Workshop HALC-LHC 23

Generalized Power Series: a semi-analytical technique for multi-loop Scattering Amplitudes

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Federico Ripani

PhD

U. Roma Sapienza

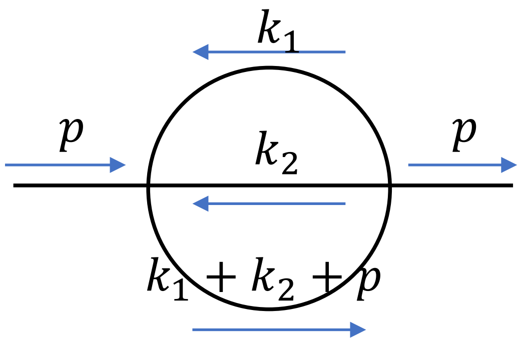
Supervisor: R. Bonciani

COMPUTATIONAL BOTTLENECKS FOR PHYSICAL AMPLITUDES

- Integration-by-parts (**IBP**) reduction of thousands of Feynman integrals to a basis of linearly independent **master integrals** (typically tens). This is implemented in a certain number of computational tools (kira, finite-flow,…).

- Efficiently evaluate the master integrals **(MI)** in the physical phase-space region.

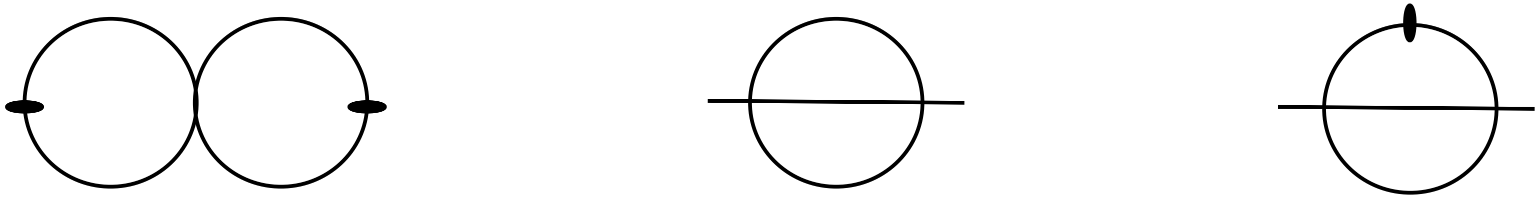
Example basis of master integrals



$$\mathcal{J}_{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5} = (m^2)^{\nu-d} \int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \frac{N_1^{-\nu_1} N_2^{-\nu_2}}{[k_1^2 - m^2]^{\nu_3} [k_2^2 - m^2]^{\nu_4} [(k_1 + k_2 + p)^2 - m^2]^{\nu_5}}$$

$$\vec{f} = \epsilon^2 \begin{pmatrix} \mathcal{J}_{0,0,2,2,0} \\ \mathcal{J}_{0,0,1,1,1} \\ \mathcal{J}_{0,0,2,1,1} \end{pmatrix}$$

$$\mathcal{D}^d k_i = \frac{d^d k_i}{i\pi^{\frac{d}{2}} \Gamma(1+\epsilon)}$$



Differential equations

Integration-by-parts allow to write a linear differential equations system in closed form w.r.t. the basis of master.

$$\frac{\partial \vec{f}(\vec{x},\epsilon)}{\partial x_j} = A_j(\vec{x},\epsilon)\vec{f}(\vec{x},\epsilon)$$

$$A(x,\epsilon) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{2\epsilon-1}{x} & -\frac{3}{x} \\ -\frac{2}{(x-9)(x-1)} & -\frac{(2\epsilon-1)(3\epsilon-2)(x-3)}{(x-9)(x-1)x} & -\frac{\epsilon x^2+10\epsilon x-27\epsilon-10x+18}{(x-9)(x-1)x} \end{pmatrix} \qquad x = \frac{p^2}{m^2}$$

Program

- Differential equations system
- Boundary conditions
- Solution
  - analytical
  - Semi-analytical
- Generalized one dimension power series expansion method
- A phenomenology application: di-photon production

Strategy (1/3)

- A set of master integrals, at a given order of ε, admit a solution in the vicinity of a point x_i (regular or singular) of the form $\sum_{j=0}^{p} \sum_{k=0}^{\infty} \vec{C}_{jk} (x - x_i)^{w+k} \log^j (x - x_i)$, $w \in \mathbb{Q}$.
- The radius of convergence of the series is determined by the nearest singularity to x_i. In our example it is the distance from $x_s = 9$, which is the only singular point.
- We know the value of the master integrals at the starting point $x_{bc} = 0$:

$$\vec{f}(0,\epsilon) = \begin{pmatrix} 1 \\ \frac{3}{2} \\ \frac{1}{2} \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ \frac{9}{2} \\ \frac{1}{2} \end{pmatrix} + \epsilon^2 \begin{pmatrix} 0 \\ \frac{21}{2} + \frac{3\sqrt{3}i}{2} \left[\mathrm{Li}_2(l_+) - \mathrm{Li}_2(l_-) \right] \\ \frac{1}{2} + \frac{\sqrt{3}i}{2} \left[\mathrm{Li}_2(l_+) - \mathrm{Li}_2(l_-) \right] \end{pmatrix} + \cdots, \quad l_\pm = \frac{-1 \pm i\sqrt{3}}{2}$$

Strategy (2/3)

- Let’s suppose we want to transport the solution until $x_{end} = 11$. **Frobenius + variation of constants** allow us to express the solution in terms of **power series** around $x_{bc} = 0$**:**

$$f_3(x,\epsilon) = \frac{1}{2} + \frac{\epsilon}{2} + \epsilon^2 \left(-0.67195362 + 0.16666667\,x + 0.00501349\,x^2 + 0.00025591\,x^3 + 0.00001628\,x^4 + \cdots\right) + \cdots$$

- Let’s evaluate it at $x_{end} = 11$: $f_3^{(2)}(11) = 4.758$ which is wrong!
- Let’s find the solution around the singular point $x_s = 9$.
- We have to match the new expansion with the previous one. Let’s choice the matching point to be $x'_{bc} = 4.5$.
- We have to evaluate $\log(x'_{bc} - 9)$. Feynman prescription on kinematical variables and masses helps us:

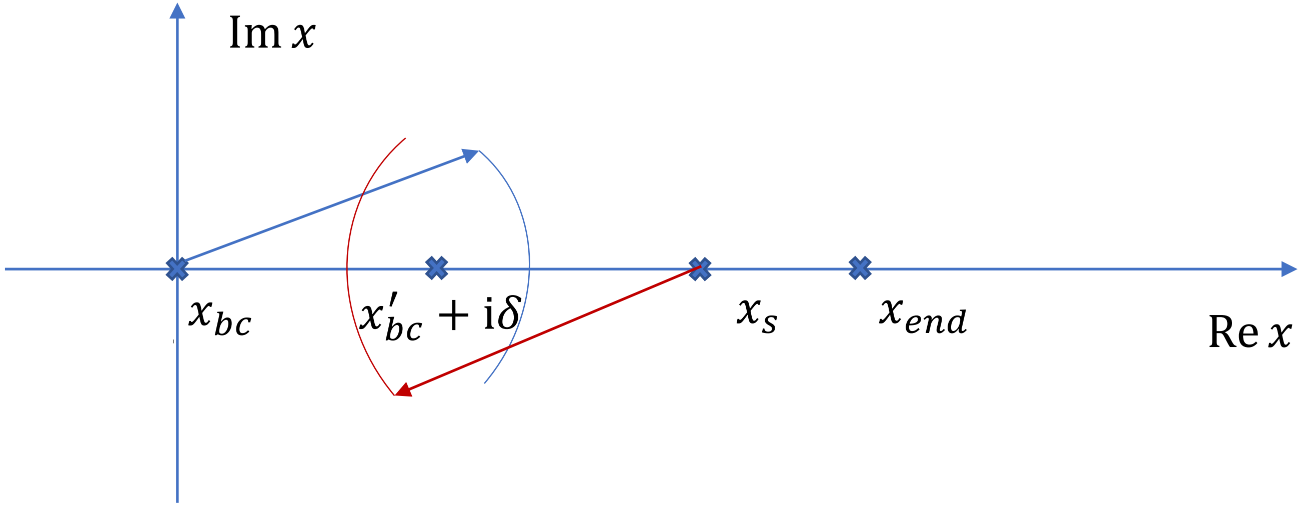
$$\log\left(x'_{bc} - 9\right) = \log\left(9 - x'_{bc}\right) + i\,\pi$$

Strategy (3/3)

- The solution expanded around the singularity is:

$$f_3(x,\epsilon) = \frac{1}{2} + \frac{\epsilon}{2} + \epsilon^2\Big[(-0.463213 - 3.141593i) + (0.580281 + 0.633135i)(x-9) +$$
$$- (0.035561 + 0.043968i)(x-9)^2 + (0.002850 + 0.003664i)(x-9)^3 + \cdots$$
$$+ \log(x-9)\Big(1 - 0.201533(x-9) + 0.013995(x-9)^2 - 0.001166(x-9)^3 + \cdots\Big)\Big] + \cdots$$

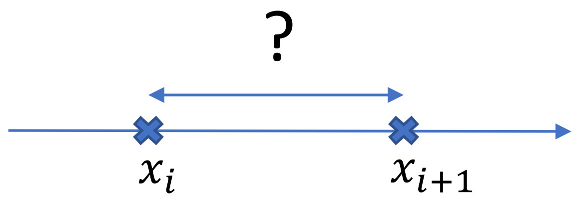
- Now we get the correct result $f_3^{(2)}(11) = 2.71007\ldots + 1.11531\ldots\, i$



Code implementation

- DiffExp arXiv:2006.05510 (M. Hidding, 2020)

  Segmentation strategies:



- **Dynamic**: $x_{i+1} = x_i + r_i y_i^{\delta}$,

  r_i is the distance of the nearest singularity to x_i, and $y_i^{\delta} = \min\left\{\left(\frac{\delta}{\left|s^{(n;i)}(A_{x,lj}^{(k)})r_i\right|}\right)^{\frac{1}{n}}, (l,j,k)\right\}$.

- **Predivision**: $x_{i+1} = x'_{bc} + \frac{s}{k}$ $\quad$ $x'_{bc} = x_i + \frac{r_i}{k}$

  $\tilde{x}_L$: left singularity $\quad$ $\tilde{x}_R$: right singularity

$$s = \begin{cases} \frac{k(x'_{bc} - \tilde{x}_L)}{k-1}, & x'_{bc} < \frac{\tilde{x}_L(1+k) + \tilde{x}_R(k-1)}{2\text{k}} \\ \frac{k(\tilde{x}_R - x'_{bc})}{k+1}, & x'_{bc} > \frac{\tilde{x}_L(1+k) + \tilde{x}_R(k-1)}{2\text{k}} \end{cases}$$

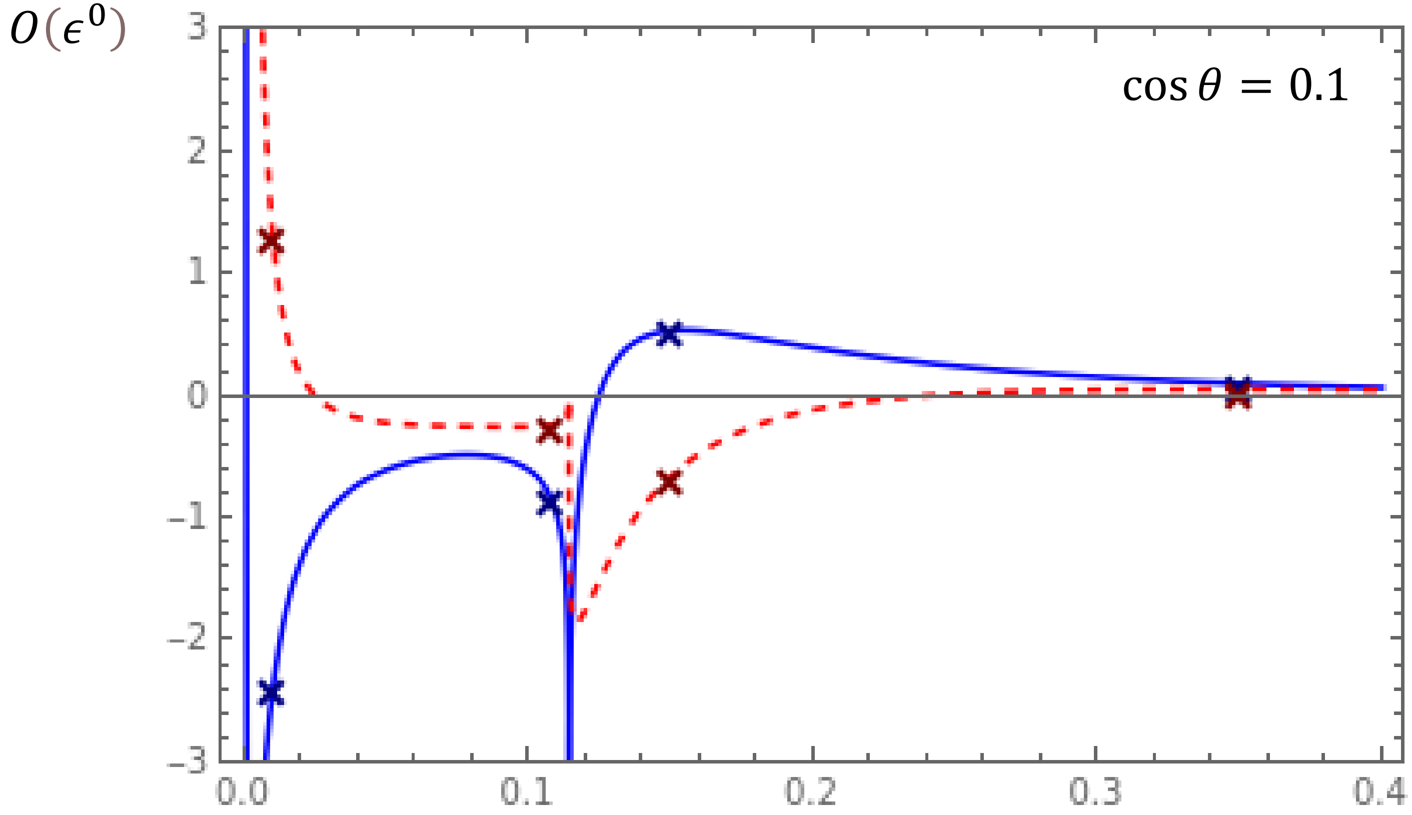
- SeaSyde arXiv:2205.03345 (T. Armadillo, R. Bonciani, S. Devoto, N. Rana, A. Vicini, 2022)
  - Expansion in the x complex plane
  - Allows evaluations for complex values of the kinematical invariants
- AMFlow arXiv:2201.11669 (X. Liu, Y. Ma, 2022)
  - Uses auxiliary mass flow to numerically evaluate Feynman integrals without any input
  - It is a very powerful tool to determine boundary conditions

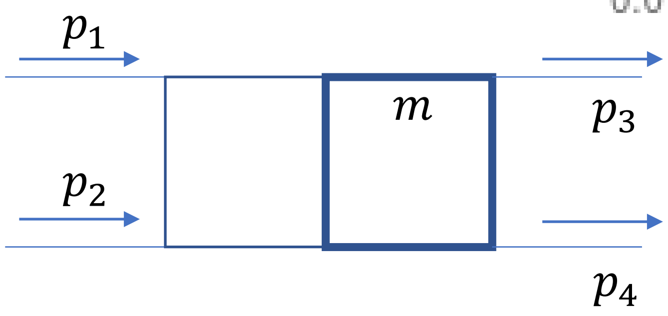
A phenomenology application (di-photon production)

$$\frac{|\mathcal{A}^{(\text{fin})}_{q\bar{q},\gamma\gamma}|^2}{|\mathcal{A}^{(0)}_{q\bar{q},\gamma\gamma}|^2} = 1 + \left(\frac{\alpha_S}{\pi}\right)\mathcal{H}^{(1)}_{\gamma\gamma} + \left(\frac{\alpha_S}{\pi}\right)^2\left(\mathcal{H}^{(2;5\text{f})}_{\gamma\gamma} + \mathcal{H}^{(2;\text{top})}_{\gamma\gamma}\right) + \cdots$$

Topology	# master	D.E. system	start point	region covered	# points	ExpansionOrder	# digits	Time
Planar	32	$\epsilon\,\mathrm{d}\log$	$s = t = 0$	$52\text{GeV} \leq \sqrt{s} \leq 619\text{GeV}$ $-0.99 \leq \cos\theta \leq 0.99$	816	50	25	$\sim$ 25min
Non-planar	36	$\epsilon\,\mathrm{d}\log$ +2 elliptic sectors	$s = t = 0$	$52\text{GeV} \leq \sqrt{s} \leq 619\text{GeV}$ $-0.99 \leq \cos\theta \leq 0.99$	816	50	18	$\sim$ 3h

M. Becchetti, R. Bonciani, L. Cieri, F. Coro, F. Ripani





$$x = \frac{\frac{s}{m^2} - x_1}{x_2 - x_1}$$

$$x_1 = \frac{1}{100}$$

$$x_2 = 35$$

Conclusions

- Generalized power series is a semi-analytical method to find solution to D.E.
- It can provide an arbitrary number of significant digits as far as we wait long enough
- High accurate numerical evaluation of Scattering Amplitudes
- It can be improved (parallelization)
- It has two intrinsic limits: 1) solution only along a countour (interpolation)
  2) I have to solve D.E. many times (wait time)
- There is a more foundamental problem: topologies that we are not able to reduce

Thank you!