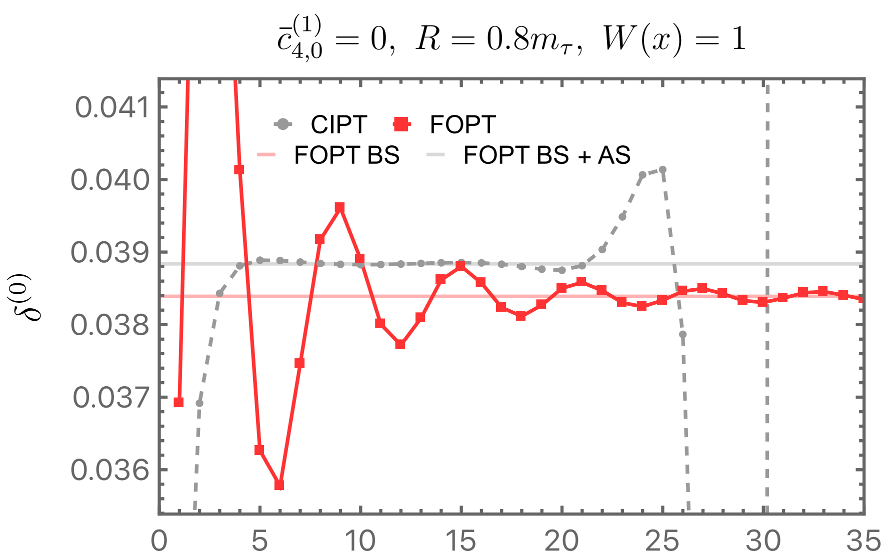
$$W(x) = 1$$

Pure O(Λ^4_{QCD}) renormalon in Adler function

→ Gluon condensate corrections vanishes

→ Per. series should be convergent

$$\delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0} \frac{ds}{s}\left(\frac{s}{s_0}\right)^m \frac{\Lambda^4_{\rm QCD}}{s^2} = \frac{\Lambda^4_{\rm QCD}}{s_0^2}\,\delta_{m2}$$



$s_0 = m_\tau^2$

- CIPT series is **divergent** ! FOPT series convergent.

This fact was overlooked in the past

- CIPT not compatible with standard OPE !

CIPT Borel representation should not be considered as ”true”, but it correctly characterizes the CIPT expansion

- Excellent description of CIPT-FOPT discrepancy by asymptotic separation $\Delta_W(s_0)$

- Moments with small asymptotic separation can be identified.

Mathematical Perspective

Gracia, AHH, Mateu, arXiv:2305.10288

Can we identify the mathematical reason, why CIPT is inconsistent with the OPE?

→ Inconsistency means: CIPT series is divergent for cases where OPE demands convergence

CIPT: $$\sum_{n=1}^{\infty} c_n H_{n,\ell}(a)$$

FOPT: $$\sum_{n=1}^{\infty} d_n a^n$$

$$H_{n,\ell}(a) \equiv \frac{1}{2i\pi} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^\ell a^n(-x s_0)$$

$$a \equiv \frac{\beta_0 \alpha_(s_0)}{4\pi}$$

Series in non-trivial functions of a

Power series in a

Is CIPT a consistent asymptotic expansion?

→ Yes because the following relation holds:

$$\lim_{a\to 0} \frac{|H_{n+1,\ell}(a)|}{|H_{n,\ell}(a)|} = 0 \quad \text{for all } n$$

„The { $H_{n,l}(a)$ } form an asymptotic sequence“

Ensures that coefficients c_n can be determined unambiguously.

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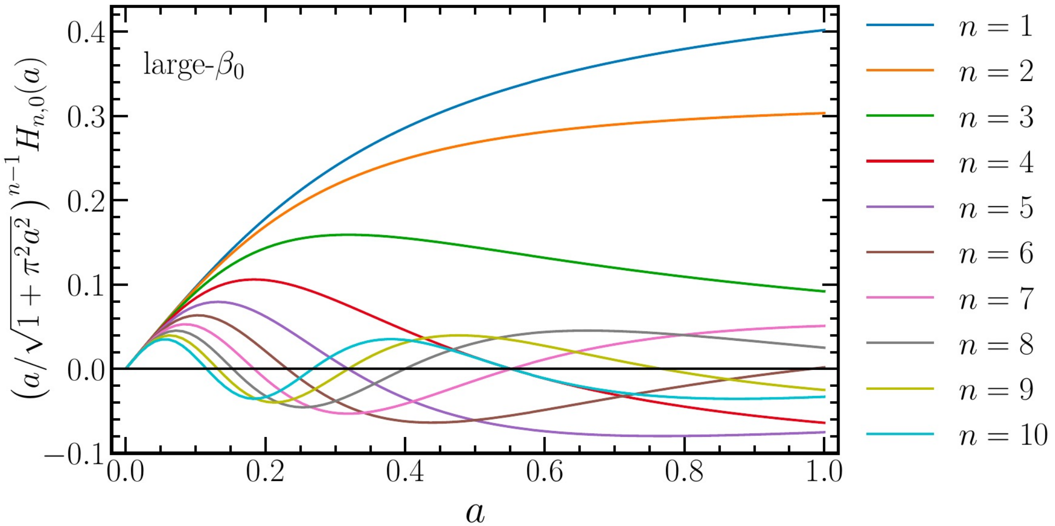
Mathematical Perspective

But the asymptotic sequence { $H_{n,l}(a)$ } it non-uniform because the $H_{n,l}(a)$ has zeroes for real a

$$\frac{H_{n,0}(a)}{(a/\sqrt{1+a^2\pi^2})^{n-1}}$$

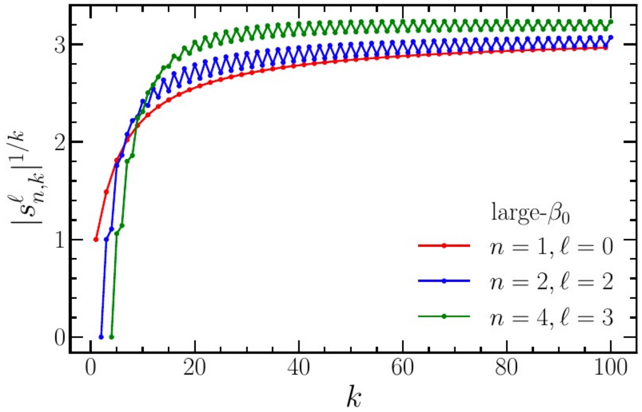


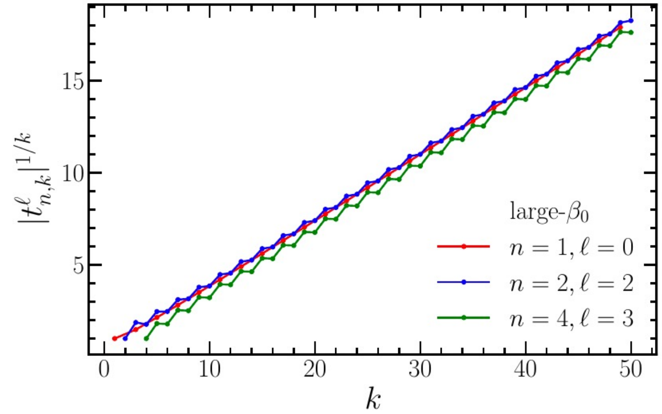
→ Zeros approach zero for large n

$$H_{n,\ell}(a) = \sum_{k=n}^{\infty} s^{\ell}_{n,k} a^k$$ ← Has finite radius of convergence

$$a^n = \sum_{k=n}^{\infty} t^{\ell}_{n,k}\, H_{k,\ell}(a)$$ ← Divergent for any a !

Mathematical Perspective





Theorem A.6 (Weierstrass' Double Series Theorem). *Consider an infinite set of functions $f_i(x)$ that are analytic for $|x| < r$, so that the power expansions $f_i(x) = \sum_{k=0}^{\infty} a_k^{(i)} x^k$ exist and converge at least for $|x| < r$ for all i. Furthermore, consider a convergent series of these functions $F(x) = \sum_{i=0}^{\infty} f_i(x)$ that is uniformly convergent for $|x| \le \rho$ for every $\rho < r$, so that the series converges in particular everywhere within the interval $|x| < r$ and defines the function $F(x)$ there. Then, the infinite sums $A_k = \sum_{i=0}^{\infty} a_k^{(i)}$ are convergent and the infinite sum $\sum_{k=0}^{\infty} A_k x^k$ converges to $F(x)$ for $|x| < r$, so that $F(x) = \sum_{i=0}^{\infty}(\sum_{k=0}^{\infty} a_k^{(i)} x^k) = \sum_{k=0}^{\infty}(\sum_{i=0}^{\infty} a_k^{(i)}) x^k$ and is analytic for $|x| < r$.*

⇒ Any convergent CIPT series will be convergent in FOPT and sums to the same value

⇒ Any convergent FOPT series will be in general divergent in CIPT !

Mathematical Perspective

Models for expansion function

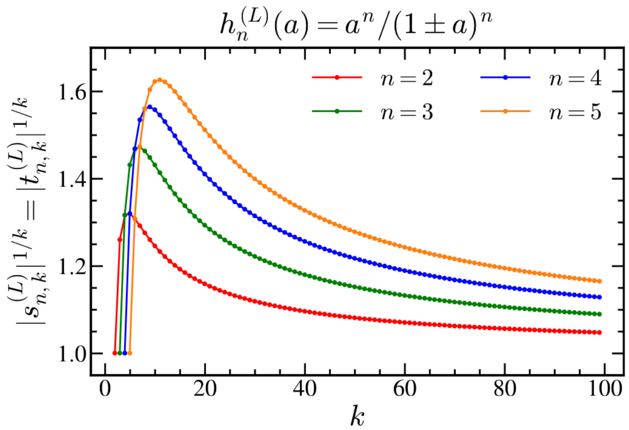
Change of renormalization scale:

zeros at a=-1/L

$$h_n^{(L)}(a) = [a(\mu^2)]^n = \frac{a^n}{(1+aL)^n} \qquad L = \log(\mu^2/s_0)$$

Expansion of a^n in terms of $h_n^{(L)}(a)$ has same convergence radius as expansion of $h_n^{(L)}(a)$ in powers of n.

Change of renormalization scale does not change the radius of convergence.



Mathematical Perspective

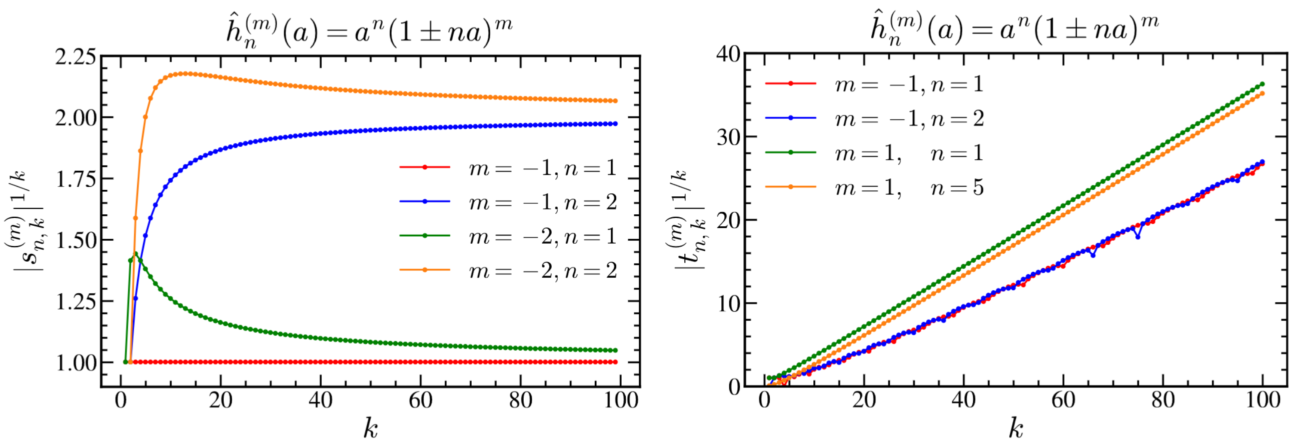
Models for expansion function

Model with decreasing zeros/singuarity points:

$$\hat{h}_n^{(m)}(a) = a^n (1 - \xi n a)^m$$

m=-1: singularities at a=1/ξn

m=1: zeros at a=1/ξn



Expansion of a^n in terms of $h_n^{(m)}(a)$ is divergent for any a !

The zeros are one reason why CIPT has good apparent convergence at low orders, but they are likely also the reason why CIPT has the bad divergence property.

The fact that the Adler function contains IR renormalons ensures that the inconsistency of CIPT is unavoidable.

Renormalon-Free GC Scheme

Conclusion from the asymptotic separation:

AHH, Regner 2008.00578

- Asymptotic separation/CIPT inconsistency vanishes if IR renormalons are absent
- CIPT and FOPT should become consistent for IR-subtracted perturbation theory
- Gluon condensate renormalon is mostly responsible for the asymptotic separation

Renormalon-Free GC Scheme

Conclusion from the asymptotic separation:

AHH, Regner 2008.00578

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Idea: "short-distance" scheme for the gluon condensate

Benitez-Rathgeb, Boito, Jamin, AHH 2202.10957

$$\langle \bar{G}^2 \rangle^{\overline{\rm MS},(n)}$$

Original $\overline{\rm MS}$ GC contains pure O($\Lambda^4_{\rm QCD}$) renormalon (scale invariant)

$$\langle \bar{G}^2 \rangle^{\overline{\rm MS},(n)} \equiv \underbrace{\langle G^2 \rangle (R^2)}_{\text{renormalon-free R-dependent}} - R^4 \sum_{\ell=1}^{n} \underbrace{N_g}_{\text{renormalon norm (approximately known)}} \, r_\ell^{(4,0)} \, \bar{a}^\ell(R^2) \, ,$$

Expand perturbatively with Adler function

IR factorization scale R

$$r_\ell^{(4,0)} = \left(\frac{1}{2}\right)^{\ell+4\hat{b}_1} \frac{\Gamma(\ell+4\hat{b}_1)}{\Gamma(1+4\hat{b}_1)} \qquad \bar{a}(R^2) = \frac{\beta_0 \, \bar{\alpha}_s(R^2)}{4\pi}$$

C-scheme (C=0)

Boito, Jamin, Miravitllas 2016

Renormalon-Free GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH 2202.10957

$$\langle G^2\rangle(R^2) - \langle G^2\rangle({R'}^2)$$

Renormalon-free (convergent series)

$$\frac{\mathrm{d}}{\mathrm{d}\ln R^2}\,\langle G^2\rangle(R^2) \;=\; \frac{N_g}{2^{4\hat{b}_1}}\,\frac{R^4\,\bar{a}(R^2)}{1-2\hat{b}_1\bar{a}(R^2)}$$

(R-evolution equation)

Convergent series!

We can define an R-independent „short-distance“ GC:

treated like a tree-level term
(Do not expand !)

$$\langle G^2\rangle(R^2) \;\equiv\; \langle G^2\rangle^{\mathrm{RF}} + N_g\,\bar{c}_0(R^2)\,.$$

→ R-invariance of scheme at infinite truncation order

$$\bar{c}_0(R^2) \equiv R^4\;\;\mathrm{PV}\int_0^\infty \frac{\mathrm{d}u\;e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}} = -\frac{R^4\,e^{-\frac{2}{\bar{a}(R^2)}}}{(\bar{a}(R^2))^{4\hat{b}_1}}\,\mathrm{Re}\left[e^{4\pi\hat{b}_1 i}\,\Gamma\Big(-4\hat{b}_1,-\frac{2}{\bar{a}(R^2)}\Big)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln R^2}\,\langle G^2\rangle^{\mathrm{RF}} \;=\; 0$$

Scale-invariant “short-distance“ scheme for the gluon condensate

→ „true“ Borel sum value unchanged (i.e. N_g-independent) ! („minimal scheme“)

CIPT and FOPT: RF GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

Single renormalon model:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

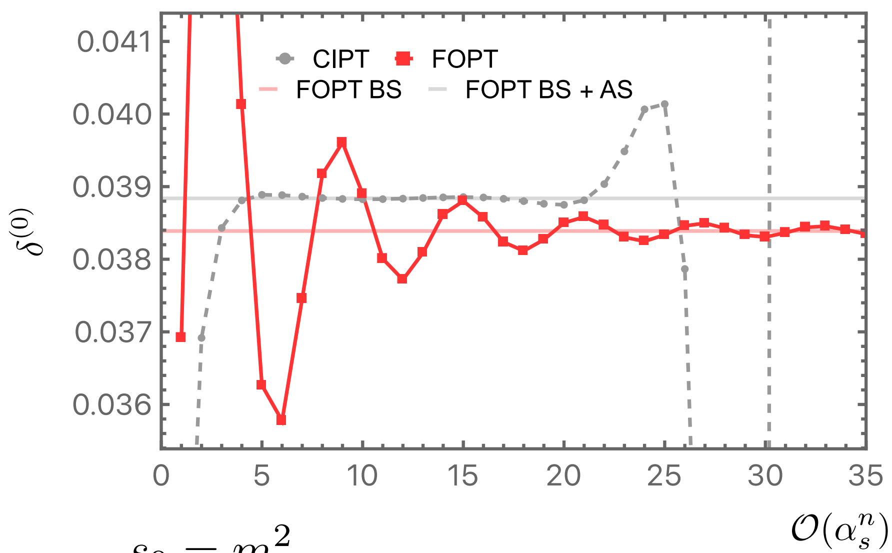
$$W(x) = 1 \qquad N_g = \frac{3}{2\pi^2}$$

Pure O(Λ^4_{QCD}) renormalon in Adler function

→ Gluon condensate corrections vanishes !

→ Nevertheless dramatic impact of changing to the RF GC scheme

$\bar{c}^{(1)}_{4,0} = 0,\ R = 0.8 m_\tau,\ W(x) = 1$



$s_0 = m_\tau^2$

- FOPT same as in the original GC scheme
- CIPT^{RS} series is convergent
- CIPT^{RS} consistent with FOPT !
- CIPT^{RS} compatible with standard OPE !
- CIPT^{RS} Borel sum = FOPT Borel sum
- CIPT^{RS} converges much faster than FOPT (oscillating behavior absent)

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Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957

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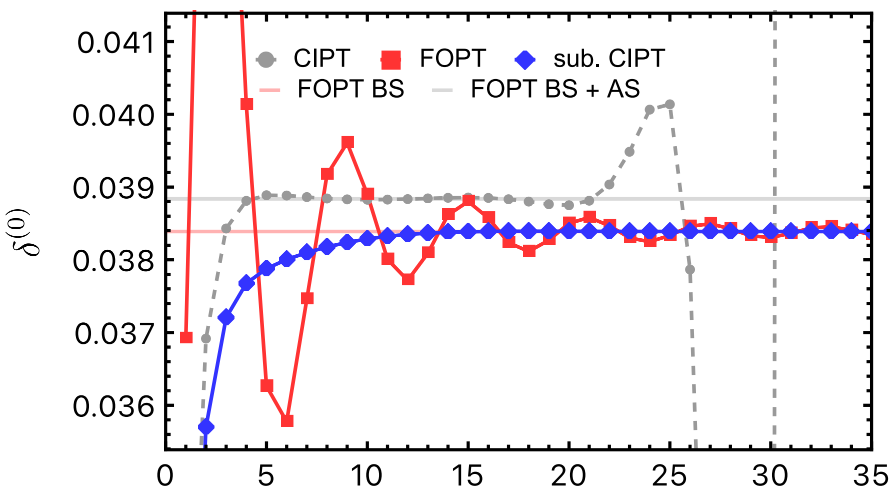

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CIPT and FOPT: RF GC Scheme

Realistic Multi renormalon model: GC, O(Λ^4_{QCD}, Λ^6_{QCD}) + UV renormalons in Adler function

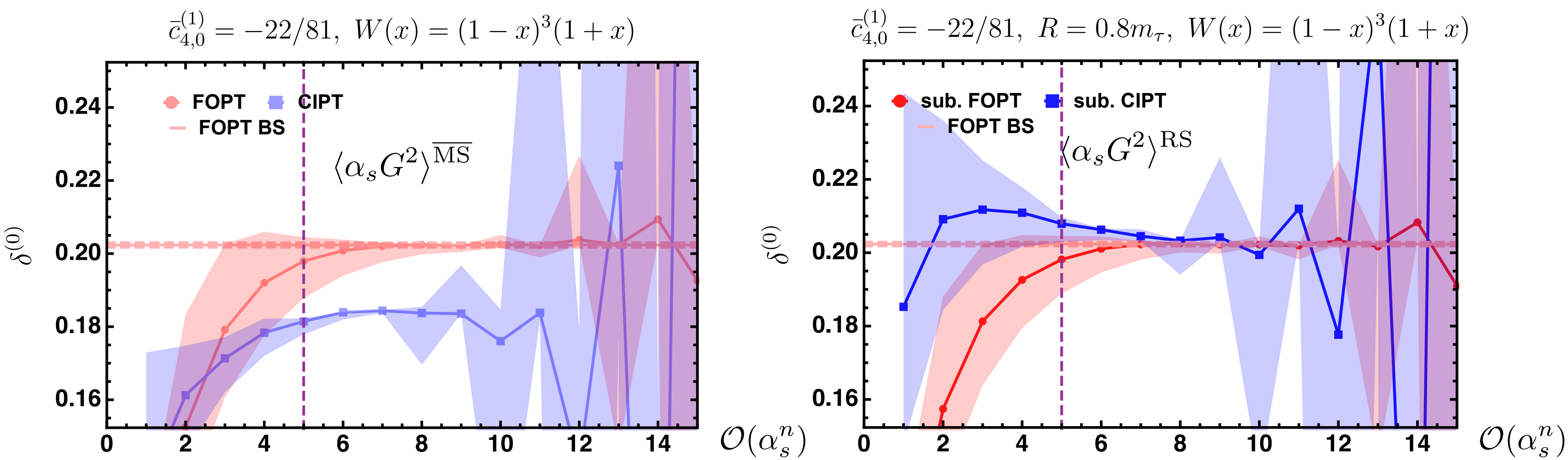
Beneke, Jamin 2008

$$W(x) = 1 - 2x + 2x^3 - x^4$$

→ GC suppressed

$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \le \xi \le 2\,, \quad N_g = 0.64$$

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957



New RF GC Scheme !

- Discrepancy between CIPT and FOPT removed
- CIPT becomes consistent with FOPT (which is only slightly modified)
- Higher precision for α_s determinations from hadronic tau decays achievable
- Additional uncertainty from uncertainties in N_g

CIPT and FOPT: RF GC Scheme

Realistic Multi renormalon model: GC, O(Λ^4_{QCD}, Λ^6_{QCD}) + UV renormalons in Adler function

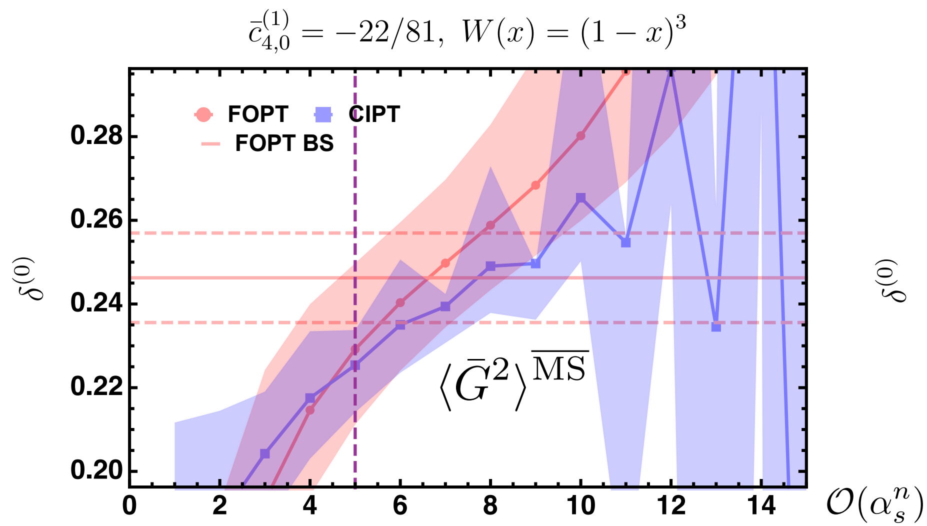
Beneke, Jamin 2008

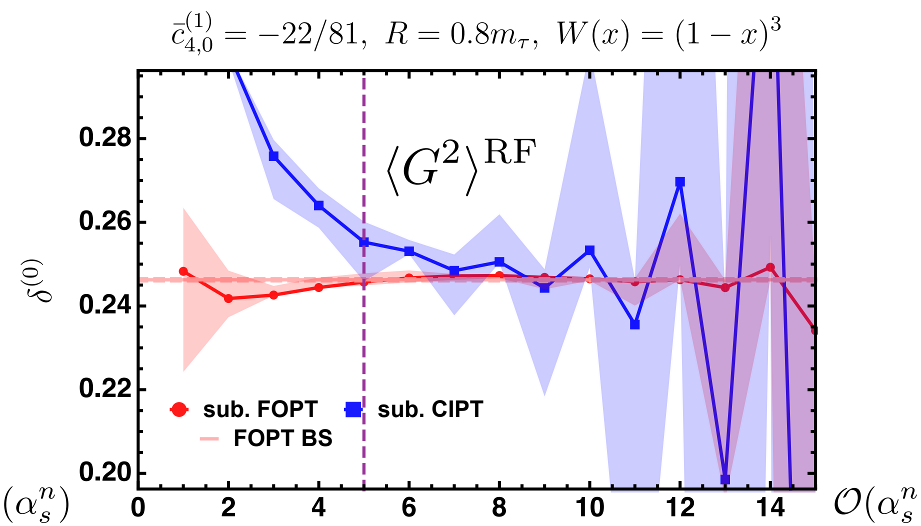
$$W(x) = (1-x)^3$$

→ GC enhanced

$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \le \xi \le 2\,, \quad N_g = 0.64$$

Benitez-Rathgeb, Boito, Jamin, AHH: 2202.10957





New RF GC Scheme !

- FOPT and CIPT expansions both get improved substantially
- Spectral function moments with high sensitivity to the GC can now be used for high-precision determinations of the strong coupling and the GC

GC Renormalon Normalization

Benitez-Rathgeb, Boito, Jamin, AHH: 2206.xxxxx

GC Norm in the Adler function's Borel function:

$$B[\hat{D}(s)]_{\rm GC}(u) = \frac{2\pi^2}{3}\,\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} \qquad \bar{a}(\mu^2) \equiv \frac{\beta_0\,\bar{\alpha}_s(\mu^2)}{4\pi} \qquad \hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

Multi-renormalon model approach

Beneke, Jamin 2008

$$B[\hat{D}(s)]_{\rm mr}(u) = b^{(0)} + b^{(1)}u + \frac{2\pi^2}{3}\,\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} + \frac{N_6}{(3-u)^{1+6\hat{b}_1}} + \frac{N_{-2}}{(1+u)^{2-2\hat{b}_1}}$$

$$\begin{aligned} c_{0,1} &= c_{1,1} = 1\,, \quad c_{2,1} = 1.640 \\ c_{3.1} &= 6.371 \\ c_{4,1} &= 49.076 \\ c_{5,1} &= 280 \pm 140 \end{aligned}$$

→ $N_g = 0.64 \pm 0.27$

Conformal mapping approach

Lee 2012

$$\tilde{B}(u) \equiv \frac{3\,(2-u)^{1+4\hat{b}_1}}{2\pi^2}\,B[\hat{D}(s)](u)$$

GC renormalon-free

$$w(u,p) = \frac{\sqrt{1+u}-\sqrt{1-\frac{u}{p}}}{\sqrt{1+u}+\sqrt{1-\frac{u}{p}}}$$

u=2 closest to the origin in the w plane $N_g = \tilde{B}(w(2,p))$

Use $c_{1,1}$ to $c_{5,1}$ and w-expansion → $N_g = 0.71 \pm 0.26$

GC Renormalon Normalization

Benitez-Rathgeb, Boito, Jamin, AHH: 2206.xxxxx

GC Norm in the Adler function's Borel function:

$$B[\hat{D}(s)]_{\rm GC}(u) = \frac{2\pi^2}{3}\,\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} \qquad \bar{a}(\mu^2) \equiv \frac{\beta_0\,\bar{\alpha}_s(\mu^2)}{4\pi} \qquad \hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

Optimal subtraction approach

New !

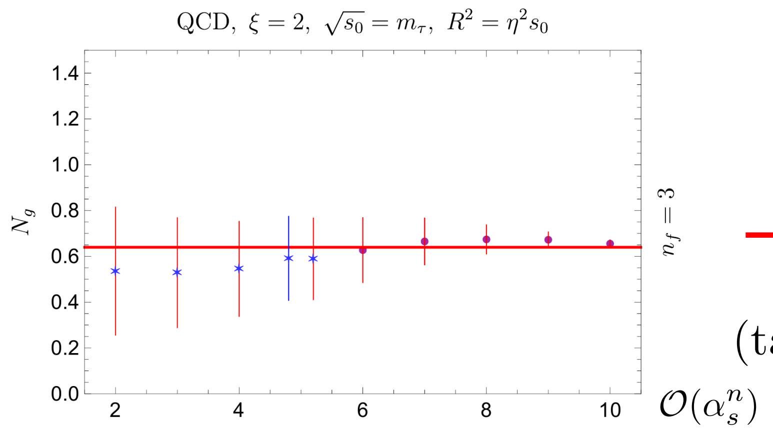
Use quantiative measure for improvements for GC suppresed and GC enhanced moments in the RF GC scheme

$$\chi^2_m(N_g) = \chi^2_{m,\rm GCS}(N_g) + \chi^2_{m,\rm GCE}(N_g)$$

Good convergence of 5 GC enhanced moments

Small discrepancy for 5 GC supressed moments

Can precisely determine N_g for the Beneke-Jamin model



$$\longrightarrow \quad N_g = 0.57 \pm 0.23$$

(take $\mathcal{O}(\alpha_s^4)$ result)

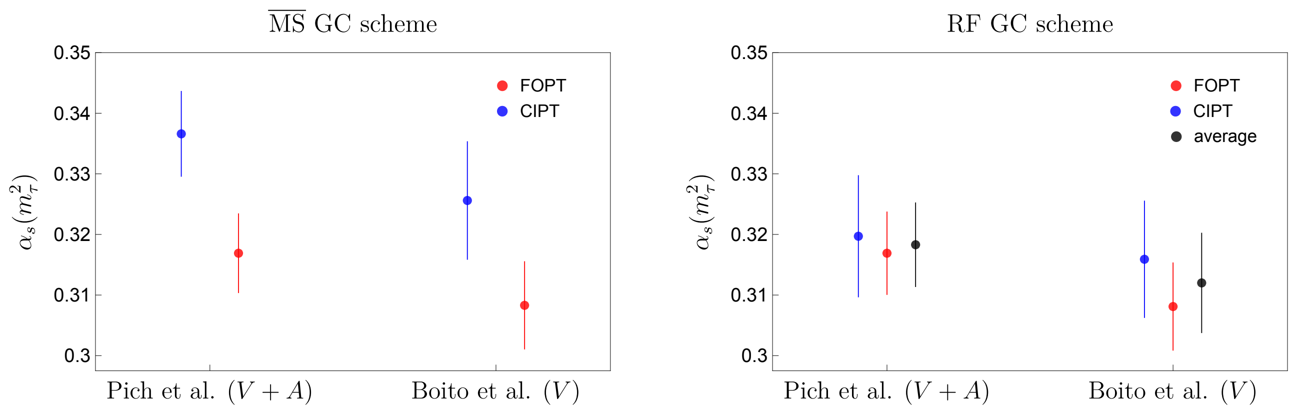
Strong Coupling Determinations

We repeat (in detail!) two state-of-the-art determination methods in the RF GC scheme:

Truncated OPE approach: Pich, Rodriguez-Sanchez 2016

Duality Violation model approach: Boito, Golterman, Maltman, Peris, Rodriguesz, Scharf 2021

→ Include uncertainties: $N_g = 0.57 \pm 0.23$ $\quad 0.7 m_\tau \leq R \leq m_\tau$



- FOPT-CIPT for GC suppressed moments remedied
- Taking average of FOPT and CIPT results now meaningful
- Spectral function moments with high sensitivity to the GC can now be used for high-precision determinations of the strong coupling and the GC
- Uncertainties due GC renormalon norm N_g and R variations very small !

Summary and Conclusions

- CIPT Borel representation different from FOPT Borel representation in the presence of IR renormalons. ⟶ **Asymptotic Separation**

  ⟶ **CIPT expansion NOT consistent with standard OPE approach**

- **Mathematica reason for CIPT inconsistency identified**

  → General mathematical criterion applicable for other non-standard expansions

  → Zeros in CIPT functions: reason for apparent good behavior AND OPE inconsistency

- Problems of CIPT resolved largely in renormalon-free RF gluon condensate (GC) scheme. ⟶ $\mathrm{CIPT}^{\mathrm{RF}}$ “cured” and still useful

- We have devised such a GC scheme in the most minimalistic and transparent way.

  (Additional uncertainty from N_g (GC renormalon norm), and factorization scale R.)

- RF GC scheme: Disparity between CIPT and FOPT strongly suppressed

- RF GC scheme: Moments with high sensitivity to the GC can be used for high precision analyses

- Excellent prospects for new high-precision determinations of the strong coupling

CIPT Borel Sum Contour Integration

The contour integration for the CIPT Borel representation must be deformed away from |x| = 1.

(Leaves FOPT Borel sum unchanged!)

Do the Borel-u-integral first:

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0)$$

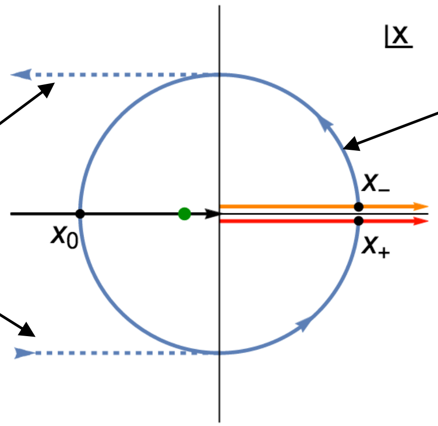
$$= \frac{1}{2\Gamma(\gamma)} \oint\limits_{\mathcal{C}_x} \frac{\mathrm{d}x}{x} (-x)^m \,\mathrm{sig}[\mathrm{Im}[a(-x)]]\, (a(-x))^{1-\gamma}\, e^{-\frac{p}{a(-x)}} \,.$$

„Asymptotic Separation“

Cut along the negative real s-axis!

Power-suppressed $\sim \left(\frac{\Lambda^2_{\mathrm{QCD}}}{s}\right)^p$

Remaining contour integration must be deformed (to negative real infinity in the x-plane)



Computation of the CIPT Borel Sum

An analytic continuation is mandatory to compute the CIPT Borel sum for m>p

$$W(x) \sim x^m$$

$$B(u) \sim \frac{1}{(p-u)^\gamma}$$

$$\Delta(m,p,\gamma,s_0) \equiv \delta^{(0),\mathrm{CIPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0) - \delta^{(0),\mathrm{FOPT}}_{\{(-x)^m,p,\gamma\},\mathrm{Borel}}(s_0)$$

$$= \frac{1}{2\Gamma(\gamma)} \oint_{\mathcal{C}_x} \frac{\mathrm{d}x}{x}\,(-x)^m\,\mathrm{sig}[\mathrm{Im}[a(-x)]]\,(a(-x))^{1-\gamma}\,e^{-\frac{p}{a(-x)}}\,.$$

$$\sim x^{-p}$$

Properties of the asymptotic separation:

- Renormalization scheme invariant
- **Much larger than canonical FOPT Borel sum ambiguity estimate if the Borel function has a sizeable gluon condensate cut**
- Fully analytic results
- **Properties of CIPT Borel representation imply that OPE corrections for CIPT do not have the common standard form C x <condensate> / s^p**