Hydrodynamic model for particle beam-driven wakefield in CNT

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1. Introduction

• The current state-of-the-art of the RF techniques for particle acceleration is limited to gradients on the order of 100 MV/m

• To obtain higher energies, we can increase the length of the accelerators... or use new techniques of acceleration with higher gradients

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1. Introduction

• Plasma-based accelerators $\rightarrow$ GeV/cm accelerating gradients (wakefield amplitude $\propto$ plasma density):

$$E_0 [\text{V/m}] = \frac{m_e c \omega_p}{e} \approx 96 \sqrt{n_0 [\text{cm}^{-3}]}$$

• Density of charge carriers in solids $\rightarrow$ 4–5 orders of magnitude higher than those in a gaseous plasma

• Solid-based acceleration media, such as crystals or nanostructures, could enable ultra-high gradients, on the order of $E_0 \sim 1$–10 TV/m

• If compared to natural crystals, 2D carbon-based structures could help achieving more realistic regimes:
  • larger dimensional flexibility / thermomechanical strength
  • transverse acceptances of up to $\sim 100$ nm (3 orders of magnitude higher than a typical silicon channel)
  • lower dechanneling rate

• 2D carbon-based materials (graphene, CNT) are good candidates to be used as ultra-compact accelerating structures
1. Introduction

- **Plasmonic acceleration**
  - Excitation of surface plasmonic modes by laser (laser-driven) or charged particle beam (beam-driven)
  - Collective motion of wall electrons acting like a structured plasma
  - To properly excite wakefields laser or beam driving parameters need to be in the time and space scale of the plasmon wave

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<th>RF cavities</th>
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**2D plasmon**

- **Dielectric**
- **Metal**
- **E_d**
- **E_m**
- **ε_d**
- **ε_m**
- **TM wave**
2. Theoretical background

• **Geometry**
  
  • *2D electron gas* confined in the cylindrical surface of a nanotube or microtube (made of CNTs, metallic nature)
  
  • Let us assume a *point-like charge* $q$ travelling with speed $v \approx c$ inside a single walled tube

\[
\begin{align*}
\mathbf{r}_0 &= (r_0, \varphi_0, z_0) \\
\mathbf{r}_a &= (a, \varphi_a, z_a)
\end{align*}
\]

2. Theoretical background

• Electronic excitation on the tube wall:
  
  • Continuity equation:

  \[
  \partial_t n_1(r_a, t) + n_0 \nabla_\parallel \cdot \mathbf{u}(r_a, t) = 0
  \]

  Perturbed surface plasma density  \quad \text{velocity of the plasma  \quad Unperturbed surface plasma density}

  • Momentum-balance equation:

  \[
  \partial_t \mathbf{u}(r_a, t) = \frac{e}{m_e} \nabla_\parallel \Phi(r_a, t) - \frac{\alpha}{n_0} \nabla_\parallel n_1(r_a, t) - \gamma \mathbf{u}(r_a, t) + \frac{\beta}{n_0} \nabla_\parallel [\nabla_\parallel^2 n_1(r_a, t)]
  \]

  Thomas-Fermi electron fluid pressure  \quad Frictional force  \quad Quantum correction to the semiclassical Thomas-Fermi theory

  \quad Acoustic modes
2. Theoretical background

• Electronic excitation on the tube wall:

  • Poisson’s equation:

    \[ \nabla^2 \Phi(r, t) = \frac{1}{\varepsilon_0} \left[ en_1(r_a, t) - q\delta(r - r_0) \right] \]

    Contribution from plasma perturbation  Contribution from driving charge

  • Boundary conditions:
    - Vanishing normal component of the velocity perturbation at the surface of the cylinder
    - Vanishing of the potential as \( r \rightarrow \infty \)
    - The finiteness of the electric potential at the origin

These boundary conditions will allow the expansion of the potential and the charge density disturbance in terms of modified Bessel functions \( I_m \) and \( K_m \) of integer order \( m \)
2. Theoretical background

• The total potential inside the tube: \( \Phi_{in} = \Phi_0 + \Phi_{ind} \)

  – Coulomb potential from the driving charge:

\[
\Phi_0(r, \varphi, z, t) = \frac{1}{4\pi \varepsilon_0} \frac{q}{||r - r_0||} = \frac{q}{4\pi^2 \varepsilon_0} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk e^{ik\zeta + im(\varphi - \varphi_0)} I_m(|k|r_<) K_m(|k|r_>) 
\]

  Modified Bessel functions

  – Induced potential from the tube surface electron fluid:

\[
\Phi_{ind}(r, \varphi, z, t) = \frac{q}{4\pi^2 \varepsilon_0} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk e^{ik\zeta + im(\varphi - \varphi_0)} I_m(|k|r_<) I_m(|k|r) A_m(k) 
\]

\( \zeta = z - vt \) : comoving coordinate; \( r_< = \min(r, r_0), \ r_> = \max(r, r_0) \)
2. Theoretical background

- The total potential outside the tube:

\[ \Phi_{out}(r, \varphi, z, t) = \frac{q}{4\pi^2\varepsilon_0} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk e^{ikz+i(m-\varphi_0)} I_m(|k|r_0) K_m(|k|a)B_m(k) \]

Boundary conditions at \( r = a \):

\[ \Phi_{in}(r, \varphi, z, t) \bigg|_{r=a} = \Phi_{out}(r, \varphi, z, t) \bigg|_{r=a} \]

\[ \frac{\partial \Phi_{out}(r, \varphi, z, t)}{\partial r} \bigg|_{r=a} - \frac{\partial \Phi_{in}(r, \varphi, z, t)}{\partial r} \bigg|_{r=a} = n_1(\varphi, z, t) \]

\[ A_m(k) = \frac{\Omega^2 a^2 (k^2 + m^2/a^2) K^2_m(ka)}{kv(kv + i\gamma) - \omega^2_m(k)} \]

\[ B_m(k) = \frac{kv(kv + i\gamma) - \alpha(k^2 + m^2/a^2) - \beta(k^2 + m^2/a^2)^2}{kv(kv + i\gamma) - \omega^2_m(k)} \]

Resonance at \( kv = \omega_m(k) \)
2. Theoretical background

- **Resonance frequencies:**

\[
\omega_m(k) = \left[ \alpha \left( \frac{k^2 + m^2}{a^2} \right) + \beta \left( \frac{k^2 + m^2}{a^2} \right)^2 + \Omega_p^2 a^2 \left( \frac{k^2 + m^2}{a^2} \right) K_m(|k|a) I_m(|k|a) \right]^{1/2}
\]

- **Correction for acoustic modes in the plasma**
- **Quantum correction to the semiclassical Thomas-Fermi theory**

\[
\Omega_p = \sqrt{\frac{e^2 n_0}{\epsilon_0 m_e a}} \quad \text{Plasma frequency}
\]
2. Theoretical background

- Resonance frequencies: 
  \[ \omega_m(k) = \left[ \alpha \left( k^2 + \frac{m^2}{a^2} \right) + \beta \left( k^2 + \frac{m^2}{a^2} \right)^2 + \Omega_p^2 a^2 \left( k^2 + \frac{m^2}{a^2} \right) K_m(|k|a)I_m(|k|a) \right]^{1/2} \]

Density of electrons: \( n_0/a = 10^{26} \text{ m}^{-3} \)
Nanotube radius: \( a = 0.2\lambda_p \approx 0.67 \mu\text{m} \)
Correction parameters:
\[ \alpha = \frac{\hbar}{m_e} \left( \frac{3\pi^2 n_0}{a} \right)^{1/3}, \beta = \frac{1}{4} \left( \frac{\hbar}{m_e} \right)^2, \gamma = 0.05\Omega_p \]

- If the radius \( a \) or the density \( n_0/a \) increases, then \( \omega_m \) increases and the resonance condition moves to higher \( k \).
- If the velocity decreases, the resonance condition moves to higher \( k \).
- If \( v \) is non-relativistic all modes have the same resonance \( k \).

The resonance condition for particle acceleration is achieved whenever the dashed particle velocity lines intercept the dispersion curves.
2. Theoretical background

• Longitudinal electric wakefield inside the tube:

\[ W_z = -\partial_z \Phi_{in} \]

\[
W_z(r, \varphi, z, t) = \frac{q}{4\pi^2\varepsilon_0} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk k e^{im(\varphi - \varphi_0)}
\]

\[
\times \left[ I_m(|k| r_<) K_m(|k| r_>) \sin(k\zeta)
+ I_m(|k| r_0) I_m(|k| r) \left( \Re[A_m(k)] \sin(k\zeta)
+ \Im[A_m(k)] \cos(k\zeta) \right) \right].
\]

Coulomb term

2nd term

3rd term

\[ \zeta = z - vt : \text{comoving coordinate}; \ r_0 = \min(r, r_0), \ r_> = \max(r, r_0) \]
3. Results

• Longitudinal electric wakefield:

Amplitude at different transverse position $r$

For $m = 0$; nanotube radius $a = 0.2\lambda_p \approx 0.7$ $\mu$m
Density of electrons $n_0/a = 1 \times 10^{26}$ $m^{-3}$

Driver parameters: point-like charge $q = -2.75$ pC; $r_0 = 0$ (particle on axis); $v = 0.6c$

Correction parameters: $\alpha = \frac{\hbar}{m_e} (3\pi^2 n_0/a)^{1/3}$; $\beta = \frac{1}{4} \left( \frac{\hbar}{m_e} \right)^2$; $\gamma = 0.05\Omega_p$

$m = 0$ is the only mode excited

Electric fields increase near to the nanotube surface
3. Results

- **Longitudinal electric wakefield:**

   For different driver velocities

   For $m = 0$; nanotube radius $a = 0.2 \lambda_p \simeq 0.7 \mu m$

   Density of electrons $n_0 / a = 1 \times 10^{26} m^{-3}$

   Driver parameters: point-like charge $q = -2.75 \text{ pC}$; $r_0 = 0$ (particle on axis); $r = 0$

   Correction parameters: $\alpha = \frac{\hbar}{m_e} (3\pi^2 n_0 / a)^{1/3}$; $\beta = \frac{1}{4} \left( \frac{\hbar}{m_e} \right)^2$; $\gamma = 0.05 \Omega_p$

   m=0 is the only mode excited

   The plasmon wavelength increases as $v$ increases. If $v$ is too small there is no excitation.

   Electric fields increase as $v$ increases, although for $v > 0.8c$ there is no big change (for the density used).
3. Results

• Longitudinal electric wakefield:

Comparison with PIC codes
3. Results

- **Longitudinal electric wakefield:**

  Depending on friction parameter $\gamma$

  For $m = 0$; nanotube radius $a = 0.2\lambda_p \simeq 0.7 \mu m$
  Density of electrons $n_0/a = 1 \times 10^{26} \text{ m}^{-3}$
  Driver parameters: point-like charge $q = -2.75 \text{ pC}$; $r_0 = 0$ (particle on axis); $v = 0.6c$; $r = 0$

  Correction parameters:
  $\alpha = \frac{\hbar}{m_e} (3\pi^2 n_0/a)^{1/3}$; $\beta = \frac{1}{4} \left( \frac{\hbar}{m_e} \right)^2$; $\gamma = \cdots$

  $\gamma = 0.0005\Omega_p$

  $\gamma = 0.05\Omega_p$

  $\gamma = 0.5\Omega_p$

For $\gamma < 0.005\Omega_p$, electric fields are practically non-attenuated (hollow plasma).
If $\gamma$ increases there is no excitation of plasmons.
3. Results

• Longitudinal electric wakefield:

If $\gamma \to 0^+$, the third term of the longitudinal electric wakefield can be calculated via the residue theorem:

$$W_{z}^{3rd}(r, \varphi, z, t) = -\frac{q}{4\pi^2 \varepsilon_0} (2\pi) \sum_{m=-\infty}^{+\infty} e^{im(\varphi-\varphi_0)} k_m l_m(|k_m| r_0) l_m(|k_m| r) \Omega_p^2 a^2 \left( \frac{k_m^2 + m^2}{a^2} \right) K_m(|k_m| a) \left| \frac{\partial Z_m}{\partial k} \right|^{-1}_{k=k_m} \cos(k_m \zeta),$$

$L_m$: amplitude of the wakefield due to the third term for $\gamma \to 0^+$

(The total amplitude is approximately twice because of the second term)

We have defined the quantity:

$$Z_m(k) = (k \nu)^2 - \omega_m^2(k),$$

and $k_m$ is determined by the condition of the plasma resonance,

$$Z_m(k_m) = (k_m \nu)^2 - \omega_m^2(k_m) = 0$$
3. Results

• Longitudinal electric wakefield:

If $\gamma \to 0^+$:

$v = 0.6c, \quad r = r_0 = 0$

Correction parameters:

$\alpha = \frac{\hbar}{m_e} \left( \frac{3\pi^2 n_0}{a} \right)^{1/3}, \quad \beta = \frac{1}{4} \left( \frac{\hbar}{m_e} \right)^2$
3. Results

- Longitudinal electric wakefield:

If $\gamma \to 0^+$:

\[ v = 0.99c, \quad r = r_0 = 0 \]

For higher velocities, the maximum wakefield is lower and the best radius increases (in units of the plasmon wavelength)
3. Results

- Longitudinal electric wakefield:

If $\gamma \to 0^+$:

Despite the discrepancies, the analytical model allows us to predict an optimal radius.
3. Results

• Transversal electric wakefield:

\[
W_r = -\partial_r \Phi_{in} = -\frac{q}{4\pi \varepsilon_0} \partial_r \left( \frac{1}{||r - r_0||} \right) + \\
- \frac{q}{4\pi^2 \varepsilon_0} \sum_{m=-\infty}^{+\infty} e^{im(\phi - \phi_0)} \int_{-\infty}^{+\infty} dkk [l_m(|k| r_0)l_m(|k| r)(\text{Re}[A_m(k)]) \cos(k\zeta) - \text{Im}[A_m(k)] \sin(k\zeta))] = \\
= W_r^{\text{Coulomb}}(r, \phi, z, t) + W_r^{2nd}(r, \phi, z, t) + W_r^{3rd}(r, \phi, z, t)
\]

If \(\gamma \to 0^+\):

\[
W_z^{3rd}(r, \phi, z, t) = \frac{q}{4\pi^2 \varepsilon_0} \sum_{m=-\infty}^{+\infty} e^{im(\phi - \phi_0)} \int_{-\infty}^{+\infty} dkk l_m(|k| r_0)l'_m(|k| r)\text{Im}[A_m(k)] \sin(k\zeta) = \\
= -\frac{q}{4\pi^2 \varepsilon_0} (2\pi) \sum_{m=-\infty}^{+\infty} e^{im(\phi - \phi_0)} k_m l_m(|k_m| r_0)l'_m(|k_m| r) \Omega_p^2 a^2 \left( k_m^2 + \frac{m^2}{a^2} \right) K_m^2(|k_m| a) \left| \frac{\partial Z_m}{\partial k} \right|^{-1}_{k=k_m} \sin(k_m \zeta),
\]
3. Results

• Transversal electric wakefield:

If $\gamma \rightarrow 0^+$:

\[ n_0 = 10^{26} \text{m}^{-3}, \quad v = 0.99c, \quad r = 0.5a, \quad r_0 = 0, \quad a = 0.2\lambda_p \]
4. Conclusions and outlook

- We have presented a **linear hydrodynamic model for wakefield excitation** by charges in nanotubes and micro-tubes.
- A driver particle can excite **plasmons** in nanotubes.
- The electric fields are **higher** if the particle travels **near** to the CNT **surface**. For this reason, it is better to use CNT with small radius (about $0.1\lambda_p$).
- The results show **>GV/m fields** and are **qualitatively similar to the PIC simulations** and the order of magnitude is similar. Systematic comparisons with PIC simulations are ongoing.
- Changing the values of $\alpha$ and $\beta$ 2-3 orders of magnitude (respect to the typical values of these parameters) does not affect to the results.
- All the calculations have been made for the **mode m=0** (since if $r_0=0$ is the only mode excited). **Higher modes are less important**, but they are not negligible if the driver is off-axis near to the surface.
- In the limit $\gamma \to 0^+$ we can **calculate easily the excited wakefield** and the dependences on the different parameters.
Acknowledgments

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Thank you for your attention
BACKUP SLIDES
3. Results

• Longitudinal electric wakefield:

If $\gamma \to 0^+$:

$$a = 0.2\lambda_p; \quad r = 0.5a; \quad r_0 = 0.2a$$