Gravitational waves from phase transitions

2. Dynamics of first order phase transitions

Mark Hindmarsh[1,2]

[1] Department of Physics & Astronomy
University of Sussex

[2] Department of Physics and Helsinki Institute of Physics
Helsinki University

EuCAPT Cosmology School
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Outline

Section 1

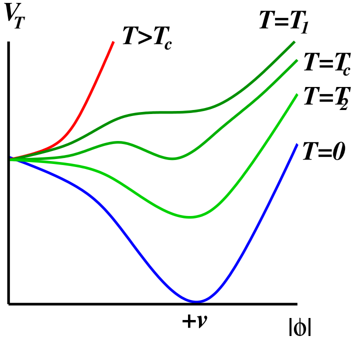
Recap

First order phase transition

Effective potential $V_T(\bar{\phi}) = V_0 + \Delta V_T$

$$\Delta V_T \simeq \frac{D}{2}(T^2 - T_2^2)|\bar{\phi}|^2 - \frac{A}{3}T|\bar{\phi}|^3 + \frac{\lambda}{4!}|\bar{\phi}|^4$$

- Second minimum develops at T_1
- Critical temperature T_c:
  free energies are equal.
- System can supercool below T_c.
- Symmetric ($\bar{\phi} = 0$) unstable for $T < T_2$ (spinodal temperature)
- First order transition
  discontinuity in equilibrium free energy
  discontinuity in equilibrium field value



Scalar field coupled to relativistic fluid

Model of coupled order-parameter ϕ and fluid $T_{\rm f}^{\mu\nu}$

$$\Box\phi - V_T'(\phi) = \frac{dm^2}{d\bar\phi}\int \frac{đ^3p}{2E}\Delta f(p,x) \qquad \simeq \tilde\eta\frac{\phi^2}{T}(U\cdot\partial\phi)$$

$$\partial_\mu T_{\rm f}^{\mu\nu} + \partial^\nu\phi\frac{\partial V_T(\phi)}{\partial\phi} = -\partial^\nu\phi\frac{dm^2}{d\bar\phi}\int\frac{đ^3p}{2E}\Delta f(p,x) \simeq \tilde\eta\frac{\phi^2}{T}(U\cdot\partial\phi)\partial^\nu\phi$$

Where $p = g_{\rm eff}\pi^2T^4/90 - V_T(\phi)$, $\Delta f(p,x) = f(p,x) - f^{\rm eq}(p,x)$

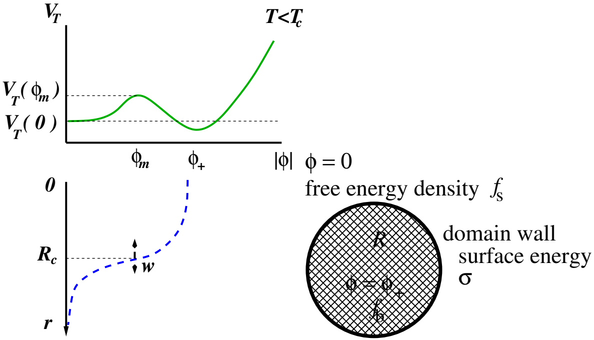
Section 2

Dynamics of first-order phase transitions: outline

Dynamics of first order phase transitions

$$V_T(0) \simeq V_0(0)+\frac{D}{2}(T^2-T_2^2)|\bar{\phi}|^2-\frac{1}{3}AT|\bar{\phi}|^3+\frac{1}{4}\lambda|\bar{\phi}|^4$$

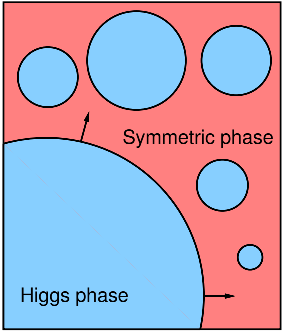
- Below T_c, state $\phi = 0$ metastable
- Separated from equilibrium state by $B = V_T(\phi_m) - V_T(0)$
- Lowest energy path to equilibrium state via critical bubble
- Energy of critical bubble E_c
- Nucleation rate per unit volume (high T): $\Gamma/V \simeq T^4 \exp(-E_c/T)$



Field profile

Transition via growth and merger of bubbles

- Thermal fluctuations produce bubbles at rate/volume $\Gamma/V \simeq T^4 \exp(-E_c/T)$
- Bubbles growth speed v_w set by interaction with medium
- Bubble merger completes phase transition



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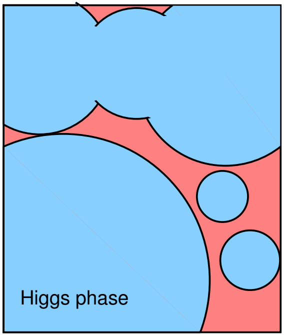

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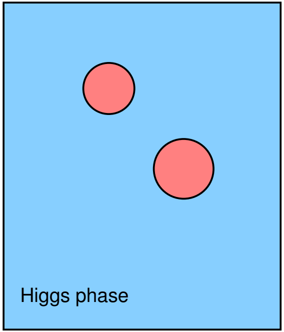
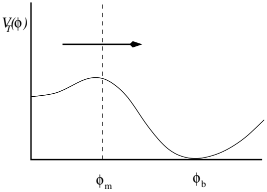


Section 3

Bubble nucleation in detail

Thermal activation: Kramers escape problem



Simplify: particles in a potential V_T.

- Position ϕ
- momentum π
- Hamiltonian $H = \frac{1}{2}\pi^2 + V_T(\phi)$

What is flux across barrier Γ?

$$\Gamma = \frac{1}{Z}\int d\pi d\phi e^{-H/T}\delta(\phi - \phi_{\rm m})\pi\theta(\pi) = \frac{T}{Z}e^{-V_T(\phi_{\rm m})/T}$$

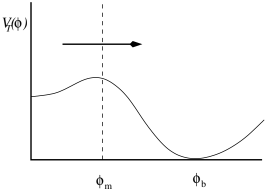
Evaluate Z by steepest descent; assume no particles near $\phi_{\rm b}$

$$Z = \int d\pi d\phi e^{-H/T} \qquad = \frac{2\pi T}{\sqrt{V_T''(0)}}e^{-V_T(0)/T}$$

$$\boxed{\Gamma = \frac{\omega}{2\pi}e^{-E_b/T}}$$

Barrier height: $E_b = V_T(\phi_{\rm m}) - V_T(0)$, Attempt rate: $\omega = \sqrt{V_T''(0)}$

Thermal activation: imaginary part of the free energy



- Position ϕ
- momentum π
- Hamiltonian $H = \frac{1}{2}\pi^2 + V_T(\phi)$

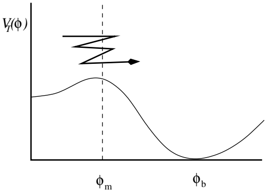
What is free energy $F = -T \ln Z$?

Evaluate Z by steepest descent, taking into account particles near $\phi_{\rm m}$

$$Z = \int d\pi d\phi e^{-H/T} = 2\pi T \left(\frac{1}{\sqrt{V_T''(0)}} e^{-V_T(0)/T} + \frac{1}{2} \frac{1}{\sqrt{V_T''(\phi_{\rm m})}} e^{-V_T(\phi_{\rm m})/T} \right)$$

Second term is imaginary: $\text{Im}\, F = \frac{T}{2} \frac{\sqrt{V_T''(0)}}{|V_T''(\phi_{\rm m})|} e^{-E_b/T}$ Thermal activation rate $\Gamma = \frac{\sqrt{|V_T''(\phi_{\rm m})|}}{\pi T} \text{Im}\, F$

Overdamped (diffusive) thermal activation



- Position ϕ
- momentum π
- Diffusion constant γ

Diffusion modelled by Langevin equation

$$\gamma\pi + V_T'(\phi) = \xi(t)$$

Random force

$$\langle \xi(t)\xi(t')\rangle = 2\gamma T\delta(t-t')$$

Diffusive thermal activation rate[1] $\Gamma = \frac{\omega_{\rm diff}}{2\pi} e^{-E_b/T}$, with $\boxed{\omega_{\rm diff} = \sqrt{V_T''(0)|V_T''(\phi_{\rm m})|}/\gamma}$

[1] arxiv:1906.08684

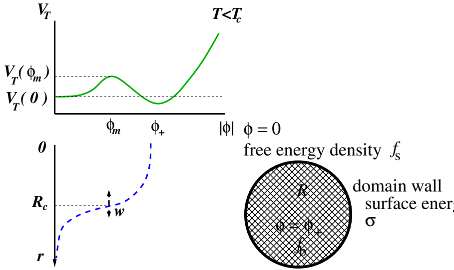
The critical bubble

Evaluate Z by steepest descent: $H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V_T(\phi)\right)$

$$Z = \int \mathcal{D}\pi\mathcal{D}\phi e^{-H[\pi,\phi]/T} = \mathcal{N}\mathcal{D}\phi e^{-E[\phi]/T}$$

$$E[\phi] = \int d^3x \left(\tfrac{1}{2}(\nabla\phi)^2 + V_T(\phi)\right)$$

- Critical bubble $\phi_c(\mathbf{x})$ solves $\dfrac{\delta E[\phi]}{\delta\phi(\mathbf{x})} = 0$
- Spherically symmetric, radius R_c
- Energy E_c
- Activation rate $\Gamma \sim e^{-E_c/T}$



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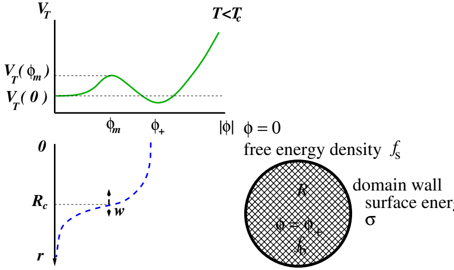
$$V_T(\phi) = V_0(0) + \frac{D}{2}(T^2 - T_2^2)\phi^2 - \frac{1}{3}AT\phi^3 + \frac{1}{4}\lambda\phi^4$$



- Phase boundary surface energy $\sigma \simeq \phi_+^3/\lambda$
- Free energy difference $B = V_T(\phi_+) - V_T(0) \propto (T_c - T)$

The critical bubble

Evaluate Z by steepest descent: $H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V_T(\phi)\right)$

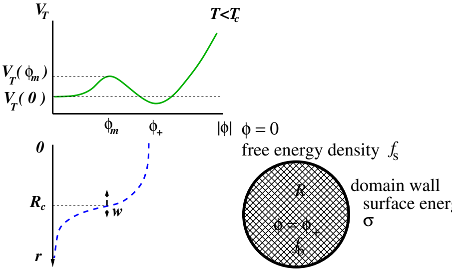
$$Z = \int \mathcal{D}\pi\mathcal{D}\phi e^{-H[\pi,\phi]/T} = \mathcal{N}\mathcal{D}\phi e^{-E[\phi]/T}$$

$$E[\phi] = \int d^3x \left(\frac{1}{2}(\nabla\phi)^2 + V_T(\phi)\right)$$

- Estimate (thin wall):

$$E_c \simeq -\frac{4\pi R^3}{3}B + 4\pi R^2\sigma$$

- Critical bubble at $dE/dR = 0$
- Critical bubble radius: $R_c = 2\sigma/B$
- Critical bubble energy: $E_c = 16\sigma^3/3B^2 \sim T_c^3/|T - T_c|^{-2}$



Nucleation rate formula

Bubble nucleation rate per unit volume[2]

$$\Gamma \sim \frac{\omega}{2\pi}\left(\frac{E_c}{2\pi T}\right)^{\frac{3}{2}} \left(V_T''(0)\right)^{3/2} e^{-E_c/T}$$

Notes

- E_c is calculated relative to the energy density of the metastable state
- Bubbles with $R > R_c$ are unstable to growth, $R < R_c$ are unstable to shrinking
- In thin wall approximation, $(Tc - T) \ll T_c$, and hence $S = E_c/T \gg 1$
- $V_T''(0) = M_H^2$, scalar (order parameter) mass squared
- Attempt rate

$$\omega = \begin{cases} |V_T''(0)|^{1/2} & \text{underdamped field} \\ |V_T''(0)|/\gamma & \text{overdamped field, diffusion const.}\gamma \end{cases}$$

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[2] Langer 1969, Linde 1976

Transition rate parameter

- Write nucleation rate density as
$$\Gamma(T) = \Gamma_0(T)e^{-S(T)}$$
- Nucleation rate density is very sensitive to the temperature:
$$S(T) = \frac{E_c}{T} \propto \frac{T_c}{T}\frac{T_c^2}{|T_c - T|^2} \quad \text{(thin wall approx)}$$
- Choose a reference time t_r:
$$\Gamma(t) = \Gamma_r e^{-\dot{S}(t_r)(t-t_r)}$$
- Write $\beta = -\dot{S}(t_r)$, the **transition rate parameter**[3] (positive)
  - $\beta \simeq -H\dfrac{dS}{d\ln(T)}$
  - $\beta \gtrsim H$, otherwise universe stays in metastable state, and inflates forever

[3] This β is not inverse temperature!

Size of transition rate parameter β

What critical bubble energy is needed to get activation rate/volume $\Gamma \sim H^4$ at $T \sim 100$ GeV?

- $\Gamma \sim M_H(T)^4 e^{-E_c/T} \sim H^4$
- Use Friedmann equation $H^2 \sim T^4/M_{\rm P}^2$
- Result for $T \sim 10^2$ GeV: $S \equiv E_c/T \simeq \ln(M_{\rm P}^4/T^4) \simeq 150$

What is the magnitude of the transition rate parameter β?

- Transition rate parameter $\beta/H \simeq -\frac{dS}{d\ln(T)}$
- Estimate from $E_c \simeq T_{\rm c}^3/|T_{\rm c} - T|^2$: $\beta/H \sim S^{3/2} \sim 10^3$
- Recall expected GW frequency today:

$$f_0 \sim (\beta/H)10^{-5} \text{ Hz}$$

- First order EW transition should emit GWs at O(10) mHz

Fraction of universe in metastable phase $h(t)$

- Once nucleated, bubbles grow with constant speed v_w (see later)
- Number density of bubbles nucleated in interval $(t', t' + dt')$ is $dn(t') = \Gamma(t')dt'$
- Total area of bubble walls at time t: $\mathcal{A} = \int_{t_c}^{t} 4\pi v_\mathrm{w}^2 (t - t')^2 \Gamma(t')dt'$
- Walls expand: in time dt eat up volume $v_\mathrm{w}\mathcal{A}dt$
- Change in fractional volume: $dh(t) = -h(t)v_\mathrm{w}\mathcal{A}dt = -v_\mathrm{w}dt \int_{t_c}^{t} 4\pi v_\mathrm{w}^2 (t - t')^2 \Gamma(t')dt'$
- Fractional volume in metastable phase, including overlaps

$$h(t) = \exp\left(-\int_{t_c}^{t} dt' 4\pi v_\mathrm{w}^3 (t - t')^3 \Gamma(t')dt'\right)$$

- Choose reference time t_f such that $h(t_f) = 1/e$
- Define $\beta = d\ln(\Gamma)/dt'|_{t_f}$, evaluate by steepest descent:

$$h(t) = \exp\left(-e^{\beta(t-t_f)}\right)$$

- Reference time satisfies $\frac{4\pi}{3} v_\mathrm{w}^3 \left(\frac{3!}{\beta^4}\right) \Gamma_0 e^{-S(t_f)} = 1$.

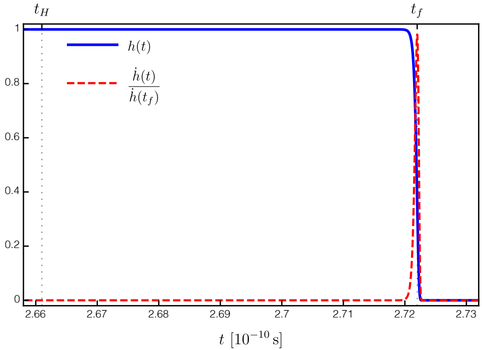
Evolution of fraction in metastable phase

$$h(t) = \exp\left(-e^{\beta(t-t_f)}\right)$$

- Define nucleation onset temperature, time T_H, t_f:
- One bubble nucleates in a a Hubble time per Hubble volume

$$\Gamma_H = \Gamma_f e^{\beta(t_H - t_f)} = H^4$$

- Example:
  - $T_H = 105$ GeV
  - Transition rate parameter $\beta/H = 8950$
  - Wall speed $v_w = 0.9$



Bubble density

- Bubbles can nucleate only in metastable phase
- In metastable phase rate/volume is $\Gamma_f = \Gamma_0 e^{-S(t_f)}$ at reference time.
- Nucleation rate averaged over both phases $\dot{n}_{\rm b}(t') = \Gamma(t')h(t')$ so

$$n_{\rm b}(t) = \int^t \Gamma(t')h(t')dt' = \Gamma_f \int^t e^{\beta(t'-t_f)} h(t')dt'$$

- Recall $\frac{4\pi}{3} v_{\rm w}^3 \left(3!/\beta^4\right) \Gamma_f = 1$, and $h(t) = \exp\left(-e^{\beta(t-t_f)}\right)$
- Hence

$$n_{\rm b}(t) = -\frac{\Gamma_f}{\beta} \int^t dt' \frac{dh(t')}{dt'} = \frac{\Gamma_f}{\beta}(1 - h(t))$$

- Final bubble density

$$n_{\rm b} = \frac{\Gamma_f}{\beta} = \frac{\beta^3}{8\pi v_{\rm w}^3}$$

- Mean bubble spacing

$$R_* = n_{\rm b}^{-\frac{1}{3}} = (8\pi)^{\frac{1}{3}} v_{\rm w}/\beta$$

Section 4

Hydrodynamics of bubble growth

Bubble wall speed

- Recall equation for scalar field

$$\Box\phi - V_T'(\phi) \simeq \tilde{\eta}\frac{\phi^2}{T}(U\cdot\partial\phi), \quad \partial_\mu T_{\rm f}^{\mu\nu} + \partial^\nu\phi\frac{\partial V_T(\phi)}{\partial\phi} = \tilde{\eta}\frac{\phi^2}{T}(U\cdot\partial\phi)$$

- Consider wall frame, where fluid moves with velocity $v^z \simeq -v_{\rm w}$
- Look for stable time-independent solution $\phi(z)$, $v(z)$, $w(z)$: constant wall speed
- E.g. scalar field equation

$$\partial_z^2\phi - V_T'(\phi) \simeq -\tilde{\eta}\frac{\phi^2}{T}\gamma_{\rm w}v_{\rm w}\partial_z\phi$$

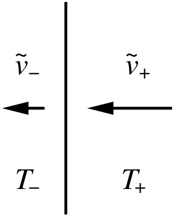
  - Good approximation: $\phi(z) = \frac{1}{2}\phi_+\left[\tanh(M_\phi z/2) + 1\right]$
- Multiply both sides by $\partial_z\phi$ and integrate $\int dz$:

$$\Delta V_T = \tilde{\eta}\gamma_{\rm w}v_{\rm w}\frac{1}{T}\int dz\phi^2(\partial_z\phi)^2$$

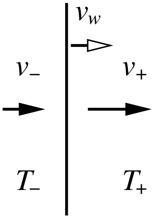
- Solve to get $v_{\rm w}$ (need to calculate $\tilde{\eta}$ from Boltzmann equation).

Fluid flow at bubble wall

- Approximate planar symmetry near wall. Wall motion in $+z$ direction



Wall frame

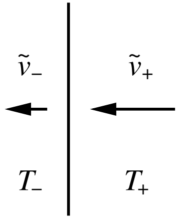


Universe frame

- Fluid motion in $-z$ direction
- Speeds $v_\pm > 0$

- Fluid motion in $+z$ direction
- $\tilde{v}_\pm = \dfrac{v_w - v_\pm}{1 - v_w v_\pm}, \quad v_\pm = \dfrac{v_w - \tilde{v}_\pm}{1 - v_w \tilde{v}_\pm}$

Energy-momentum conservation at bubble wall



Wall frame

- Fluid motion in $-z$ direction
- Speeds $v_\pm > 0$
- $T^{\mu\nu} = wU^\mu U^\nu + pg^{\mu\nu}$

- Energy-Momentum conservation:

$$\partial_t T^{tt} + \partial_z T^{zt} = 0$$
$$\partial_t T^{tz} + \partial_z T^{zz} = 0$$

- Assume steady state and integrate $\int dz$

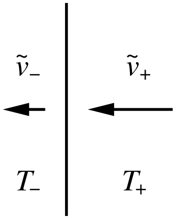
$$T^{zt}_- = T^{zt}_+$$
$$T^{zz}_- = T^{zz}_+$$

- Giving:

$$w_-\gamma_-^2 \tilde{v}_- = w_+\gamma_+^2 \tilde{v}_+$$
$$w_-\gamma_-^2 \tilde{v}_-^2 + p_- = w_+\gamma_+^2 \tilde{v}_+^2 + p_+$$

Bubble wall junction conditions

EM conservation: $w_-\gamma_-^2\,\tilde{v}_- = w_+\gamma_+^2\,\tilde{v}_+, \quad w_-\gamma_-^2\,\tilde{v}_-^2 + p_- = w_+\gamma_+^2\,\tilde{v}_+^2 + p_+$



Wall frame

- $T^{\mu\nu} = wU^\mu U^\nu + pg^{\mu\nu}$
- Enthalpy $w = e + p$

- Rearrange

$$\tilde{v}_+\tilde{v}_- = \frac{p_+ - p_-}{e_+ - e_-}, \quad \frac{\tilde{v}_+}{\tilde{v}_-} = \frac{e_- + p_+}{e_+ + p_-}$$

- Define[a] $\theta_\pm = \frac{1}{4}(e_\pm - 3p_\pm)$, $\Delta\theta = \theta_+ - \theta_-$
- Define $\alpha_+ = \frac{4\Delta\theta}{3w_+}$
- Define $r = w_+/w_-$

$$\tilde{v}_+\tilde{v}_- = \frac{1-(1-3\alpha_+)r}{3-3(1+\alpha_+)r},$$

$$\frac{\tilde{v}_+}{\tilde{v}_-} = \frac{3+(1-3\alpha_+)r}{1+3(1+\alpha_+)r}$$

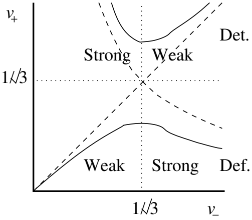
[a] Trace anomaly, or “vacuum” energy

Solution: strong and weak, deflagrations and detonations

$$\tilde{v}_+\tilde{v}_- = \frac{1-(1-3\alpha_+)r}{3-3(1+\alpha_+)r}, \qquad \frac{\tilde{v}_+}{\tilde{v}_-} = \frac{3+(1-3\alpha_+)r}{1+3(1+\alpha_+)r}$$

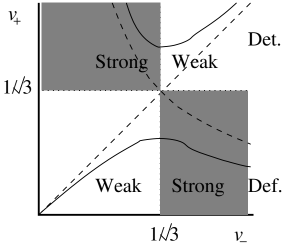
Solve for $\tilde{v}_+ = \tilde{v}_+(\tilde{v}_-, \alpha_+)$ [similar for $\tilde{v}_-(\tilde{v}_+, \alpha_+)$]

$$\tilde{v}_+ = \frac{1}{1+\alpha_+}\left[\frac{\tilde{v}_-}{2} + \frac{1}{6\tilde{v}_-} \pm \sqrt{\left(\frac{\tilde{v}_-}{2} - \frac{1}{6\tilde{v}_-}\right)^2 + \frac{2}{3}\alpha_+ + \alpha_+^2}\right]$$



- Strong: $\tilde{v}_+$ and $\tilde{v}_-$ on opposite sides of $\frac{1}{\sqrt{3}}$
- Weak: $\tilde{v}_+$ and $\tilde{v}_-$ on same side of $\frac{1}{\sqrt{3}}$
- Detonation: $\tilde{v}_+ > \frac{1}{\sqrt{3}}$
- Deflagration: $\tilde{v}_+ < \frac{1}{\sqrt{3}}$

Deflagrations and detonations: general remarks



- Recall
$$\alpha_+ = \frac{4\Delta\theta}{3w_+}, \qquad r = \frac{w_+}{w_-}$$
- No strong deflagrations or detonations
- No deflagrations for $\alpha_+ > 1/3$
- Turning points at $\tilde{v}_- > \frac{1}{\sqrt{3}}$
- In bulk fluid $\alpha_+ = 0$, shocks obey
$$\tilde{v}_+\tilde{v}_- = \frac{1}{3}, \qquad \frac{\tilde{v}_+}{\tilde{v}_-} = \frac{3+r}{1+3r}$$

Similarity solution: equations for v and T

- Recall: $T^{\mu\nu} = wU^\mu U^\nu + pg^{\mu\nu}$
- Recall: EM conservation (away from wall): $\partial_\mu T^{\mu\nu} = 0$
- Project onto $U^\mu = \gamma(1, \mathbf{v})$ and $\bar{U}^\mu = \gamma(v, \hat{\mathbf{v}})$ ($\bar{U}^2 = +1$, $\bar{U} \cdot U = 0$)

$$U_\nu \partial_\mu T^{\mu\nu} = -\partial_\mu (wU^\mu) + U \cdot \partial p = 0$$
$$\bar{U}_\nu \partial_\mu T^{\mu\nu} = w\bar{U}^\nu U \cdot \partial U_\nu + \bar{U} \cdot \partial p = 0$$

- Bubbles spherical, radius $R = v_\mathrm{w} t$ (take nucleation time $t' = 0$)
- Fluid velocity $\mathbf{v} = v(r,t)\hat{\mathbf{r}} \to v(\xi)\hat{\mathbf{r}}$, with $\xi = r/t$
- Speed of sound $c_\mathrm{s}^2 = \frac{\partial p}{\partial T} / \frac{\partial e}{\partial T}$

$$\frac{dv}{d\xi} = 2\frac{v}{\xi} \frac{1}{\gamma^2(1-\xi v)(\mu^2/c_\mathrm{s}^2 - 1)}$$
$$\frac{dw}{d\xi} = w \frac{\gamma^2(\xi - v)}{1 - \xi v} \frac{dv}{d\xi}$$

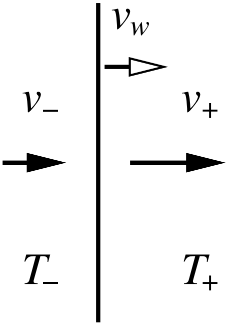
- $\mu = \frac{\xi - v}{1 - \xi v}$, fluid speed in expanding frames

Similarity solution: invariant profiles (solutions for $v(\xi)$, $T(\xi)$)

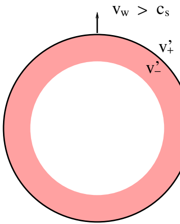
- Boundary conditions:
  - $v \to 0$ for $\xi \to 0, \infty$
  - $v \to v_\pm$ for $\xi \to \xi_{\mathrm{w}}$ (wall speed)
- $v_\pm$ are fluid speeds just ahead/behind of bubble wall in universe frame

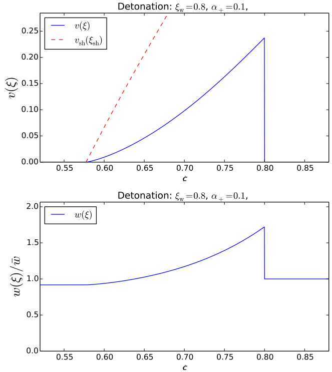
$$v_+ = \mu(\xi_{\mathrm{w}}, \tilde{v}_+) = \frac{\xi_{\mathrm{w}} - \tilde{v}_+(\xi_{\mathrm{w}}, \alpha_+)}{1 - \xi_{\mathrm{w}}\tilde{v}_+(\xi_{\mathrm{w}}, \alpha_+)}$$

- Intricate reasoning leads to three classes of solution:
  - Detonations
  - Deflagrations
  - Supersonic deflagrations ("hybrids")



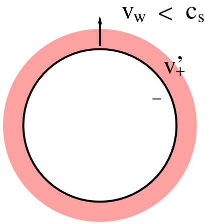
Detonations

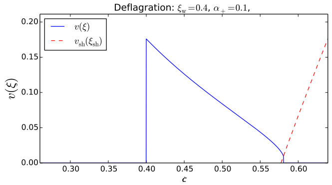


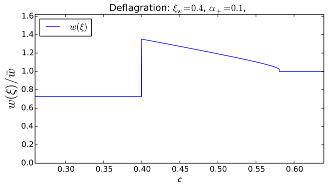


- Fluid at rest in front of wall $v_+ = 0$
- Fluid catching up behind (rarefaction wave):
- $v \to 0$ at $\xi \to c_s^+$
- $v = 0$ at $\xi < c_s^+$

Deflagrations

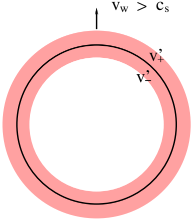


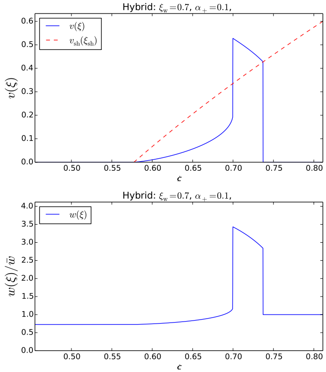




- Fluid at rest in behind wall $v'_- = 0$
- Fluid pushed out in front (compression wave):
- $v \to 0$ discontinously at $\xi = \xi_{sh}$
- Shock also obeys junction conditions.

Supersonic deflagrations (hybrids)





- Behind wall $\tilde{v}_- = c_s$ (wall frame)
- Both compression and rarefaction

Energy conversion: potential energy to kinetic energy

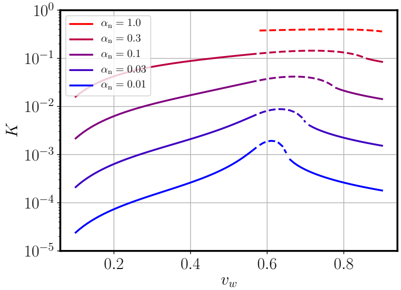
- Total energy density of fluid: $-T^0{}_0 = w\gamma^2 - p$.
- Kinetic energy density of fluid: $e_K = w(\gamma^2 - 1) = w\gamma^2 v^2$.
- Kinetic energy fraction: ratio of kinetic energy to energy of bubble

$$K \equiv \frac{\int d^3x \, w\gamma^2 v^2}{\frac{4\pi}{3} R^3 \bar{e}}$$

- Define $T_{\rm N}$, temperature when bubble nucleates (highest nucleation rate at t_f, when $h = 1/e$)
- **Transition strength parameter**

$$\alpha_{\rm n} = \frac{4[\theta_s(T_{\rm N}) - \theta_b(T_{\rm N})]}{3w(T_{\rm N})}$$

Kinetic energy fraction of bubbles (Dashed: hybrid)



Kinetic energy is source for GWs

Sound waves

Sound shell model: kinetic energy becomes sound waves
Consider EM tensor for perturbations with z dependence only

$$T^{tt} = w\gamma^2 - p, \qquad T^{tz} = w\gamma^2 v^z, \qquad T^{zz} = w\gamma^2 (v^z)^2 + p$$

Perturbations: $\delta e = e - \bar{e}$, $\delta p = p - \bar{p}$, v^z all $\ll 1$

$$\partial_t T^{tt} + \partial_z T^{zt} = 0 \implies \partial_t(\delta e) + \bar{w}\partial_z v^z = 0 \qquad (1)$$

$$\partial_t T^{tz} + \partial_z T^{zz} = 0 \implies \bar{w}\partial_t v^z + \partial_z(\delta p) = 0 \qquad (2)$$

NB δp and δe both depend on temperature T: $\delta p = \left(\frac{\partial p}{\partial T} / \frac{\partial e}{\partial T}\right) \delta e = c_s^2 \delta e$
Combine equations (1) and (2) by differentiating:

$$\left(\partial_t^2 - c_s^2 \partial_z^2\right) v^z = 0, \qquad \left(\partial_t^2 - c_s^2 \partial_z\right) \delta e = 0$$

Sound wave: longitudinal mode of fluid velocity **v** and energy density e.

Section 5

Summary

Summary

- Bubble nucleation rate/volume

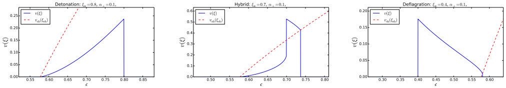
$$\Gamma(T) = \Gamma_0(T)e^{-S(T)}$$

- Transition rate parameter β

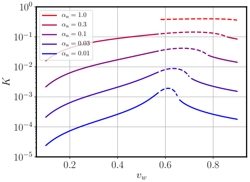
$$\Gamma(t) = \Gamma_f e^{\beta(t-t_f)}$$

with $\frac{4\pi}{3} v_\mathrm{w}^3 \left(3!/\beta^4\right) \Gamma_f = 1$

- Wall speed v_w
- Mean bubble spacing $R_* = (8\pi)^{\frac{1}{3}} v_\mathrm{w}/\beta$
- Transition strength $\alpha_\mathrm{n} = \frac{4\Delta\theta}{3w_\mathrm{n}}$



- Similarity solution for bubble growth: detonation, deflagration, hybrid
- Kinetic energy fraction K into sound waves



Reading

Relativistic hydrodynamics

- *Relativistic Hydrodynamics*, L. Rezzolla and O. Zanotti (OUP, 2013)
- *Fluid Mechanics*, L. Landau and Lifshitz (Butterworth-Heinemann, 1980)

Bubble nucleation and growth

- *Gravitational waves from first order cosmological phase transitions in the Sound Shell Model*, M.H., M. Hijazi [arXiv:1909.10040]
- *Energy Budget of Cosmological First-order Phase Transitions*, J.R. Espinosa, T. Konstandin, J.M. No, G. Servant [arXiv:1004.4187]
- *Bubble growth and droplet decay in cosmological phase transitions*, H. Kurki-Suonio, M. Laine [arXiv:hep-ph/9512202]
- *Nucleation and bubble growth in cosmological electroweak phase transitions*, K. Enqvist, J. Ignatius, K. Kajantie, K. Rummukainen [Phys.Rev.D 45 (1992) 3415-3428]
- *Growth of bubbles in cosmological phase transitions*, J. Ignatius, K. Kajantie, H. Kurki-Suonio, M. Laine [arXiv:astro-ph/9309059]