0.- Consider a generic term in a Lagrangian with e_i fields (bosons or fermions)

$$\mathcal{L} = c_i \quad (\phi_1 \cdots \phi_{e_i})$$

a) Using that $[\phi]_{4-2\epsilon} = [\phi]_4 - \epsilon$ and $[\mathcal{L}]_{4-2\epsilon} = [\mathcal{L}]_4 - 2\epsilon$

show that $[c_i]_{4-2\epsilon} = [c_i]_4 + (e_i - 2)\epsilon$

b) Show that derivatives in the Lagrangian do not change this counting.

c) Using that a Feynman diagram at L loops with E external lines satisfies (V_n is the # of vertices with n legs)

$$\boxed{E + 2L - 2 = \sum_n (n-2) V_n}$$

show that at one loop we have the relation

$$n_i c_i' - \sum_j n_j c_j \frac{\partial c_i'}{\partial c_j} = -2 c_i'$$

where $n_i \equiv \dfrac{[c_i]_{4-2\epsilon} - [c_i]_4}{\epsilon}$ and c_i' can be written as

$$c_i' = \alpha \prod_j c_j^{p_j}$$

a) $[\mathcal{L}]_4 - 2\epsilon = [C_i]_{4-2\epsilon} + \sum_{i=1}^{e_i} [\phi_i]_4 - n_i \epsilon$

$\Rightarrow [C_i]_{4-2\epsilon} = [\mathcal{L}]_4 - \sum_{i=1}^{e_i} [\phi_i]_4 + (n_i - 2)\epsilon$

$= [C_i]_4 + (n_i - 2)\epsilon$

b) $[D]=1$ so they do not change the scaling.

c) At one loop we have $E = \sum_n (n-2) V_n$

In the language of the problem $e_i = \sum_j (e_j - 2) p_j$

Also $n_i = (e_i - 2)$ and

$$n_i C_i' - \sum_j n_j c_j \frac{\partial C_i'}{\partial c_j} = (e_i - 2) C_i' - \sum_j (e_j - 2) p_j C_i'$$

$$= \sum_j (e_j - 2) p_j C_i' - 2 C_i' - \sum_j (e_j - 2) p_j C_i' = -2 C_i' .$$

note that $c_j \frac{\partial C_i'}{\partial c_j} = p_j C_i'$

1: Scaleless integrals vanish.

Write
$$\int \frac{1}{k^4} = \int \frac{1}{k^2(k^2-M^2)} - \int \frac{M^2}{k^4(k^2-M^2)} \equiv I_1 - I_2$$

Compute both integrals and show that they are identical in dim reg.

Discuss whether they are UV or IR divergent.

You can use the general formula
$$I_{n,m} \equiv \int \frac{1}{(k^2)^n (k^2-M^2)^m} = \frac{(-1)^{n+m}\, i}{(4\pi)^{2-\epsilon}\,(M^2)^{n+m-2+\epsilon}} \frac{\Gamma(n+m-2+\epsilon)\,\Gamma(2-n-\epsilon)}{\Gamma(m)\,\Gamma(2-\epsilon)}$$

Also $M^{\alpha\epsilon} = 1 + \alpha\epsilon \log M + \dots$,

$\Gamma(x) = \frac{1}{x} - \gamma + O(x)$ near $x=0$

$\Gamma(x) = \frac{(-1)^n}{n!}\left(\frac{1}{x+n} - \gamma + 1 + \dots + \frac{1}{n} + O(x+n)\right)$ near $x=-n$.

Solution:

I_1 is UV divergent, IR finite., I_2 is the opposite | Mathematica

$$I_1 = I_{1,1} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon_{UV}} + 1 + \log\frac{\mu^2}{M^2}\right)$$

$$I_2 = M^2 I_{2,1} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon_{IR}} + 1 + \log\frac{\mu^2}{M^2}\right)$$

$$\Rightarrow I = I_1 - I_2 = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right) = 0$$

Technically $\epsilon_{UV} > 0$, $\epsilon_{IR} < 0$ but we analitycally continue $\epsilon_{IR} \to -\epsilon_{IR}$.

2.- Prove the integration by parts identity

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2)^{n+1}} = \frac{D-2n}{2n} \frac{1}{M^2} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2)^n}$$

by using $\quad 0 = \int \frac{d^Dk}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2-M^2)^n}$

Sol:
$$\frac{D}{(k^2-M^2)^n} - \frac{2n\,k^2}{(k^2-M^2)^{n+1}} = \frac{D}{(k^2-M^2)^n} - \frac{2n(k^2-M^2+M^2)}{(k^2-M^2)^{n+1}}$$

$$= \frac{D-2n}{(k^2-M^2)^n} - \frac{2n\,M^2}{(k^2-M^2)^{n+1}} \Rightarrow I_{n+1} = \frac{D-2n}{2n} \frac{1}{M^2} I_n$$

3.- Compute the integral $I_F = \mu^{2\epsilon} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-M^2)} \frac{1}{(k^2-m^2)}$

Solution:

$$I_F = \frac{\mu^{2\epsilon}}{M^2-m^2} \int \left(\frac{1}{k^2-M^2} - \frac{1}{k^2-m^2} \right) =$$

$$= \frac{1}{M^2-m^2} \frac{i}{16\pi^2} \left[M^2 \left(\frac{1}{\epsilon} + 1 - \log \frac{M^2}{\mu^2} \right) - m^2 \left(\frac{1}{\epsilon} + 1 - \log \frac{m^2}{\mu^2} \right) \right]$$

$$= \frac{i}{16\pi^2} \left\{ \frac{1}{\epsilon} + 1 + \log\mu^2 - \frac{M^2}{M^2-m^2} \log M^2 + \frac{m^2}{M^2-m^2} \left(\log m^2 + \log M^2 - \log M^2 \right) \right\}$$

$$= \frac{i}{16\pi^2} \left\{ \frac{1}{\epsilon} + 1 - \log \frac{M^2}{\mu^2} + \frac{m^2}{M^2-m^2} \log \frac{m^2}{M^2} \right\}$$

4.- Compute the integrals $I_\epsilon = -\frac{\mu^{2\epsilon}}{M^2}\int \frac{1}{k^2-m^2}\left(1+\frac{k^2}{M^2}+\frac{k^4}{M^4}+\dots\right)$

Solution

$$I_{EFT} = \mu^{2\epsilon}\int \frac{1}{k^2-m^2}\left(-\frac{1}{M^2}\right)\left(1+\frac{k^2}{M^2}+\frac{k^4}{M^4}+\dots\right) = \frac{i}{16\pi^2}\left[-\frac{m^2}{M^2}-\frac{m^4}{M^4}-\frac{m^6}{M^6}\dots\right]$$

$$\int \frac{k^2}{k^2-m^2} = \int \frac{k^2-m^2+m^2}{k^2-m^2} = \int \overset{\text{scaleless}}{\overset{0}{\cancel{1}}} + \frac{m^2}{k^2-m^2} = \frac{i m^4}{16\pi^2}\left(\frac{1}{\epsilon}+1-\log\frac{m^2}{\mu^2}\right)$$

$$\int \frac{k^4}{k^2-m^2} = \int \frac{k^2}{k^2-m^2}\left(k^2-m^2+m^2\right) = \int \overset{0 \text{ scaleless}}{\cancel{k^2}} + \frac{k^2 m^2}{k^2-m^2} = \frac{i m^6}{16\pi^2}\left(\frac{1}{\epsilon}+1-\log\frac{m^2}{\mu^2}\right)$$

$$= \frac{-i}{16\pi^2}\left(\frac{1}{\epsilon}+1-\log\frac{m^2}{\mu^2}\right)\left(\frac{m^2}{M^2}\right)\left[1+\left(\frac{m^2}{M^2}\right)+\dots\right] =$$

$$= -\frac{i}{16\pi^2}\left(\frac{1}{\epsilon}+1-\log\frac{m^2}{\mu^2}\right)\frac{m^2}{M^2-m^2}$$

5.- Compute, using the expansion by regions,

$$I_F = \mu^{2\epsilon}\int \frac{1}{k^2-M^2}\,\frac{1}{k^2-m^2}$$

Solution:

$$I_F = \frac{\mu^{2\epsilon}}{M^2-m^2}\int\left(\frac{1}{k^2-M^2}-\frac{1}{k^2-m^2}\right)$$

$$I_F^{(s)} = \frac{\mu^{2\epsilon}}{M^2-m^2}\int\left[\overbrace{-\frac{1}{M^2}\left(1+\frac{k^2}{M^2}+\dots\right)}^{\text{scaleless}} - \frac{1}{k^2-m^2}\right] = -\frac{i}{16\pi^2}\left(\frac{1}{\epsilon}+1+\log\frac{\mu^2}{m^2}\right)\frac{m^2}{M^2-m^2}$$

$$I_F^{(h)} = \frac{\mu^{2\epsilon}}{M^2-m^2}\int\left[\frac{1}{k^2-M^2} - \underbrace{\frac{1}{k^2}\left(1+\frac{m^2}{k^2}+\dots\right)}_{\text{scaleless}}\right] = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon}+1+\log\frac{\mu^2}{M^2}\right)\frac{M^2}{M^2-m^2}$$

$$I_F^{(s)}+I_F^{(h)} = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon}+1+\log\frac{\mu^2}{M^2}+\frac{m^2}{M^2-m^2}\log\frac{m^2}{M^2}\right] = I_F$$

6.- Show that a basis of scalar 4-fermion operators with 2 derivatives can be chosen as

$$O_1 = \bar\psi\psi\,\bar\psi\partial^2\psi + \text{h.c.}$$

$$O_2 = \bar\psi\psi\,\partial_\mu\bar\psi\partial^\mu\psi$$

$$O_3 = \bar\psi\partial_\mu\psi\,\partial_\mu\bar\psi\psi$$

Perform the complete off-shell tree level matching using the $\psi\psi\to\psi\psi$ scattering and show it agrees with the functional result.

Solution

$$O_1 = \bar\psi\psi\,\bar\psi\partial^2\psi + \text{h.c.}$$

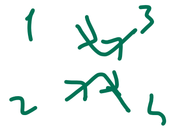
$$O_2 = \bar\psi\psi\,\partial_\mu\bar\psi\partial^\mu\psi$$

$$O_3 = \bar\psi\partial_\mu\psi\,\partial_\mu\bar\psi\psi$$

$$O_4 = \bar\psi\partial_\mu\psi\,\bar\psi\partial_\mu\psi + \text{h.c.}$$

In total we have 6 real operators.

Ibp relates one to the others



Let's remove O_4 via Ibp.

$$O_4 = \bar\psi\partial_\mu\psi\,\bar\psi\partial_\mu\psi = -\partial_\mu\bar\psi\psi\,\bar\psi\partial_\mu\psi - \bar\psi\psi\,\partial_\mu\bar\psi\partial_\mu\psi - \bar\psi\psi\,\bar\psi\partial^2\psi$$
$$= -O_3 - O_2 - O_1$$

Feynman rules

$$\alpha_1 O_1 \to -i\alpha_1(p_1^2+p_2^2) - (3\leftrightarrow 4), \quad \alpha_1^* O_1^\dagger \to -i\alpha_1^*(p_3^2+p_4^2) - (3\leftrightarrow 4)$$

$$\alpha_2 O_2 \to +i\alpha_2(p_1\cdot p_3 + p_2\cdot p_4) - (3\leftrightarrow 4)$$

$$\alpha_3 O_3 \to +i\alpha_3(p_1\cdot p_4 + p_2\cdot p_3) - (3\leftrightarrow 4)$$

p_3 and p_4 are outgoing (p_1 and p_2 incoming).

The full theory amplitude at this order is

$$i\mathcal{M}_F = \frac{i d^2}{M^4}\left(p_1^2+p_3^2-2p_1\cdot p_3\right)\bar{u}_3 u_1 \bar{u}_4 u_2 \quad -(3\leftrightarrow 4)$$

$$i\mathcal{M}_E = -i\left[\alpha_1\left(p_1^2+p_2^2\right)+\alpha_1^*\left(p_3^2+p_4^2\right) - \alpha_2\left(p_1\cdot p_3+p_2\cdot p_4\right)-\alpha_3\left(p_1\cdot p_4+p_2\cdot p_3\right)\right]-(3\leftrightarrow 4)$$

Let's use momentum conservation $p_4 \to p_1+p_2-p_3$

$$= -i\left[(\alpha_1+\alpha_1^*-\alpha_3)p_1^2+(\alpha_1+\alpha_1^*-\alpha_2)p_2^2+2\alpha_1^* p_3^2 + (2\alpha_1^*-\alpha_2-\alpha_3)p_1\cdot p_2+(-2\alpha_1^*-\alpha_2+\alpha_3)p_1\cdot p_3 + (-2\alpha_1^*+\alpha_2-\alpha_3)p_2\cdot p_3\right]$$

Equating we get

$$\alpha_1+\alpha_1^*-\alpha_3 = -\frac{d^2}{M^4} \quad (p_1^2)$$

$$2\alpha_1^* = -\frac{d^2}{M^4} \quad (p_3^2)$$

$$-2\alpha_1^*-\alpha_2+\alpha_3 = \frac{2d^2}{M^4} \quad (p_1\cdot p_3)$$

$$\alpha_3 = 0$$

$$\alpha_2 = -\frac{2d^2}{M^4}-2\alpha_1^* = -\frac{d^2}{M^4}$$

Then

$$i\mathcal{M}_E = \frac{i d^2}{M^2}\left(p_1^2+p_3^2-2p_1\cdot p_3\right) \checkmark$$

Thus

$$\mathcal{L}_{EFT} = -\frac{d^2}{2M^4}\left(\bar{\psi}\psi\bar{\psi}\partial^2\psi + h.c.\right) - \frac{d^2}{M^4}\bar{\psi}\psi\partial_\mu\bar{\psi}\partial^\mu\psi$$

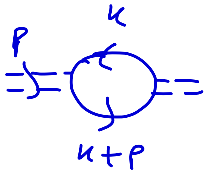
8.- Compute the λ^4 contribution to $\psi\psi \to \psi\psi$ in the full theory (use Package-X in Mathematica).

Solution

$$
\begin{aligned}
(a) &= (-i\lambda)^4 \int \frac{d^dk}{(2\pi)^d} \overline{u}_3 \frac{i(\not{k}+\sigma)}{k^2-\sigma^2} u_1 \, \overline{u}_4 i \frac{i(-\not{k}+\sigma)}{k^2-\sigma^2} u_2 \frac{i^2}{(k^2-M^2)^2} \\
&= \lambda^4 \Bigg[-\overline{u}_3 \gamma^\alpha u_1 \, \overline{u}_4 \gamma^\beta u_2 \int \frac{d^dk}{(2\pi)^d} \frac{k_\alpha k_\beta}{(k^2-\sigma^2)^2 (k^2-M^2)^2} \\
&\quad + \overline{u}_3 u_1 \, \overline{u}_4 u_2 \int \frac{d^dk}{(2\pi)^d} \frac{\sigma^2}{(k^2-\sigma^2)^2 (k^2-M^2)^2} \Bigg].
\end{aligned}
$$

The rest is done in Mathematica.

9.- Compute the 1-loop renormalization of λ and M in the full theory.



$$
= (-1)(-i\lambda\mu^\epsilon)^2 \int \frac{i^2 \, \mathrm{Tr}\left[(\not{k}+\not{p}+m)(\not{k}+m)\right]}{((k+p)^2-m^2)(k^2-m^2)}
$$

$$
\mathrm{Num} = 4(k^2+m^2+k\cdot p)
$$

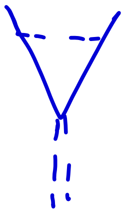
$$
\begin{aligned}
\frac{1}{(k+p)^2-m^2} &= \frac{1}{k^2-m^2}\left[1-\frac{p^2+2k\cdot p}{k^2-m^2}\left(1-\frac{p^2+2k\cdot p}{k^2-m^2}+\dots\right)\right] \\
&= \frac{1}{k^2-m^2} - \frac{p^2-2k\cdot p}{(k^2-m^2)^2} + \frac{4(k\cdot p)^2}{(k^2-m^2)^3}
\end{aligned}
$$

$$
\Rightarrow \frac{1}{4}\frac{\mathrm{Num}}{\mathrm{Den}} = \frac{k^2+m^2}{(k^2-m^2)^2} - \frac{(k^2+m^2)p^2+2(k\cdot p)^2}{(k^2-m^2)^3} + \frac{4(k^2+m^2)(k\cdot p)^2}{(k^2-m^2)^4} + O(p^3)
$$

$$
I = \frac{4\lambda^2 i}{16\pi^2}\left\{ p^2\left[\frac{1/2}{\epsilon}+\frac{1}{2}\log\frac{\mu^2}{m^2}-\frac{1}{3}\right] - m^2\left[\frac{3}{\epsilon}+1+3\log\frac{\mu^2}{m^2}\right]+\dots\right\}
$$

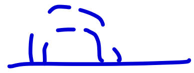
$$
\Rightarrow \mathcal{L}_\Phi^{\text{quad.}} = \left(1+\frac{2\lambda^2}{16\pi^2\epsilon}\right)\frac{1}{2}(\partial_\mu\Phi)^2 - M^2\left(1+\frac{12\lambda^2}{16\pi^2\epsilon}\frac{m^2}{M^2}\right)\Phi^2+\dots
$$

We also need λ. (we can set $p=0$ since we are interested in the UV divergence).



$$= (-i\lambda)(-iy)^2 \int \frac{ik\!\!\!/ \; ik\!\!\!/ \; i}{(k^2-m^2)^3} \sim \lambda y^2 \int \frac{1}{k^4} = \frac{i\lambda y^2}{16\pi^2}\frac{1}{\epsilon} + \dots$$

$$= \frac{i\lambda^3}{16\pi^2}\frac{1}{\epsilon} + \dots$$

We already did . We have to add  but they are the same. (up to couplings).

Thus

$$\mathcal{L} = \left(1 + \frac{1}{2}\frac{y^2+\lambda^2}{16\pi^2\epsilon}\right)\bar\Psi i\partial\!\!\!/ \Psi$$

$$+ \left(1 + \frac{2\lambda^2}{16\pi^2\epsilon}\right)\frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}m^2\Phi^2$$

we neglect $O(\frac{m^2}{M^2})$.

$$-\lambda\left(1 - \frac{y^2+\lambda^2}{16\pi^2\epsilon}\right)\bar\Psi\Psi\Phi + \dots$$

canonical normalization.

$$\longrightarrow \bar\Psi i\partial\!\!\!/ \Psi + \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}m^2\left(1 - \frac{2\lambda^2}{16\pi^2\epsilon}\right)\Phi^2$$

$$-\lambda\left[1 - \frac{1}{16\pi^2\epsilon}\left(y^2+\lambda^2+\frac{y^2+\lambda^2}{2}+\lambda^2\right)\right]\bar\Psi\Psi\Phi$$

$$= \bar\Psi i\partial\!\!\!/ \Psi + \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}m^2\left(1 - \frac{2\lambda^2}{16\pi^2\epsilon}\right)\Phi^2$$

$$-\lambda\left[1 - \frac{1}{16\pi^2\epsilon}\left(\frac{3}{2}y^2 + \frac{5}{2}\lambda^2\right)\right]\bar\Psi\Psi\Phi$$

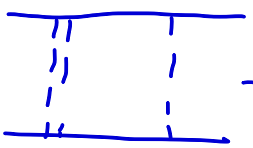
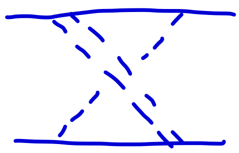
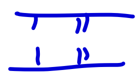
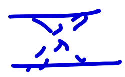
$$\Rightarrow \kappa_{M^2} = \frac{2\lambda^2}{16\pi^2}\,, \qquad \kappa_\lambda = \frac{1}{16\pi^2}\left(\frac{3}{2}y^2 + \frac{5}{2}\lambda^2\right)$$

$$\frac{d}{d\ln\mu} M^2 = M^2 \frac{2\lambda^2}{16\pi^2}\left[2\, \overbrace{n_\lambda}^{1}\right] = M^2 \frac{4\lambda^2}{16\pi^2}$$

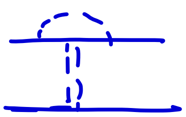
$$\frac{d}{d\ln\mu}\lambda = \lambda \frac{1}{16\pi^2}\left[3y^2 \underbrace{n_y}_{1} + 5\lambda^2 \underbrace{n_\lambda}_{1}\right] = \frac{\lambda}{16\pi^2}\left(3y^2 + 5\lambda^2\right)$$

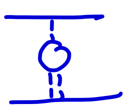
10.- Compute the $\lambda^2 y^2$ contribution to the 4-fermion operator at dim 6 and one-loop (use the efficient way).

Solution

 +  $= O\left(\frac{m^2}{M^4}\right)$ dim 8 (same for  + ).

All others are either scaleless or not 1ℓPI.

 scaleless

 ,  not 1ℓPI.

So there is no contribution at dim 6.