Gravitational waves from phase transitions

1. Thermodynamics and hydrodynamics in the early Universe

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Outline

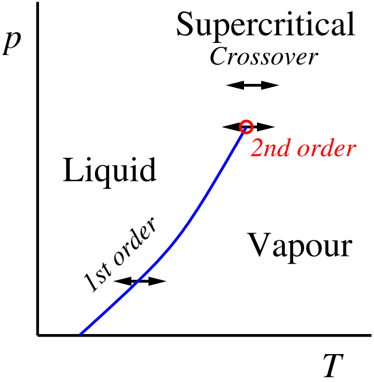
Section 1

Introduction: phase transitions in the early universe

Phase transitions

- Types of phase transition:
  - **1st order** latent heat, phase boundaries, mixed phases, metastable states
  - **2nd order (continuous):** diverging susceptibilities & correlation lengths, power laws with $|T - T_c|$
  - **Cross-over:** smooth changes in all thermodynamic quantities, local maxima in susceptibilities
- Phases distinguished by an **order parameter**. Discontinuous in a 1st order PT.
- Non-zero order parameter may “break" a symmetry.

Example: part of **Water** phase diagram (critical point: $T = 647$ K, $p = 220$ bar)



Phase transitions & cosmology

Phase transitions in early Universe:

Thermal Changing $T(t)$ | Vacuum (inflation) Changing field $\sigma(t)$

- **QCD transition** 100 MeV
  - Confinement of strong interactions: quarks & gluons $\rightarrow$ hadrons
- **Electroweak transition** 100 GeV
  - Generation of elementary particle masses
  - Spontaneous symmetry-breaking
- **Higher-scale transitions e.g. Grand Unified Theory** up to 10^{15} GeV
  - Confinement of new strong interactions?
  - Breaking new symmetries?

Phase transitions and gravitational waves

- First order transition $\implies$ departure from equilibrium $\implies$ shear stress $\implies$ GWs[(1)]
- What frequency GWs can we expect from a phase transition?
- Suppose process happens at a rate β at time t. Causality: $(H/\beta) \lesssim 1$

Event	T	t	f_0 (freq. today)
QCD transition	100 MeV	10^{-3} s	$10^{-8}(\beta/H)$ Hz
Electroweak transition	100 GeV	10^{-11} s	$10^{-5}(\beta/H))$ Hz
GUT/Hybrid inflation	$< 10^{15}$ GeV	$> 10^{-35}$ s	$< 10^{8}(\beta/H)$ Hz

- Electroweak-scale transition most interesting for LISA ($0.3 - 30$ mHz)
- QCD-scale transition most interesting for Pulsar Timing Arrays ($0.1 - 1$ nHz)

[(1)] Witten 1984; Hogan 1986

Section 2

Thermodynamics of free relativistic particles

Free energy (density) of an ideal gas $\mathcal{F}$

Free relativistic particles of mass m in equilibrium (zero chemical potential)

$$\mathcal{F}_\eta = -T \ln Z/\mathcal{V} = -\eta T \int đ^3 p \ln(1 + \eta e^{-E_{\mathbf{p}}/T})$$

where $E_{\mathbf{p}}^2 = \mathbf{p}^2 + m^2$, $\eta = +1$ (Fermi-Dirac/F), $\eta = -1$ (Bose-Einstein/B).

- Entropy density: $s = -\dfrac{\partial \mathcal{F}_\eta}{\partial T}$
- Energy density: $e = \mathcal{F}_\eta + Ts$
- Pressure: $p = Ts - e$ (Note $p = -\mathcal{F}$)

To find equilibrium state at temperature T we minimise free energy

- Dimensions: $\mathcal{F}_\eta = T^4 J_\eta(m/T)$

Defines dimensionless functions $J_\eta(m/T)$

Free energy: exact formulae in high T expansion ($m/T \ll 1$)

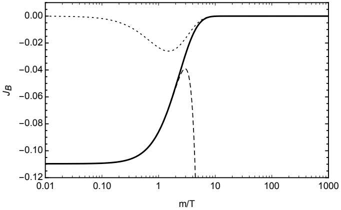
Bosons:

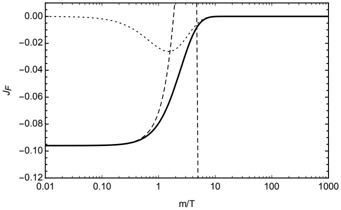
$$\mathcal{F}_B = -\frac{\pi^2}{90}T^4 + \frac{m^2T^2}{24} - \frac{(m^2)^{\frac{3}{2}}T}{12\pi} - \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{a_BT^2}\right) - \frac{m^4}{16\pi^{\frac{5}{2}}}\sum_\ell (-1)^\ell \frac{\zeta(2\ell+1)}{(\ell+1)!}\left(\frac{m^2}{4\pi^2T^2}\right)^\ell$$

Fermions:

$$\mathcal{F}_F = -\frac{\pi^2}{90}\frac{7}{8}T^4 + \frac{m^2T^2}{48} + \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{a_FT^2}\right) + \frac{m^4}{16\pi^{\frac{5}{2}}}\sum_\ell (-1)^\ell \frac{\zeta(2\ell+1)}{(\ell+1)!}(1-2^{-2\ell-1})\Gamma(\ell+\tfrac{1}{2})\left(\frac{m^2}{4\pi^2T^2}\right)^\ell$$

$a_B = 16\pi^2\ln(\frac{3}{2} - 2\gamma_E)$, $a_F = a_B/16$,
$\gamma_E = 0.5772\ldots$ (Euler's constant)





Approximations to $O(m/T)^4$ (dashed), $T/m \to 0$ (dotted)

Model field theory: particle masses from a real scalar field

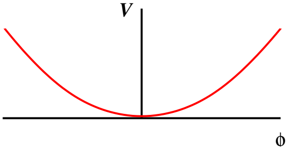
Real scalar field ϕ

$$\mathcal{L} = \frac{1}{2}\partial\phi\cdot\partial\phi - V_0(\phi)$$

- $V_0(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$

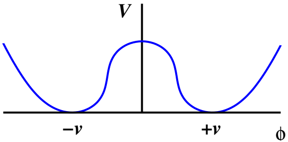
Symmetry: $\phi \to -\phi$ (Z_2).

$\mu^2 > 0$



- Minimum (ground state) at $T = 0$: $\phi = 0$
- Scalar field mass $M_\phi = \mu$
- Ground state respects symmetry

$\mu^2 < 0$



- Minimum (ground state) at $T = 0$: $\phi = v = \sqrt{|\mu^2|/\lambda}$
- Scalar field mass $M_\phi = \sqrt{V_0''(v)} = 2|\mu^2|$
- Ground state "breaks" symmetry.

Effective potential for scalar field

Let scalar field give masses to ($\bar{\phi} \equiv |\phi|$)
Neglect $M_\phi(\bar{\phi}) = \sqrt{V_0''(\bar{\phi})}$ – needs special treatment

- scalar bosons ($M_S(\bar{\phi}) = g_S \bar{\phi}$)
- vector bosons ($M_V(\bar{\phi}) = g_V \bar{\phi}$)
- Dirac fermions ($M_F(\bar{\phi}) = g_F \bar{\phi}$)

$$\begin{aligned}
\mathcal{F}(\bar{\phi}, T) &= -g_{\text{eff}} \frac{\pi^2}{90} T^4 + \frac{T^2}{24} \left(\sum_S M_S^2(\bar{\phi}) + 3 \sum_V M_V^2(\bar{\phi}) + 2 \sum_F M_F^2(\bar{\phi}) \right) \\
&- \frac{T}{12\pi} \left(\sum_S (M_S^2(\bar{\phi}))^{\frac{3}{2}} + 3 \sum_V (M_V^2(\bar{\phi}))^{\frac{3}{2}} \right) \\
&+ \frac{1}{64\pi^2} \sum_S M_S^4(\bar{\phi}) \ln \left(\frac{M_S^2}{a_B T^2} \right) + \frac{3}{64\pi^2} \sum_V M_V^4(\bar{\phi}) \ln \left(\frac{M_V^2}{a_B T^2} \right) - \frac{2}{64\pi^2} \sum_F M_F^4(\bar{\phi}) \ln \left(\frac{M_F^2}{a_F T^2} \right) + \cdots
\end{aligned}$$

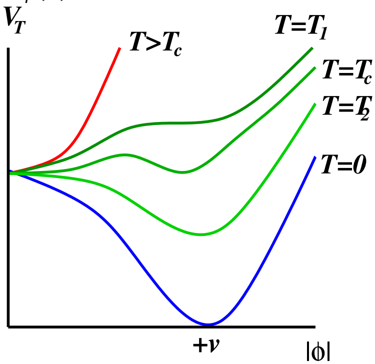
- $g_{\text{eff}} = \sum_B 1 + \sum_F \frac{7}{8}$
- Define $\Delta V_T(\bar{\phi}) = \mathcal{F}(\bar{\phi}, T) + g_{\text{eff}} \pi^2 T^4/90$ (i.e. field-dependent part of free energy)
- Define **effective potential** $\boxed{V_T(\bar{\phi}) = V_0(\bar{\phi}) + \Delta V_T(\bar{\phi})}$

Phase transition (weakly coupled field theory, $M/T \ll 1$)

Equilibrium state: minimum of effective potential, require $V'_T(\bar\phi) = 0$, $V''_T(\bar\phi) > 0$.

$$V_T(\bar\phi) \simeq V_0(0) + \frac{D}{2}(T^2 - T_2^2)\bar\phi^2 - \frac{A}{3}T\bar\phi^3 + \frac{\lambda_T}{4!}\bar\phi^4$$

- High temperature: equilibrium at $\bar\phi = 0$.
- Second minimum develops at T_1, $\phi_b(T)$.
- Critical temperature T_c: $V_{T_c}(0) = V_{T_c}(\bar\phi_b)$.
- System supercools below T_c
- Unstable below $T < T_2$ (spinodal temperature)
- First order transition (apparently)
- Latent heat $\mathcal{L} = T_c\,\Delta s(T_c)$
- 1st order from cubic term (bosons only)



Section 3

The Standard Model plasma

Standard Model effective potential in weak coupling approximation

Standard Model in unitary “gauge": $\bar{\phi} = \sqrt{H^\dagger H/2}$

	H	$W^\pm$	Z	t
$M(\bar{\phi})$	$\sqrt{V_0''(\bar{\phi})}$	$\frac{1}{2} g_w \bar{\phi}$	$\frac{1}{2}\sqrt{g_w^2 + g'^2}\bar{\phi}$	$\sqrt{2} y_t \bar{\phi}$
M/GeV	125	80.4	91.2	174
d.o.f.	1	6	3	$\frac{7}{8}12$

Form of effective potential: $V_T \simeq \frac{D}{2}(T^2 - T_0^2)\bar{\phi}^2 - \frac{A}{3} T \bar{\phi}^3 + \frac{\lambda_T}{4}\bar{\phi}^4$

$$D = \frac{1}{12\bar{\phi}^2}\left(6M_W^2 + 3M_Z^2 + 6M_t^2\right) \qquad A = \frac{1}{12\pi\bar{\phi}^2}\left(6M_W^3 + 3M_Z^3\right)$$

$$\lambda_T = \lambda - \frac{1}{16\pi^2\bar{\phi}^4}\left(6M_W^4 \ln\left(\frac{M_W^2}{a_B T^2}\right) + 3M_Z^4 \ln\left(\frac{M_Z^2}{a_B T^2}\right) - 4M_t^4 \ln\left(\frac{M_t^2}{a_F T^2}\right)\right)$$

Predicts: $T_c = 166$ GeV, $T_2 = 165$ GeV

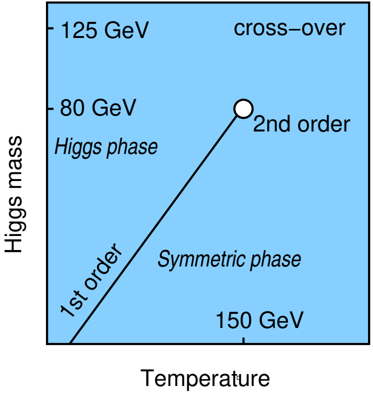
First order, but little supercooling is possible.

Electroweak transition in the Standard Model: cross-over

Interactions at high T are important!

- BE distribution function of massless bosons of momentum p: $f_B = (e^{p/T} - 1)^{-1}$
- Goes as T/p as $p \to 0$
- Interactions important when coupling*$f_B \gtrsim 1$
- So interactions are important for massless bosons $p \lesssim gT$ at any temperature and coupling.
- "Infrared problem" or "Linde problem"
- Need improved methods
  - Dimensional Reduction 4D $\to$ 3D
  - 3D lattice evaluation of partition function

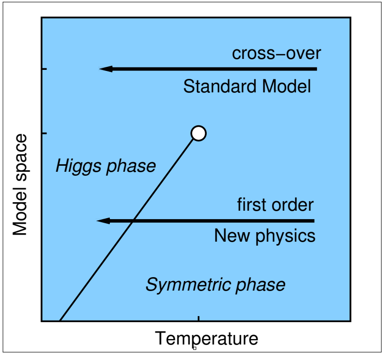
Standard Model phase diagram



Kajantie et al 1996

1st order electroweak phase transitions in SM extensions

- 2HDM (2 Higgs doublet model)
  - Extra scalars (A^0, H^0, $H^{\pm}$) increase cubic term.
  - Strong phase transition when $m_{A^0} \gtrsim 400$ GeV
- Extra singlet scalar(s)
  - Tree level first order phase transition
  - SM-like phenomenology allowed
- Extra triplet scalar(s)
  - Only three more parameters (mass and two quartic couplings)
  - SM-like phenomenology allowed
- Composite Higgs
  - Higgs is like a pion
  - new QCD-like interactions at O(1) TeV



Section 4

Relativistic hydrodynamics for phase transitions

Standard Model plasma: stable particle approximation

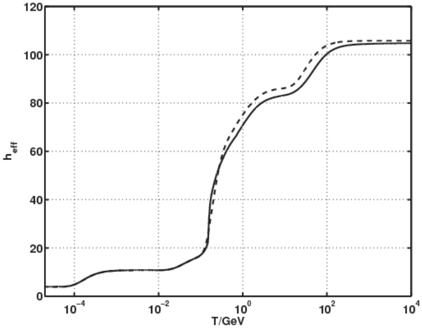
In SM extensions, SM particles expected to dominate at $T \sim 100$ GeV

	H	$W^{\pm}$	Z	t
M/GeV [PDG]	125	80.4	91.2	174
Γ/GeV [PDG]	$(3 \pm 2) \times 10^{-3}$ (*)	2.1	2.5	1.4
d.o.f.	1	6	3	$\frac{7}{8}12$

(*) assumes equal on-shell and off-shell effective couplings

- W, Z, t, H have largest mass change: $g_{\text{eff}} = 20.5$
- Each have frequent scatterings with "light" particles $g_{\text{eff}} = 86.25$
- Each have relatively narrow width
- Scattering with thermal bath of light particles more rapid than decays
- Treat W, Z, t, H as stable particles
- Doesn't work so well for symmetric phase (IR problem)

Standard Model effective degrees of freedom



$$h_{\text{eff}} = s(T) / \frac{2\pi^2}{45}$$

Temp.	Event
100 GeV	t non-relativistic
1 GeV	b non-relativistic
500 GeV	c, τ non-relativistic
200 MeV	QCD phase transition
30 MeV	μ non-relativistic
2 MeV	ν freeze-out
0.2 MeV	e non-relativistic
1 eV	matter = radiation
0.1 eV	photon decoupling

Ideal gas, model QCD transition[2] (dashed)
With interactions, lattice QCD[3] (solid)

[2] Olive 1981
[3] Hindmarsh & Philipsen 2005, Laine & Schroder 2006, Borsanyi et al 2016

Relativistic Boltzmann equation

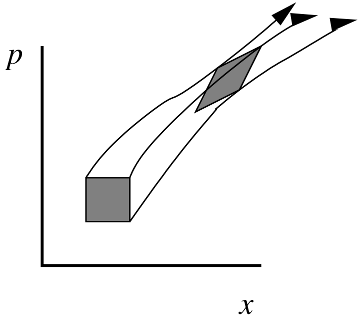
- **Distribution function** (Lorentz scalar): $f(p, x)$
- Stable particles: evaluate on-shell $p^0 = E_{\mathbf{p}} = \sqrt{(\mathbf{p}^2 + m^2)}$
- Average number of particles in phase space volume element at $(\mathbf{p}, \mathbf{x})$ at time t.

Particle flux 4-vector and **energy-momentum** tensor:

$$j^\mu = \int \frac{\bar{d}^3p}{E_{\mathbf{p}}} p^\mu \ f(p,x)|_{E_{\mathbf{p}}} \qquad T^{\mu\nu} = \int \frac{\bar{d}^3p}{E_{\mathbf{p}}} p^\mu p^\nu \ f(p,x)|_{E_{\mathbf{p}}}$$

- Manifestly covariant integration: $\int \frac{\bar{d}^3p}{2E_{\mathbf{p}}} = \int \bar{d}^4p \, \theta(p^0) \, \bar{\delta}(p^2 + m^2)$
- Enforce on-shell condition $p^0 = E_{\mathbf{p}} = \sqrt{(\mathbf{p}^2 + m^2)}$: $f(p,x)|_{E_{\mathbf{p}}} = \int \bar{d}p^0 \, \theta(p^0) \, \bar{\delta}(p^2 + m^2) f(p,x)$

Particle flow in phase space with forces



$$x^\mu \rightarrow x^\mu + \frac{dX^\mu}{d\tau}\Delta\tau$$
$$p^\mu \rightarrow p^\mu + F^\mu \Delta\tau$$

Force must preserve $p^2 + m^2 = 0$

- $F^\mu p_\mu = 0$
- or $F^\mu + \partial^\mu m(x) = 0$

Exercise!

Without collisions: $f(p + F\Delta\tau, x + \frac{1}{m}p\Delta\tau) = f(p, x)$. Hence

$$\left(p^\mu \partial_\mu + mF^\mu \frac{\partial}{\partial p^\mu}\right) \theta(p^0)\delta(p^2 + m^2) f(p, x) = 0$$

where p^μ are independent in $f(p, x)$.

Fluid energy-momentum tensor

Distribution function for system in local equilibrium:

$$f^{\mathrm{eq}}(p,x) = \frac{1}{e^{(U_\mu p^\mu - \mu)/T} + \eta}$$

$U^\mu(x)$ 4-velocity – $T(x)$ temperature – $\mu(x)$ chemical potential (= 0 here)
Energy-momentum tensor:

$$T^{\mu\nu} = \int \frac{đ^3 p}{E} p^\mu p^\nu \ f^{\mathrm{eq}}(p,x)\Big|_{E_\mathbf{p}} = (e+p)U^\mu U^\nu + p g^{\mu\nu}$$

where

$$e = \int đ^3 p\, E \ f_0^{\mathrm{eq}}(p,x)\Big|_{E_\mathbf{p}} \quad \text{rest frame energy density}$$

$$p = \int đ^3 p\, \frac{\mathbf{p}^2}{3E} \ f_0^{\mathrm{eq}}(p,x)\Big|_{E_\mathbf{p}} \quad \text{rest frame (kinetic) pressure}$$

EM (non)-conservation for particles with field-dependent mass

With collisions, $f(p,x)$ can increase (scattering in) and decrease (scattering out)

$$\left(p^\mu \partial_\mu + mF^\mu \frac{\partial}{\partial p^\mu} \right) \theta(p^0)\delta(p^2+m^2) f(p,x) = C[f]$$

- $\times$ both sides by p^ν and integrate over momenta
- Assume collisions occur "at a point" and still conserve momentum: $\int đ^4 p\, p^\nu C[f] = 0$

$$\tfrac{1}{2}\partial_\mu T^{\mu\nu} + mF^\mu \int đ^4 p\, p^\nu \tfrac{\partial}{\partial p^\mu} \theta(p^0)\delta(p^2+m^2) f(p,x) = 0$$

- Integrate by parts
- $F^\mu = -\partial^\mu m = \partial^\mu \bar\phi \, dm/d\bar\phi$

$$\partial_\mu T^{\mu\nu} = -\partial^\nu \bar\phi \frac{dm^2}{d\bar\phi} \int \frac{đ^3 p}{2E} f(p,x)$$

Fluid of particles coupled to scalar field through mass 1

Model for the system near phase transition[4] (good assumption: scattering rates much higher than $|\dot{T}/T| = H$)

$$\begin{aligned} \text{fluid} \quad T^{\mu\nu}_{\rm f} &= (e+p)U^\mu U^\nu + p g^{\mu\nu} \\ \text{field} \quad T^{\mu\nu}_{\phi} &= \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left[\tfrac{1}{2}(\partial\phi)^2 + V_0(\phi)\right] \end{aligned}$$

- Note: $p(T,\phi) = g_{\rm eff}\pi^2 T^4/90 - \Delta V_T(\phi)$ i.e. minus free energy of the fluid
- Conservation of total energy-momentum: $\partial_\mu\left(T^{\mu\nu}_{\rm f} + T^{\mu\nu}_{\phi}\right) = 0$

Hence non-conservation of $T^{\mu\nu}_{\rm f}$ must appear in $T^{\mu\nu}_{\phi}$

$$\partial_\mu T^{\mu\nu}_{\phi} = +\partial^\nu\bar{\phi}\,\frac{dm^2}{d\bar{\phi}}\int\frac{d\!\!{}^-{}^3p}{2E}f(p,x)$$

Implies for scalar field equation[5]

$$\Box\phi - V_0'(\phi) = \frac{dm}{d\bar{\phi}}\int\frac{d\!\!{}^-{}^3p}{2E}f(p,x)$$

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[4] Ignatius, Kajantie, Kurki-Suonio, Rummukainen 1991
[5] Also derivable from field theory, see Moore & Prokopec 1996

Fluid of particles coupled to scalar field through mass 2

$$\Box\phi - V_0'(\phi) = \frac{dm}{d\bar{\phi}} \int \frac{đ^3 p}{2E} f(p,x)$$

Write $f = f^{\rm eq} + \Delta f$

$$\Box\phi - V_0'(\phi) = \Delta V_T'(\phi) + \frac{dm}{d\bar{\phi}} \int \frac{đ^3 p}{2E} \Delta f(p,x)$$

Put equilibrium part on LHS:

$$\boxed{\Box\phi - V_T'(\phi) = \frac{dm^2}{d\bar{\phi}} \int \frac{đ^3 p}{2E} \Delta f(p,x)}$$

Move scalar potential $V_0(\phi)$ into fluid EM: $p_{\rm tot} = g_{\rm eff}\pi^2 T^4/90 - V_T(\phi)$

$$\boxed{\partial_\mu T_{\rm f}^{\mu\nu} + \partial^\nu\phi \frac{\partial V_T(\phi)}{\partial\phi} = -\partial^\nu\phi \frac{dm^2}{d\bar{\phi}} \int \frac{đ^3 p}{2E} \Delta f(p,x)}$$

IKKR model and entropy generation

$$\Box\phi - V_T'(\phi) = \frac{dm^2}{d\bar{\phi}} \int \frac{d\!\!{}^-{}^3 p}{2E} \Delta f(p, x)$$

- Near equilibrium RHS a function of dynamical variables T, U^μ, ϕ
- Field gradients disturb eqm: expect RHS $\sim \partial_\mu \phi$
- Lorentz invariance: expect RHS $\sim U^\mu \partial_\mu \phi$
- Field comes from $m^2(\phi)$ so $\partial_\mu \phi \to \partial_\mu m^2 / T$

Suggests:

$$\boxed{\Box\phi - V_T'(\phi) = \eta_T(\phi) U \cdot \partial\phi \qquad \text{with} \qquad \eta_T(\phi) = \tilde{\eta}\phi^2 / T}$$

Can show that entropy generation is always non-negative Exercise!:
Entropy current $S^\mu = sU^\mu$, $s = dp/dT$

$$\partial^\mu S^\mu = \tilde{\eta}(\phi/T)^2 (U \cdot \partial\phi)^2 \geq 0$$

Summary

- Electroweak symmetry is broken at $T \simeq 100$ GeV
- Standard Model plasma at $T \simeq 100$ GeV (first approximation): weakly-interacting and long-lived W, Z, t, h + "bath" of light particles
- In simple picture SM phase transition is 1st order (just)
- Interactions $(W, Z) \to$ cross-over
- Beyond the Standard Model: more scalars $\to$ 1st order phase transition?
- Model of coupled order parameter ϕ and fluid $T_{\rm f}^{\mu\nu}$

$$\Box\phi - V_T'(\phi) = \frac{dm^2}{d\bar{\phi}} \int \frac{d\!\!^{-3}p}{2E} \Delta f(p,x) \qquad \simeq \tilde{\eta}\frac{\phi^2}{T}(U\cdot\partial\phi)$$

$$\partial_\mu T_{\rm f}^{\mu\nu} + \partial^\nu\phi \frac{\partial V_T(\phi)}{\partial\phi} = -\partial^\nu\phi\frac{dm^2}{d\bar{\phi}} \int \frac{d\!\!^{-3}p}{2E} \Delta f(p,x) \simeq \tilde{\eta}\frac{\phi^2}{T}(U\cdot\partial\phi)\partial^\nu\phi$$

Where $p = g_{\rm eff}\pi^2 T^4/90 - V_T(\phi)$, $\Delta f(p,x) = f(p,x) - f^{\rm eq}(p,x)$

Reading

Statistical physics

- *Statistical Mechanics*, K Huang (Wiley, 1987)

Thermal quantum field theory

- *Basics of Thermal Field Theory*, M. Laine and A. Vuorinen (Springer, 2016) [arXiv:1701.01554]

Relativistic hydrodynamics

- *Relativistic Hydrodynamics*, L. Rezzolla and O. Zanotti (OUP, 2013)

Scalar field coupled to a fluid

- *Growth of bubbles in cosmological phase transitions*, J. Ignatius, K. Kajantie, H. Kurki-Suonio, M. Laine [arXiv:astro-ph/9309059]
- *How fast can a wall move? ...*, G. Moore, T. Prokopec [arXiv:hep-ph/9506475]
- *Energy Budget of Cosmological First-order Phase Transitions*, J.R. Espinosa, T. Konstantin, J.M. No, G. Servant [arXiv:1004.4187]
- *From Boltzmann equations to steady wall velocities*, T. Konstantin, G. Nardini, I. Rues [arXiv:1407.3132]
- *Bubble wall velocities in local equilibrium*, W-Y. Ai, B. Garbrecht, C. Tamarit [arXiv:2109.13710]
- *First principles determination of bubble wall velocity*, B. Laurent, J. Cline [arXiv:2204.13120]