Rigidity and flexibility for low-dimensional manifolds

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Research lines at ICMAT

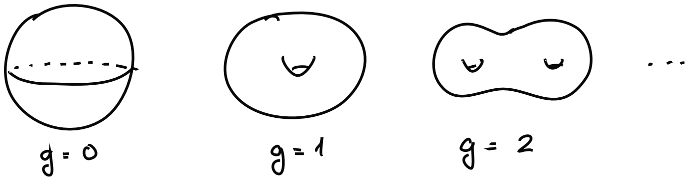
Research at ICMAT is channeled through three lines:

- **Algebra and Geometry**. Geometry in various flavours; geometric mechanics; group theory; number theory.
- **Analysis and Differential Equations**. Harmonic analysis, fluid mechanics, Calderón-Zygmund theory.
- **Applied Mathematics**. Quantum information theory; data science; modelling.

I. Surfaces and their deformation spaces

Classification of surfaces

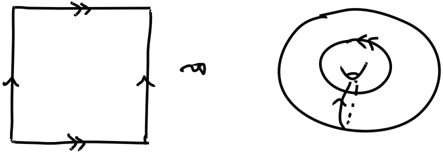
Compact surfaces are (topologically) classified by their *genus* $g \geq 0$.



Metrics of constant curvature

S_g admits a metric of constant curvature

- $+1$, when $g = 0$;
- 0, when $g = 1$;
- -1, when $g \geq 2$.



Metrics of constant curvature

Moreover, any such constant-curvature metric is obtained as

$$X/\Gamma,$$

where X is either $\mathbb{S}^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$, and Γ is a discrete subgroup of the relevant isometry group. Moreover, $\Gamma \cong \pi_1(S_g)$.

Moduli space of surfaces

- $\mathcal{M}_g$: the space of *distinct* constant curvature metrics on S_g.
- When $g \geq 2$, it has (real) dimension $6g - 6$.
- Several other (equivalent) interpretations, in terms of *Riemann surfaces*, *algebraic curves*...
- Of great importance in many different areas.

Moduli space of representations

(The universal cover of) $\mathcal{M}_g$ is a subset of

$$\mathrm{Hom}(\pi_1(S_g), \mathsf{PSL}(2, \mathbb{R}))$$

i.e. those that are *discrete* and *injective*.

Hot topic: higher-dimensional representations

Understand structure of

$$\mathrm{Hom}(\pi_1(S_g), G),$$

for G is a higher-rank Lie group, e.g. $\mathrm{SL}(n, \mathbb{R})$.

García-Prada *et al* (*Invent. Math.* 2019)

Determine number of connected components, when $G = \mathrm{SO}(p, q)$.

II. Three-dimensional manifolds

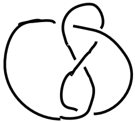
Thurston's Geometrization Conjecture

Perelman after Thurston

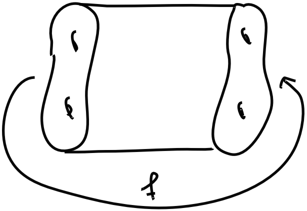
Any compact three-dimensional manifold may be *decomposed* into *geometric pieces*, each modeled on one of eight *geometries*.

Hyperbolic 3-manifolds

- Thurston: generically, *knot complements* in $\mathbb{S}^3$ are hyperbolic.



- Thurston: generically, *surface bundles* over $\mathbb{S}^1$ are hyperbolic.



$$M_g = S_g \times [0,1] / (x,0) \sim (f(x),1)$$

Rigidity of hyperbolic 3-manifolds

Mostow rigidity

Suppose M, N are compact hyperbolic 3-manifolds. If M and N are homeomorphic, then they are isometric.

In particular, volume is a topological invariant!

Hot topic: Profinite rigidity

Let M be a 3-manifold. What properties are recognized by the set of finite quotients of $\pi_1(M)$?

Liu (*Invent. Math.* 2020)

If M is hyperbolic, the set of finite quotients of $\pi_1(M)$ determines M, up to finite ambiguity.

Jaikin-Zapirain (*Geom. Top.* 2019)

The set of finite quotients of $\pi_1(M)$ determines whether M fibers over $\mathbb{S}^1$.