Lectures on PBH

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1 Lecture notes and plan

- Lectures by Byrnes and Cole arxiv.org/abs/2112.05716, from where I got most of material for first part.
- Reviews:
  - Sasaki et al arxiv.org/abs/1801.05235
  - Green-Kavanagh arxiv.org/abs/2007.10722
  - Escrivà et al arxiv.org/abs/2211.05767
  - Özsoy et al arxiv.org/abs/2301.03600
  - Scalar-induced second order GW from PBH formation: Domenech's reviews arxiv.org/abs/2307.06964, arxiv.org/abs/2109.01398. I got from these works most of material for last part.
- We set $c = \hbar = 1$
- We shall keep the discussion mostly qualitative, focussing on physical ideas. You will learn technicalities going through the literature and try the computations yourself.
- We start with motivations, and a description of relevant quantities needed to characterize PBH and their cosmic abundance. Then we pass to discuss model building from inflation. Then briefly discuss existing constraints on PBH, and describe possible avenue for future (?) detection of PBH through their produced SGWB.

2 Motivations: Dark Matter, and GW events

- Late Universe: galaxy rotation curves, distribution and behaviour of galaxy clusters, lensing
- Early Universe.
  Cosmic Microwave Background, $\simeq 4 \times 10^4$ yrs after big bang (amplitude and position of peaks).
  Measurement of baryon-to-photon ratio at BBN epochs, $\simeq 1$ min after big bang.
- Open question: we do not know what DM is. Many hypothesis wrt particles with direct gauge-type interactions with SM. So far, no hints of such interactions through DM direct searches.

  But what if DM interacts *only gravitationally* with SM ? A possibility is that DM might be made (partially or totally) of black holes produced in the early universe.

  - Primordial BHs are a very economical option: no new particles, only GR+SM, only (as we will see) specific initial conditions from inflation: Put together QM with physics of curved space-time.
- Another set of motivations come from GW detection from LVK collaboration. So far, detections are compatible with astrophysically produced BHs.
  But what about the possibility to detect GW from merging BHs produced by some mechanism in the early universe? E.g. BH with masses smaller than Chandrasekhar mass ($\simeq 1.4$ solar masses) that can not be astrophysically produced? Need to be theoretically ready to these possibilities, to be able to distinguish astro BHs from something else.

3 PBH formation

- How do PBH form? Collapse of primordial overdensities $\delta\rho/\bar{\rho}$ in the early universe.

  Suppose that in early universe, during RD, distribution of energy density is homogeneous with background value $\bar{\rho}(t)$ independent from time, plus inhomogeneities $\delta\rho/\bar{\rho}$ of different sizes. Homogeneous and isotropic space-time described by FLRW metric

  $$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 \tag{3.1}$$

  Size of observable universe controlled by horizon scale $1/H(t)$. For us, convenient to work with comoving horizon $1/(a(t)\,H(t))$ since we can compare its size with comoving scales. During RD, the comoving horizon size increases with time. Suppose at a certain point in becomes as large as the typical comoving wavelength

  $$\lambda \sim 1/k$$

  of a primordial inhomogeneity.

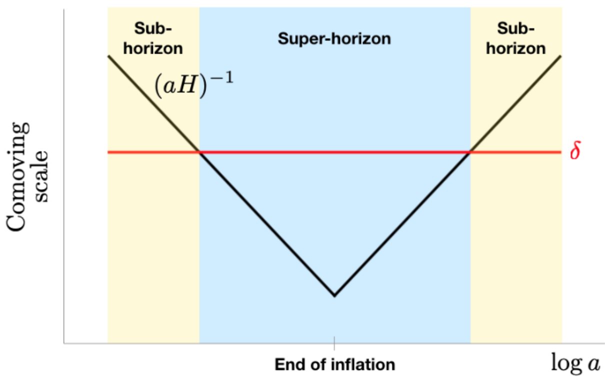

Figure 1: Behaviour of comoving horizon during cosmic history. From Byrnes-Cole

  It is convenient to think in terms of wavenumbers k corresponding to Fourier modes/transforms of fluctuations $\delta\rho$ in physical space **(recall David W lectures)**.

- At this stage, the primordial inhomogeneity enters in causal contact with observed universe: if its size is large enough to contrast RD pressure, it starts to collapse, and form a PBH.

  Roughly, the PBH mass is comparable to the total mass of the energy density within the universe horizon at that time:

$$M_{\rm PBH} \,=\, M_{\rm Hor} \,=\, \rho\,\mathcal{V} \tag{3.2}$$

- What's the threshold δ_c for formation? First estimated by Carr around 50 years ago, using Jeans-type instability arguments for fluids in expanding universe. Result is simple: it depends on the speed square c_s^2 of density fluctuations in RD, corresponding to velocity of pressure wave travels between different regions through the RD medium:

$$\delta_c \,=\, c_s^2 \tag{3.3}$$

In RD, $c_s = 1/\sqrt{3}$ hence $\delta_c \,=\, 1/3$. More refined estimates using numerical simulations give

$$\delta_c \,\simeq\, 0.45 \qquad \text{more refined value} \tag{3.4}$$

Importantly, notice that $\delta_c \sim \mathcal{O}(0.1)$ hence a very large value! We need an early universe mechanism able to:

i) Produce inhomogeneities with wavelengths larger than comoving horizon, that will re-enter the horizon at a specific scale during RD (inflation can do it)

ii) The size of these fluctuations must be large at the specific scale we're interested to (inflation can do it, with some efforts)

- Before considering early-universe model building, we need to introduce other physical quantities needed to characterize the PBH population, besides the threshold δ_c for collapse.

Question: How many PBH we need at formation, to give sizeable amount of DM today?

- First, fraction of PBH vs DM **today**

$$f_{\rm PBH} \,=\, \frac{\rho_{\rm PBH}}{\rho_{\rm DM}}\Big|_0 \tag{3.5}$$

hence if $f_{\rm PBH} = 1$ then all DM is PBH. Recall that $\rho_{\rm RD} \sim 1/a^4$, while $\rho_{\rm MD} \sim 1/a^3$, hence PBH relative fraction against total energy density $c\rho_{\rm PBH}/\rho_{\rm tot}$ increases from their formation during RD up to matter-radiation equality. Then such fraction freezes from $a_{\rm eq}$ onwards (ignoring DE).

- Let's denote with β the fraction of PBH versus total energy density at time of formation. We can write

$$f_{\text{PBH}} = \frac{\rho_{\text{PBH}}}{\rho_{\text{DM}}}\Big|_0 = \frac{\rho_{\text{PBH}}}{\rho_{\text{tot}}}\Big|_{\text{eq}} = \frac{a_{\text{eq}}}{a_{\text{form}}}\,\beta \tag{3.6}$$

  Hence if $a_{\text{eq}}/a_{\text{form}}$ is very large, we only need a **very small** β: very few PBH at time of formation can lead to totality of DM today.

- Let's put some numbers, to recollect formulas so far. We assume to work within RD where $a \propto t^{1/2}$, $\rho \propto a^{-4}$, $H \propto \rho^{1/2}$. At formation

$$M_{\text{PBH}} = M_{\text{Hor}} = \rho\,\mathcal{V} = \frac{4\pi\,\rho}{3}\,H^{-3} \propto \rho^{-1/2} \propto a^2 \propto t \tag{3.7}$$

  Hence during RD mass of PBH linearly depends on time t when it forms. Putting numbers

$$M_{\text{PBH}} = \left(\frac{a_{\text{form}}}{a_{\text{eq}}}\right)^2 M_{\text{eq}} = \left(\frac{a_{\text{form}}}{a_{\text{eq}}}\right)^2 10^{16}\,M_{\odot} \tag{3.8}$$

  hence PBH formed at equality are very massive (we do not consider them). Recall that $M_{\odot} = 2\times 10^{33}$ g.

  Moreover

$$M_{\text{PBH}} \simeq 10^{15}\text{g}\,\left(\frac{t}{10^{-23}\,\text{s}}\right) \tag{3.9}$$

  where the time pivot value is chosen to identify minimal mass to avoid Hawking evaporation (mass of a small mountain). Smaller mass BHs, produced at earlier times, are evaporated by today. For example, if we wish to produce a solar-mass PBH, we get $t \simeq 10^{-5}$ s, and $a_{\text{form}}/a_{\text{eq}} = 10^{-8}$. Hence we only need

$$\beta = 10^{-8}$$

  to produce a population of solar-mass PBH that constitutes DM. Full DM as PBH with these masses is excluded. But they can be a fraction of DM, and contribute to LVK events.

  Try yourself the computation for $M_{\text{PBH}} \sim 10^{17}$ g: asteroid-size PBH.

- We can also estimate the characteristic wavenumber k (wavelength $1/k$) corresponding to PBH of a given mass. Recall $M_{\text{PBH}} \simeq M_{\text{Hor}}$. During RD, $M_{\text{Hor}} \propto a^2$. Then, the scale of horizon re-entry in RD

$$k = aH \propto t^{1/2}\,t^{-1} \propto a^{-1} \qquad \Rightarrow \qquad M_{\text{Hor}} \sim k^{-2} \tag{3.10}$$

Putting numbers,

$$M_{\rm PBH} \; = \; 10^{13}\, M_\odot \left(\frac{\rm Mpc^{-1}}{k}\right)^2 \tag{3.11}$$

for $M_{\rm PBH} \simeq M_\odot$, we get $k \simeq 10^7$ Mpc^{-1}. A pretty small scale wrt CMB $k_{\rm CMB} \simeq 10^{-2}$ Mpc^{-1}.

Question: What does physically determine β?

- Idea: PBH form ONLY WHEN overdensities overcome a certain threshold δ_c. Call $P(\delta)$ the probability distribution function for finding certain overdensity: we assume it's Gaussian with variance σ:

$$P(\delta) \; = \; \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{\delta^2}{2\sigma^2}} \tag{3.12}$$

The variance σ will be determined by the primordial mechanism generating inhomogeneities (inflation). The parameter β, controlling the amount of PBH at the time of their formation, is expressed as

$$\beta(M_{\rm PBH}) \; = \; \frac{\rho(M_{\rm PBH})}{\rho_{\rm tot}}\,\Big|_{\rm form} \; = \; \int_{\delta_c}^{+\infty} P(\delta)\, d\delta \tag{3.13}$$

using the so-called Press-Schechter formalism. The larger the variance, the larger β is since the area under the curve is larger.

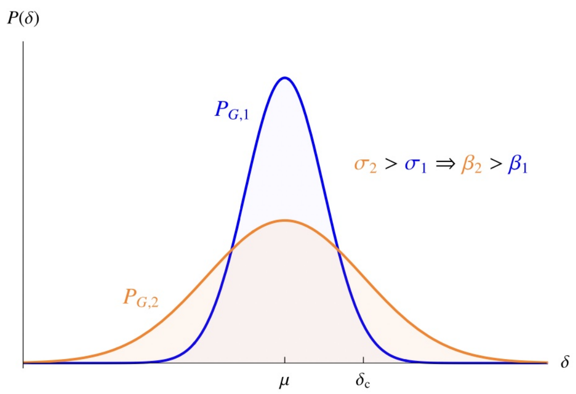

Figure 2: Examples of Gaussian distributions with different variances

- Then using definition of error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du \tag{3.14}$$

$$\text{erfc}(x) = 1 - \text{erf}(x) \tag{3.15}$$

by simple substitutions one gets

$$\beta = \frac{1}{2}\text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma}\right) \simeq \frac{\sigma}{\sqrt{2\pi}\,\delta_c}\, e^{-\delta_c^2/(2\sigma^2)} \tag{3.16}$$

where in second equality we considered the limit δ_c/σ large, justified since $\beta \ll 1$. Infact, for large x one gets $\text{erfc}(x) \equiv e^{-x^2}/(\sqrt{\pi}x)$ plus corrections.

- Then we can invert to determine σ

$$\sigma^2 \simeq \frac{\delta_c^2}{\ln(1/\beta)} \simeq \frac{0.2}{\ln(1/\beta)} \tag{3.17}$$

Hence β is exponentially sensitive to σ (theory of early universe) while σ only log-sensitive to β. For example, for solar-mass PBH, we have $\beta \sim 10^{-8}$, we get

$$\sigma^2 \sim 10^{-2} \tag{3.18}$$

We shall see how to connect this number with primordial physics.

- **brief summary:** we made the hypothesis that early universe physics is able to to produce large density fluctuations $\delta\rho/\rho$ at a specific wavelength, re-entering the horizon during RD. They've Gaussian distribution, and their variance controls the amount of PBH at formation. Next task: find the mechanism that does all this.

4 Inflation and PBH

- Use **inflation** to produce **PBH**. Inflation is a short period of quasi-exponential expansion

$$a(t) \sim e^{H_I t} \qquad \text{with} \quad H_I \text{ nearly constant} \tag{4.1}$$

Start from of QM at microscopic scales, and exponential expansion drives quantum effects at astronomical scales (larger than the size of universe horizon). In fact, the comoving horizon decreases during inflation: $1/(aH)$, $H = H_I$, and a exponentially increase. While this quantity increases during RD and MD. E.g. during RD $a \sim t^{1/2}$, $H \sim t^{-1}$, $1/(aH) \sim t^{1/2} \sim a$.

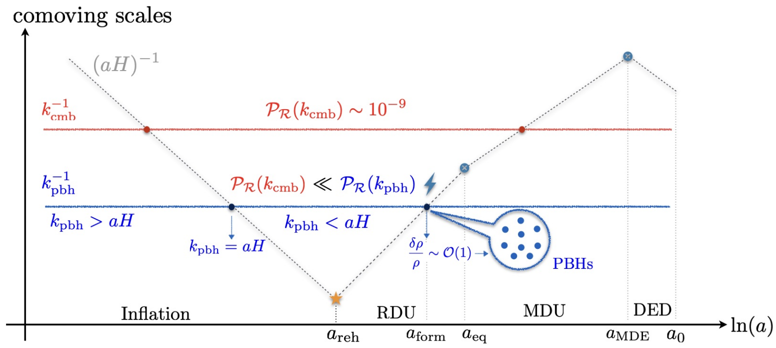

Figure 3: Pictorial representation of PBH production from inflation. From arxiv.org/abs/2301.03600

- Simplest way to get inflation: slow-roll inflation driven by single scalar field. Its EOM

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{4.2}$$

with dot derivative along time, prime derivative along field.

- Inflation requires slow-roll parameter ϵ to be small

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_{\rm Pl}^2 H^2} \ll 1 \tag{4.3}$$

Moreover, for lasting sufficiently long, also a second slow-roll parameter η should be small

$$\eta = \frac{\dot{\epsilon}}{\epsilon\, H} = 2\epsilon + \frac{2\,\ddot{\phi}}{H\,\dot{\phi}} \ll 1 \tag{4.4}$$

but this last condition can be avoided for a short period of time. Precisely this case is what we'll be interested to.

- Standard slow-roll requires ϵ and η small everytime during inflation. Since η small, $\ddot{\phi}$ small. We can then simplify EOM for scalar:

$$3H\dot{\phi} \simeq -V' \qquad \Rightarrow \qquad \dot{\phi} = -\frac{V'}{3H} \tag{4.5}$$

  Since $\dot{\phi}$ enters in ϵ, we want this small: the potential is flat, and the scalar is slowly rolling along the potential profile, with its motion is slowed down by friction. In this case, $|\dot{\phi}|$ is **nearly constant** during all inflation, since by hypothesis $\ddot{\phi}$ is small.

- But, for our PBH-production purposes, we wish to consider the possibility that $|\dot{\phi}|$ has a rapid decrease, and $|\eta|$ becomes large for a short amount of time. This will help to increase the size of density contrast/primordial fluctuations at certain scales, so to produce PBH. Among many, lets discuss two options of a phenomenon called **Ultra Slow Roll** inflation:

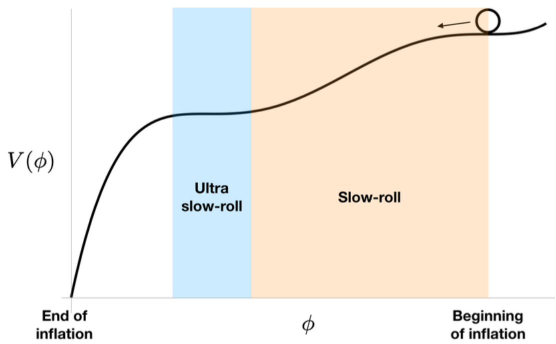

Figure 4: Representation of a potential with inflection point. From Byrnes-Cole.

1) **Inflection point inflation:** The potential has inflection point region where $V' = 0$. Then scalar EOM is

$$\ddot{\phi} + 3H\dot{\phi} = 0 \quad \Rightarrow \quad \frac{d\ln\dot{\phi}}{dt} = -\frac{d\ln a^3}{dt} \quad \Rightarrow \quad \dot{\phi} \simeq a^{-3} \tag{4.6}$$

The scalar speed is rapidly decreasing in size, reducing rapidly the value of ϵ:

$$\epsilon \simeq a^{-6}$$

The η parameter is

$$\eta = 2\epsilon + \frac{2}{\dot{\phi}\,H}\left(-3H\dot{\phi}\right) \simeq -6 \tag{4.7}$$

2) **Piece-wise linear potential:** The potential is

$$V = \begin{cases} V_0 + A_+(\phi - \phi_0) & \text{if } \phi > \phi_0 \\ V_0 + A_-(\phi - \phi_0) & \text{if } \phi < \phi_0 \end{cases} \tag{4.8}$$

Then eq for scalar velocity can be computed analytically, and one gets

$$3H_I\dot{\phi} = \begin{cases} -A_+ & \text{if } \phi > \phi_0 \\ -A_- - (A_+ - A_-)e^{-3H_I(t-t_0)} & \text{if } \phi < \phi_0 \end{cases} \tag{4.9}$$

put picture of velocity. Also in this case, there's a short period where $\dot{\phi} \sim a^{-3}$.

- The USR phase is a short phase during which $\dot{\phi}$, instead of being nearly constant, decays as $|\dot{\phi}| \sim a^{-3}$. It is very relevant for our purposes of amplifying the size of fluctuations at certain scales $k_\star$: this is the wavenumber corresponding to modes entering the horizon during RD, and will roughly correspond to the scale of modes re-entering the horizon during USR: $k_\star = a(t_*)H(t_*)$.
- We now have to work with cosmological perturbation theory: a very interesting subject, mathematically challenging but well developed, that allows us to put together theory with observations in exquisite details. Hence, although rather technical, its worth learning!
- The density contrast $\delta\rho/\rho$ can be expressed in terms of **curvature perturbation** $\mathcal{R}(t, \vec{x})$. This variable can be defined during different epochs (inflation, RD) in terms of fluctuations of energy density relevant at that stage. It's very convenient given its properties under gauge transformations, etc **see David W lectures**
- Lets start from inflation

$$\mathcal{R}(t, \vec{x}) = \frac{H\,\delta\phi(t, \vec{x})}{\dot{\phi}(t)} = \frac{\delta\phi(t, \vec{x})}{\sqrt{2\epsilon}\,M_{\rm Pl}} \tag{4.10}$$

during slow-roll.

- Observations are sensitive to correlators among curvature fluctuations. Let us convert the coordinate dependence from $(t, \vec{x})$ to conformal time

$d\tau = dt/a(t)$ and Fourier space

$$\mathcal{R}(\tau, \vec{x}) = \int d^3\vec{k}\, e^{-i\vec{k}\vec{x}}\, \mathcal{R}_{\vec{k}}(\tau) \tag{4.11}$$

where notice we use comoving momentum. The 2-point function for scalar fluctuations evaluated at horizon exit during inflation, introducing the notion of **power spectrum**

$$\langle \delta\phi_{\vec{k}}(\tau)\delta\phi_{\vec{q}}(\tau)\rangle = \delta(\vec{k}+\vec{q})\,|\delta\phi_{\vec{k}}(\tau)|^2 \quad \Rightarrow \quad \mathcal{P}_{\delta\phi} = \frac{k^3}{2\pi^2}\,|\delta\phi_{\vec{k}}(\tau)|^2$$

can be computed using techniques of QM in curved space-time

$$\mathcal{P}_{\delta\phi} = \left(\frac{H_I}{2\pi}\right)^2 \quad \Rightarrow \quad \mathcal{P}_{\mathcal{R}}(k) = \frac{1}{2M_{\rm Pl}^2\epsilon}\left(\frac{H_I}{2\pi}\right)^2 \tag{4.12}$$

where these quantities are evaluated at horizon exit during inflation, $k = aH$. **Roughly then, $\mathcal{P}_{\mathcal{R}}$ controls the size of curvature fluctuation** **put again picture comoving scales**

- Within slow-roll, the curvature power spectrum is nearly (but not exactly!) scale invariant:

  $$n_{\mathcal{R}} - 1 \equiv \frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k} = \mathcal{O}(\epsilon, \eta) \tag{4.13}$$

  in perfect agreement with observations at large CMB scales $k \sim 10^{-2} - 10^{-1}$ Mpc^{-1}. At CMB, the amplitude of the power spectrum is $\mathcal{P}_{\mathcal{R}} \sim 10^{-9}$.

- But at smaller scales, the USR period can considerably increase the size of the spectrum, and produce PBH when re-entering horizon at end of inflation. Recall that solar-mass PBH are associated with characteristic scales $k \sim 10^7$ Mpc^{-1}.

  A first hint of this phenomenon is the $1/\epsilon$ at denominator of $\mathcal{P}_{\mathcal{R}}$. But we need to be careful since this formula is derived assuming slow-roll throughout inflation.

- But we need some more equations and more pictures to follow what's going on.

- At linearized order, the evolution equation for curvature perturbation is

(we use conformal time and Fourier space)

$$\mathcal{R}_k''(\tau) + \frac{2\,z'(\tau)}{z(\tau)}\,\mathcal{R}_k'(\tau) + k^2\,\mathcal{R}_k(\tau) \;=\; 0 \tag{4.14}$$

this's relative to Mukhanov-Sasaki. The function z is called pump field:

$$z(\tau) \;=\; 2\,a^2(\tau)\,\epsilon(\tau)\,M_{\rm Pl}^2 \tag{4.15}$$

Recall that wavenumber is inverse of wavelength. We focus on large-wavelength, superhorizon fluctuations $k \ll aH$. The last term in MS eq can be neglected, and we find

$$\frac{d}{d\tau}\,\ln\,\mathcal{R}_k'(\tau) = -2\frac{d}{d\tau}\,\ln\,z(\tau) \qquad \text{superhorizon limit} \tag{4.16}$$

hence

$$\mathcal{R}_k(\tau) = C_k + D_k\,\int^\tau \frac{d\tilde\tau}{z^2(\tilde\tau)} \tag{4.17}$$

$$\mathcal{R}_k(t) = C_k + D_k\,\int^t \frac{d\tilde t}{\epsilon\,a^3(\tilde t)} \tag{4.18}$$

If slow-roll satisfied, decaying mode rapidly decays as $1/a^3$: it rapidly disappear, and only the constant mode C_k survives at superhorizon scales. As modes cross the horizon, they freeze and become time-independent.

- **recall again figure, explaining thats sufficient to compute amplitude and scale-dep of spectrum at first horizon crossing**
- **However**, the situation changes considerably if the system undergoes a short period of USR, during which $\dot\phi \simeq a^{-3}$ and $\epsilon \simeq a^{-6}$. Then,

$$\mathcal{R}_k(t) = C_k + D_k\,\int^t \frac{d\tilde t}{\epsilon\,a^3(\tilde t)} \tag{4.19}$$

$$\simeq C_k + D_k\,a^3(t) \tag{4.20}$$

hence the decaying mode **rapidly increases** and allows us to **amplify the spectrum of curvature fluctuations** at a given scale.

- Solving corresponding equations for every k is a bit difficult, and some little numerics is generally needed.

  But lets understand what happens in pictures, representing the power spectrum at the end of inflation as a function of scale k.

make parallel pictures comoving sizes vs $\log a$ and $\mathcal{P}_\mathcal{R}$ vs $\ln k$

i) Very large scales leave the horizon very early during inflation, very far from USR phase. The decaying mode has long time to decay, hence it remains always negligible. Spectrum shape does not change. **figure**

ii) We have modes that leave the horizon later: these modes start to feel the presence of USR that enhances the amplitude of the spectrum. The process of steep growth of spectrum continues until USR **figure**

iii) Then there're very small-scale modes that leave the horizon after USR; they do not feel the growth, and behave as slow-roll. Give red spectrum. **figure**

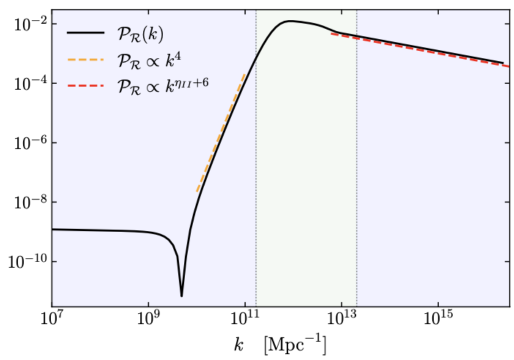

Figure 5: Plot of the resulting spectrum.

We can make more precise plot where we have maximal slope $(k/k_\star)^4$, and a dip due to disruptive interference; its located at position $[\mathcal{P}_\mathcal{R}(k_\star)]^{-1/4}$.

- After inflation ends, we can relate **density contrast in RD** $\delta\rho/\bar{\rho} = \delta$ with $\mathcal{R}$:

$$|\delta| = \frac{4}{9}\left(\frac{k}{aH}\right)^2 |\mathcal{R}| \tag{4.21}$$

Statistic of fluctuations of $\mathcal{R}$ is approximately Gaussian: its variance σ proportional to $\mathcal{P}_\mathcal{R}$. Recall that to produce solar-mass PBH we found for $k = 10^7$ Mpc^{-1}:

$$\sigma^2 \sim \frac{\delta_c}{\ln(1/\beta)} \sim 10^{-2} = \mathcal{P}_\mathcal{R} \tag{4.22}$$

hence seven orders of magnitude larger than CMB scales.

5 Constraints on PBH: present and future

Constraints on presence of PBH can be phrased in terms of f_{PBH} (or β) vs M_{PBH}. Interesting to identify mass ranges where $f_{\text{PBH}} = 1$ and PBH are totality of DM. But also other mass ranges can be phenomenologically interesting.

- **PBH evaporation** BH temperature is inversely proportional to its mass: $T_{\text{BH}} \propto 1/M_{\text{BH}}$. Too small PBH are very hot: even if not yet disappeared, their radiation might interfere with observations (CMB) etc. This sets constraints on small-mass PBH.

- **Microlensing** Observe a number of distant stars, and check whether their observed luminosity changes in time passing in front of an object.

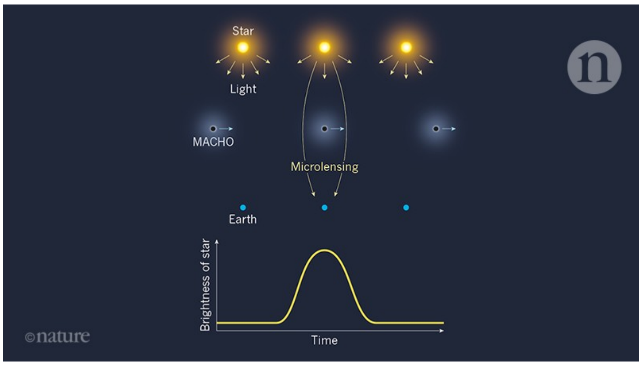

Figure 6: Microlensing phenomenon

E.g. Subaru telescope ruled out $f_{\text{PBH}} = 1$ for the mass ranges $10^{-12}\, M_\odot \leq M_{\text{PBH}} \leq 10^{-6} M_\odot$. Other experiments constrain other mass ranges.

- **LVK constraints** Current GW detections of astro sources (mostly if not all) do not favour $f_{\text{PBH}} = 1$ in the mass range $10^0\, M_\odot \leq M_{\text{PBH}} \leq 10^2 M_\odot$.

- Interestingly, there's an allowed mass range for asteroid size PBH, $10^{-16}\, M_\odot \leq M_{\text{PBH}} \leq 10^{-12} M_\odot$, or 10^{17} g $\leq M_{\text{PBH}} \leq 10^{22}$ g.

- **Gravitational waves at second order in perturbations**
This's a timely topic. Connections with Carlo's lectures. Possible observational signatures now or in the future.

- **Idea**: the starting point is the primordial stochastic background of GW produced during inflation.

  At linearized order, the spin-2 tensor modes from inflation h_{ij} (primordial GW) follow an evolution eq

$$h''_{ij}(\tau, k) + \frac{2a'(\tau)}{a(\tau)}\, h'_{ij}(\tau, k) + k^2\, h_{ij}(\tau, k) \;=\; 0 \tag{5.1}$$

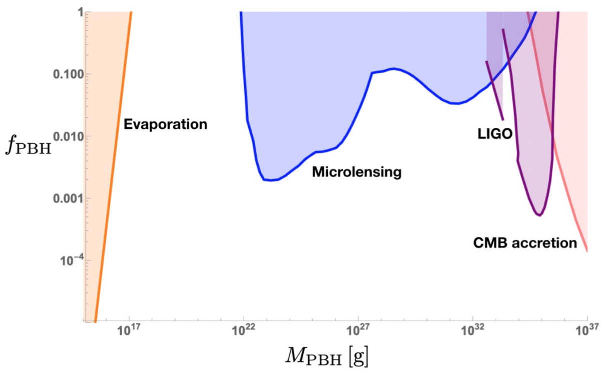

Figure 7: From arxiv.org/abs/2112.05716

However, since scalar perturbations acquire large amplitudes for a short-range of scales in PBH-forming models, it makes sense to push perturbation theory at second order. Then scalars act as **quadratic source** for tensors:

$$h''_{ij}(\tau,k)+\frac{2a'(\tau)}{a(\tau)}\,h'_{ij}(\tau,k)+k^2\,h_{ij}(\tau,k)\;=\;S_{ij}(\tau,k) \tag{5.2}$$

- This fact enhances the tensor spectrum, and the energy density in GW. The primordial stochastic background produced during inflation gets amplified for a small range of scales. Computations using perturbation theory give, for PBH produced during RD,

$$\Omega_{\rm GW}(k)\;=\;\int_0^\infty dv\int_{|1-v|}^{|1+v|}du\,\mathcal{T}_{\rm RD}(u,v)\,\mathcal{P}_{\mathcal{R}}(uk)\,\mathcal{P}_{\mathcal{R}}(vk) \tag{5.3}$$

These convolution integrals are typical when considering effects of second-order fluctuations.

- For example, for

$$\mathcal{P}_{\mathcal{R}}\;=\;A_s\,\delta\left(\ln(k/k_\star)\right) \tag{5.4}$$

one gets ($\tilde{k}=k/k_\star$)

$$\Omega_{\rm GW}\;=\;\frac{3A_{\mathcal{R}}^2}{64}\left(\frac{4-\tilde{k}^2}{4}\right)^2\,\tilde{k}^2\,\left(3\tilde{k}-2\right)^2$$
$$\times\left(\pi^2\left(3\tilde{k}^2-2\right)^2\Theta(2\sqrt{3}-3\tilde{k})+\left(4+\left(3\tilde{k}^2-2\right)\ln\left|1-\frac{4}{3\tilde{k}^2}\right|\right)^2\right)\Theta(2-\tilde{k}) \tag{5.5}$$

The resulting profile for $\Omega_{\rm GW}$ is in Fig 8, left panel.

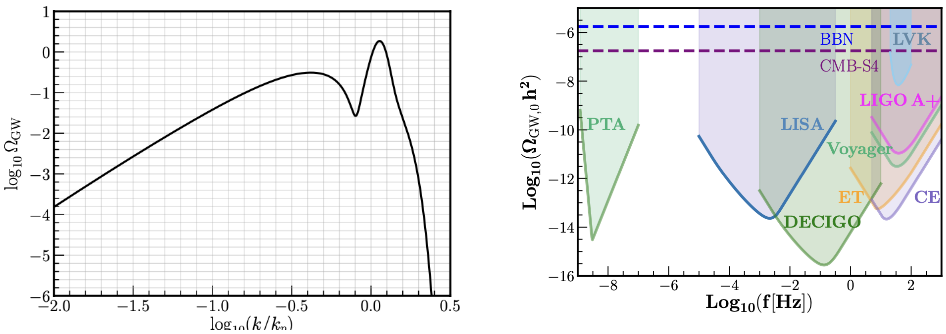

Figure 8: Left: Plot of the profile of the GW spectrum from a monochromatic delta-like scalar source. Right: sensitivity curves for different experiments. From Domenech.

- Hence the production of PBH leads to enhancement of SGWB from inflation at characteristic scales, related with PBH properties. In fact, converting to frequencies $f = 2\pi k$, and expressing in Hz, one gets

$$f_{\rm peak}^{\rm GW} = 1.2 \times 10^8 \, {\rm Hz} \left(\frac{M_{\rm PBH}}{1\,{\rm g}} \right)^{-1/2} \tag{5.6}$$

- Examples
  - Solar mass PBH $M_{\rm PBH} = 10^{34}$ g ($\simeq 5 M_\odot$)
    $\Rightarrow f_{\rm peak}^{\rm GW} = 10^8 \times 10^{-17}$ Hz , PTA frequency band.
  - Asteroid mass PBH $M_{\rm PBH} = 10^{22}$ g
    $\Rightarrow f_{\rm peak}^{\rm GW} = 10^8 \times 10^{-11}$ Hz , LISA frequency band.