Mathematical challenges for plasmas in equilibrium

Daniel Peralta-Salas

Instituto de Ciencias Matemáticas (ICMAT), Madrid

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The equations governing the plasma dynamics

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The area was initiated by Hannes Alfvén, Nobel prize in Physics 1970: "for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics."

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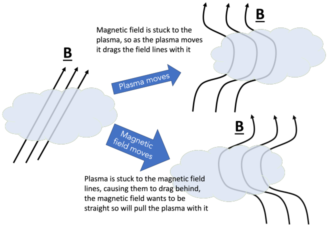
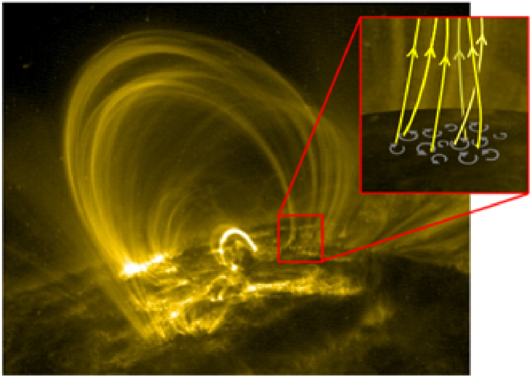
$\Longrightarrow$ the topology of the magnetic lines does not change!



Solar active regions

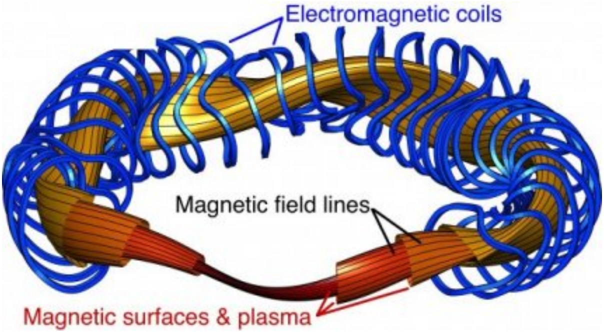
Ideal MHD is a good approximation (diffusion time is much longer than lifetime of a sunspot):



$\Longrightarrow$ It is relevant to understand the relaxation process of magnetic tubes.

Plasma confinement

Ideal MHD does not describe the fusion plasma regime. However, it is justified near equilibrium configurations:



$\Longrightarrow$ It is relevant to understand optimal geometries for plasma confinement (the stellarator program).

Magnetohydrostatic (MHS) equilibria

In several physical processes it is relevant to understand the equilibrium situation: plasma velocity $v = 0$ and B does not depend on t

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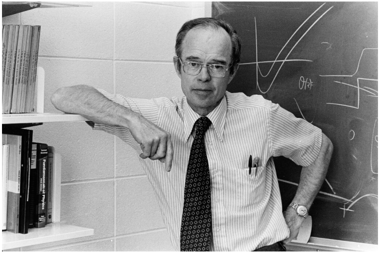
$$\begin{aligned} B \times \operatorname{curl} B + \nabla P &= 0 \\ \operatorname{div} B &= 0 \end{aligned}$$

Distinguished physicists Harold Grad and Eugene Parker stated two mathematical problems concerning MHS equilibria, with physical significance. Both problems remain widely open.

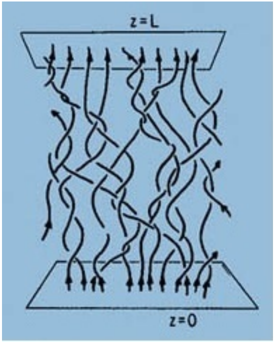


Mathematical challenge 1: Parker's hypothesis

In 1972 Eugene Parker (solar physicist) launched a hypothesis regarding the fundamental nature of equilibrium magnetic fields in astrophysical plasmas:

Parker's Hypothesis

A braided magnetic field will in general not relax towards a smooth equilibrium under ideal evolution, but rather towards a state containing discontinuities.



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Mathematical difficulties:

- A precise mathematical formulation is not clear: meaning of "in general"? It is necessary to introduce a topology in the space of braided fields. A usual model is to consider the fields in the vertical cylinder or in a toroidal domain.
- The problem involves PDEs, dynamical systems and differential topology $\Longrightarrow$ a deep interplay between very different areas of mathematics! In particular, it is unclear which magnetic line footpoint maps can be realized by MHS equilibria.

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Relation to Moffatt's relaxation hypothesis (1985)

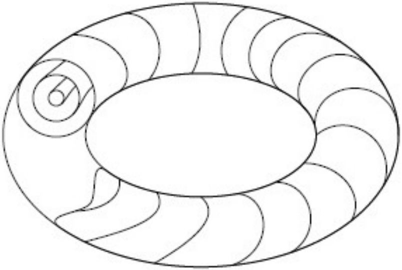
Let B_0 be a smooth magnetic field in a bounded domain. Then there is a MHS equilibrium B_E (possibly with tangential discontinuities) that is topologically accessible from B_0 (under ideal MHD relaxation).

Mathematical challenge 2: Grad's conjecture

In 1967 Harold Grad (plasma physicist) posed a conjecture regarding the topological structure of equilibrium states in confined regions:

Grad's Conjecture

Any smooth MHS equilibrium (non-isolated) on a toroidal domain whose plasma pressure defines a foliation consisting of nested toroidal surfaces is axisymmetric.



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Mathematical difficulties:

- When P is not constant, we deal with a fully nonlinear PDE $\Longrightarrow$ not much is understood about the existence of non-symmetric solutions.
- The "nested toroidal surfaces" structure is difficult to control: rational invariant tori can be easily destroyed. Again this problem requires a deep interaction between PDEs and dynamical systems.

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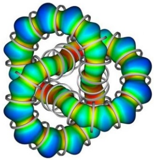

Figure: Knotatron, Stuart Hudson's dream (Princeton PPL) of magnetic confinement.

En el ICMAT trabajamos para resolver estos (y otros muchos) problemas ... esperamos aportar nuevos resultados en un futuro próximo.

¡Muchas gracias por su atención!