

J/ψ and ω decays to 3π with Khuri-Treiman equations



Miguel Albaladejo (IFIC-CSIC)

Precision tests of fundamental
physics with light mesons
ECT* (Trento) Jun. 12-16, 2023

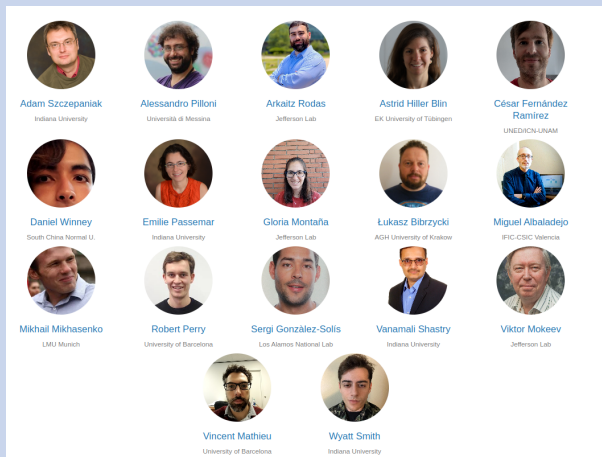


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IN NUCLEAR PHYSICS AND RELATED AREAS

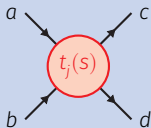


JPAC: Joint Physics Analysis Center

- Work in **theoretical/experimental/phenomenological** analysis
- Light/heavy meson **spectroscopy**
- Interaction with many **experimental collaborations**: (GlueX, CLAS, BES, ...) and **LQCD groups**
- Web site: <https://www.jpac-physics.org/>



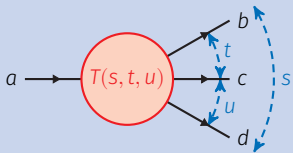
Introduction: Khuri-Treiman equations in a nutshell



$$T(s, t, u) = \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(z_s) t_{\ell}(s)$$

- Partial wave expansion **in the s-channel**:

- Two main (connected) problems:
 - ▶ Infinite number of PW
 - ▶ PW have RHC and LHC
- Only RHC: BS equation, K-matrix, DR,...
- Problem with “truncation”: $t_{\ell}(s)$ only depends on s , so singularities in the t -, u -channel can only appear summing an infinite number of PW.



- In many decay processes one wants to take into account unitarity/FSI interactions in the three possible channels.

Introduction: Khuri-Treiman equations in a nutshell

- Khuri-Treiman equations are a tool to achieve this **two-body unitarity** in the **three channels**
[N. Khuri, S. Treiman, Phys. Rev. **119**, 1115 (1960)]
- Consider three (s -, t -, u -channels) **truncated** “isobar” expansions.
- Isobars $f_\ell^{(s)}(s)$ have only RHC: amenable for **dispersion relations**.

$$\begin{aligned} T(s, t, u) &= \sum_{\ell=0}^{\infty} (2\ell + 1) P_\ell(z_s) t_\ell(s) \\ &= \sum_{\ell=0}^{n_s} (2\ell + 1) P_\ell(z_s) f_\ell^{(s)}(s) + \sum_{\ell=0}^{n_t} (2\ell + 1) P_\ell(z_t) f_\ell^{(t)}(t) + \sum_{\ell=0}^{n_u} (2\ell + 1) P_\ell(z_u) f_\ell^{(u)}(u) \end{aligned}$$

- s -channel singularities appear in the s -channel isobar, $t_\ell^{(s)}(s)$.
- Singularities in the t -, u -channel are recovered!
- The LHC of the partial waves are given by the RHC of the crossed channel isobars

$$t_\ell(s) = \frac{1}{2} \int dz P_\ell(z) T(s, t', u') = f_\ell^{(s)}(s) + \frac{1}{2} \int dz Q_{\ell\ell'}(s, t') f_{\ell'}^{(t)}(t').$$

- A different perspective: *reconstruction theorem*
[Stern, Sazdjian, Fuchs, PR,**D47**, 3814 (1993); Zdráhal, Novotný, PR,**D78**, 116016 (2008)]
- Many works in/about KT equations: Colangelo, Hoferichter, Hoid, Isken, JPAC, Kambor, Kubis, Lanz, Leutwyler, Moussallam, Niecknig, Passemar, Schneider, Wyler...

- Amplitude:

$$\mathcal{M}_+(s, t, u) = \frac{\sqrt{\phi(s, t, u)}}{2} F(s, t, u) . \quad (\phi(s, t, u) = 4sp^2(s)q^2(s) \sin^2 \theta_s)$$

- Decay width: $d^2\Gamma \sim \phi(s, t, u) |F(s, t, u)|^2$
- Dalitz plot parameters (α, β, γ) “equivalent” to bins... $(X, Y) \leftrightarrow (Z, \varphi) \leftrightarrow (s, t, u)$

$$|F(s, t, u)|^2 = |\mathcal{N}|^2 \left(1 + 2\alpha Z + 2\beta Z^{\frac{3}{2}} \sin 3\varphi + 2\gamma Z^2 + \dots \right)$$

- Why revisit $\omega \rightarrow 3\pi$?

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- Why revisit $\omega \rightarrow 3\pi$?

	Bonn (2012)		JPAC (2015)		
	Eur. Phys. J., C72, 2014 (2012)		Phys. Rev., D91, 094029 (2015)		
	w/o KT	w KT	w/o KT	w KT	
α	130(5)	79(5)	125	84	
β	31(2)	26(2)	30	28	

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- Why revisit $\omega \rightarrow 3\pi$?

	Bonn (2012)		JPAC (2015)		BESIII (2018)
	Eur. Phys. J., C72, 2014 (2012)		Phys. Rev., D91, 094029 (2015)		Phys. Rev., D98, 112007 (2018)
	w/o KT	w KT	w/o KT	w KT	Exp.
α	130(5)	79(5)	125	84	120.2(7.1)(3.8)
β	31(2)	26(2)	30	28	29.5(8.0)(5.3)

- One (or more) out of three is wrong...
 - 1) Experiment?
 - 2) KT eqs. in general?
 - 3) Something particular?

KT equations: DR, subtractions, solutions, and all that...

- PW decomposition: $F(s, t, u) = \sum_{j \text{ odd}} P'_j(\cos \theta_s) [\rho(s)q(s)]^{j-1} f_j(s) = f_1(s) + \dots$
- KT/isobar decomposition: consider only $j = 1$ (ρ) isobar, $F(s)$:

$$\underline{F(s, t, u) = F(s) + F(t) + F(u)}$$

- PW projection of the KT decomposition:

$$f_1(s) = F(s) + \hat{F}(s), \quad \hat{F}(s) = \frac{3}{2} \int_{-1}^1 dz_s (1 - z_s^2) F(t(s, z_s))$$

- Discontinuity:

$$\Delta F(s) = \Delta f_1(s) = \rho(s) t_{11}^*(s) f_1(s) = \rho(s) t_{11}^*(s) (F(s) + \hat{F}(s))$$

Unsubtracted DR

$$F(s) = a F_0(s)$$

$$F_0(s) = \Omega(s) \left[1 + \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\sin \delta(s') \hat{F}_0(s')}{|\Omega(s')|(s' - s)} \right]$$

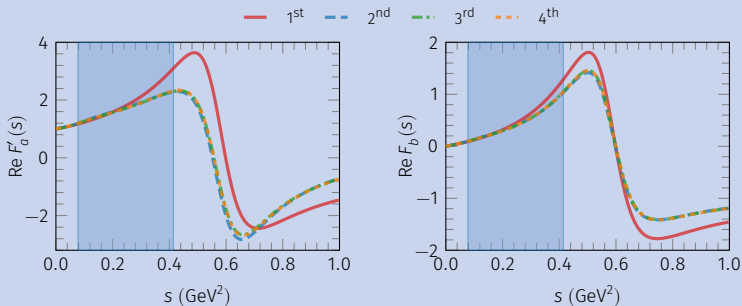
Once-subtracted DR

$$F(s) = a (F'_a(s) + b F_b(s))$$

$$F'_a(s) = \Omega(s) \left[1 + \frac{s^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s') \hat{F}'_a(s')}{|\Omega(s')|(s' - s)} \right]$$

$$F_b(s) = \Omega(s) \left[s + \frac{s^2}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2} \frac{\sin \delta(s') \hat{F}_b(s')}{|\Omega(s')|(s' - s)} \right]$$

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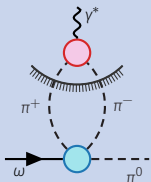
$\omega \rightarrow \pi^0$ transition form factor

- The decays $\omega(\rightarrow \pi^0 \gamma^*) \rightarrow \pi^0 \ell^+ \ell^-$ and $\omega \rightarrow \pi^0 \gamma$ governed by the TFF $f_{\omega\pi^0}(s)$.

$$\mathcal{M}(\omega \rightarrow \pi^0 \ell^+ \ell^-) = f_{\omega\pi^0}(s) \epsilon_{\mu\nu\alpha\beta} \epsilon^\mu(p_\omega, \lambda) p^\nu q^\alpha \frac{ie^2}{s} \bar{u}(p_-) \gamma^\beta v(p_+)$$

$$\Gamma(\omega \rightarrow \pi^0 \gamma) = |f_{\omega\pi^0}(0)|^2 \frac{e^2 (m_\omega^2 - m_{\pi^0}^2)^3}{96\pi m_\omega^3}$$

- Dispersive representation:



$$f_{\omega\pi^0}(s) = f_{\omega\pi^0}(0) + \frac{s}{12\pi^2} \int_{4m_\pi^2}^{\infty} ds' \frac{q_\pi(s')^3}{s'^{\frac{3}{2}}(s' - s)} \left(F(s') + \hat{F}(s') \right) F_\pi^V(s')^*$$

- $f_{\omega\pi^0}(0) = |f_{\omega\pi^0}(0)| e^{i\phi_{\omega\pi^0}(0)}$
- Only the relative phase matters in $\frac{a}{f_{\omega\pi^0}(0)} \propto \exp [i(\phi_a - \phi_{\omega\pi^0}(0))]$.

- Experimental information: $F_{\omega\pi^0}(s) = \frac{f_{\omega\pi^0}(s)}{f_{\omega\pi^0}(0)}$

- ▶ [NA60@CERN-SPS: PL,B757,437('16)]
- ▶ [A2@MAMI: PR,C95,035208('17)]

Summary of amplitudes/free parameters/exp. input

$\omega \rightarrow 3\pi$ amplitude [$F(s, t, u)$]

Free parameters: $|a|, |b|, \phi_b$

Experimental input:

- ▶ $\Gamma_{3\pi}$
- ▶ Dalitz plot parameters

$\omega \rightarrow \gamma^{(*)}\pi^0$ TFF [$f_{\omega\pi^0}(s)$]

Free parameters: $|f_{\omega\pi^0}(0)|, \phi_{\omega\pi^0}(0)$
($\oplus |a|, |b|, \phi_b$)

Experimental input:

- ▶ $\Gamma_{\gamma\pi^0}$
- ▶ $|F_{\omega\pi^0}(s)|^2$

First analysis in three steps

① Fix $|b| \simeq 2.9$, $\phi_b \simeq 1.9$ with the DP parameters.

② Fix $|a| \simeq 280 \text{ GeV}^{-3}$, $|f_{\omega\pi^0}(0)| \simeq 2.3 \text{ GeV}^{-1}$
from $\Gamma_{\omega \rightarrow 3\pi}$, $\Gamma_{\omega \rightarrow \gamma\pi}$.

③ You are left with $\phi_{\omega\pi^0}(0)$ and the TFF Data.

▶ [NA60@CERN-SPS: PL,B757,437('16)]

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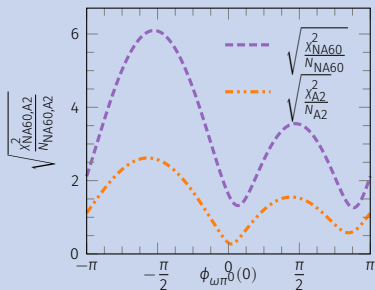
$$\textcircled{1} \chi_{\text{DP}}^2 = \left(\frac{a^{(t)} - a^{(e)}}{\sigma_a} \right)^2 + \dots$$

$$\textcircled{2} \chi_{\Gamma}^2 = \left(\frac{\Gamma_{3\pi}^{(t)} - \Gamma_{3\pi}^{(e)}}{\sigma_{\Gamma_{3\pi}}} \right)^2 + \left(\frac{\Gamma_{\gamma\pi}^{(t)} - \Gamma_{\gamma\pi}^{(e)}}{\sigma_{\Gamma_{\gamma\pi}}} \right)^2$$

$$\textcircled{3} \chi_{\text{A2,NA60}}^2 = \sum_i \left(\frac{|F_{\omega\pi}(s_i)|^2 - |F_{\omega\pi}^{(i)}|^2}{\sigma_{F_{\omega\pi}^{(i)}}} \right)^2$$

First analysis in three steps

- 1 Fix $|b| \simeq 2.9$, $\phi_b \simeq 1.9$ with the DP parameters.
- 2 Fix $|a| \simeq 280 \text{ GeV}^{-3}$, $|f_{\omega\pi^0}(0)| \simeq 2.3 \text{ GeV}^{-1}$ from $\Gamma_{\omega \rightarrow 3\pi}$, $\Gamma_{\omega \rightarrow \gamma\pi}$.
- 3 You are left with $\phi_{\omega\pi^0}(0)$ and the TFF Data.
 - ▶ [NA60@CERN-SPS: PL,B757,437('16)]
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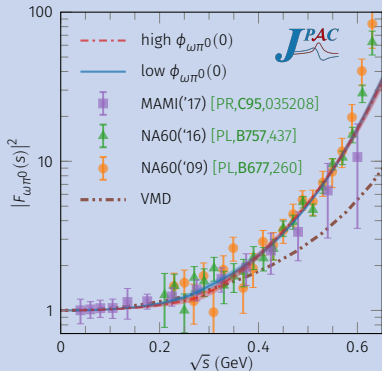


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- 2 $\chi_{\Gamma}^2 = \left(\frac{\Gamma_{3\pi}^{(t)} - \Gamma_{3\pi}^{(e)}}{\sigma_{\Gamma_{3\pi}}} \right)^2 + \left(\frac{\Gamma_{\gamma\pi}^{(t)} - \Gamma_{\gamma\pi}^{(e)}}{\sigma_{\Gamma_{\gamma\pi}}} \right)^2$
- 3 $\chi_{A2, \text{NA60}}^2 = \sum_i \left(\frac{|F_{\omega\pi}(s_i)|^2 - |F_{\omega\pi}^{(i)}|^2}{\sigma_{F_{\omega\pi}}^{(i)}} \right)^2$

- Two different minima (low and high $\phi_{\omega\pi^0}(0)$) are found.
- Both have similar χ^2 for the TFF.

Make a **global, simultaneous** analysis

$$\bar{\chi}^2 = N \left(\frac{\chi_{\text{DP}}^2}{N_{\text{DP}}} + \frac{\chi_{\Gamma}^2}{N_{\Gamma}} + \frac{\chi_{\text{NA60}}^2}{N_{\text{NA60}}} + \frac{\chi_{\text{A2}}^2}{N_{\text{A2}}} \right)$$



	α	β	γ
BESIII	111(18)	25(10)	22(29)
low	112(15)	23(6)	29(6)
high	109(14)	26(6)	19(5)

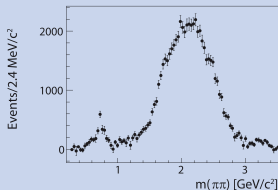
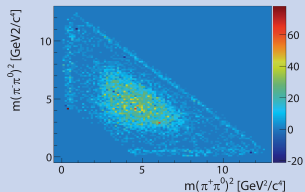
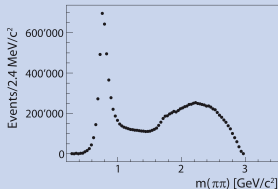
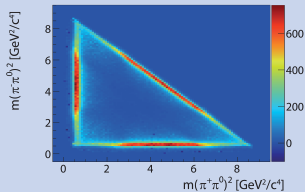
Using once-subtracted DR for KT:

- Agreement is restored with DP parameters by BESIII
- One can also describe the $\omega\pi^0$ TFF

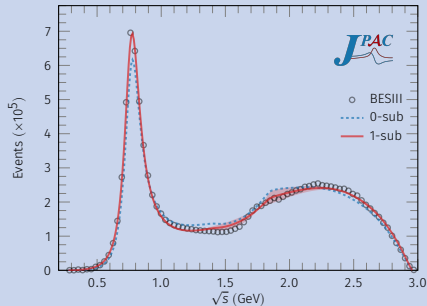
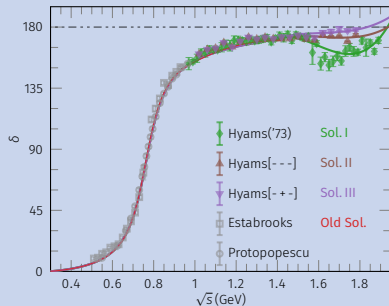
	2 par.		3 par.	
	low $\phi_{\omega\pi^0}(0)$	high $\phi_{\omega\pi^0}(0)$	low $\phi_{\omega\pi^0}(0)$	high $\phi_{\omega\pi^0}(0)$
$10^{-2} a $ [GeV ⁻³]	3.14(25)	2.63(25)	3.11(28)	2.70(30)
$ b $	3.15(22)	2.59(35)	3.25(26)	2.65(35)
ϕ_b	2.03(14)	1.61(38)	2.03(13)	1.70(27)
$ f_{\omega\pi^0}(0) $ [GeV ⁻¹]	2.314(32)	2.314(32)	2.314(32)	2.315(32)
$\phi_{\omega\pi^0}(0)$	0.207(60)	2.39(46)	0.195(76)	2.48(31)
χ^2_{DP}	0.19	< 0.01	0.10	0.03
$10^4 \chi^2_{\Gamma}$	2.4	2.4	1.1	3.5
χ^2_{A2}	2.3	3.6	2.4	3.7
χ^2_{NA60}	31	35	31	35

$J/\psi \rightarrow 3\pi$ decays

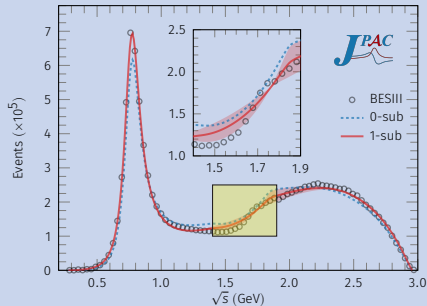
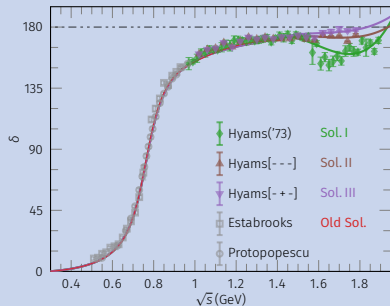
- Formalism for J/ψ is completely analogous to $\omega(V)$.
- BESIII data [Phys. Lett., B710, 594 (2012)] show ψ/ψ' puzzle:



- The J/ψ decay seems to be dominated by ρ , despite the larger phase space
- One would expect that 0-sub (prediction) would get the basic features



- $\delta_{\pi\pi}(\sqrt{s})$ taken as input:
 - ▶ Old solution: [Garcia-Martin, Kaminski, Pelaez, Ruiz de Elvira, Yndurain, Phys. Rev. D83, 074004 (2011)]
 - ▶ New solutions: [Pelaez, Rodas, Ruiz De Elvira, Eur. Phys. J. C79, 1008 (2019)]
- Take as central fit the one performed with solution I for $\delta_{\pi\pi}$
- The spread in the other solutions: theoretical uncertainty
- The J/ψ decay seems to be dominated by ρ , despite the larger phase space
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 - ▶ Kubis, Niecknig, Phys. Rev. D91,036004 (2015)
- 1-sub (fit) improves the description
- 1-sub + F-wave [$\rho_3(1690)$] describes better the movements above $\gtrsim 1.5$ GeV.

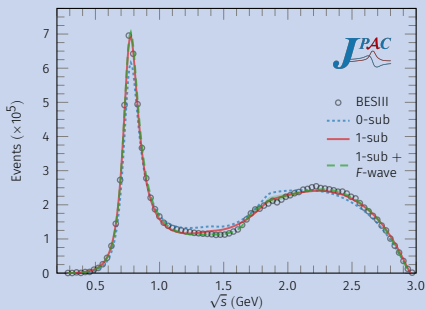


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- How to improve the description? Include F -wave, $\rho_3(1690)$ (PDG values)

$$F(s, t, u) = F_1(s) + F_1(t) + F_1(u) + \frac{k^2(s)}{16} P'_3(z_s) F_3(s) + \frac{k^2(t)}{16} P'_3(z_t) F_3(t) + \frac{k^2(u)}{16} P'_3(z_u) F_3(u)$$

- Neglect \hat{F}_3 , so that $F_3(s) = \rho_3(s) \Omega_3(s)$

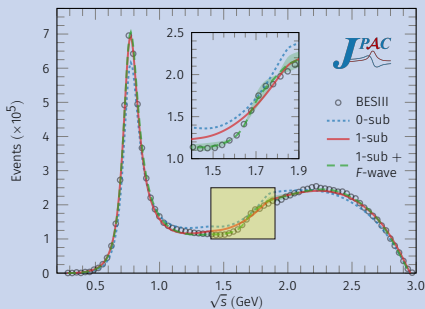


- The **fit improves** significantly, especially around $\sqrt{s} \gtrsim 1.5$ GeV, the main contribution being the **P - F -wave interference**.

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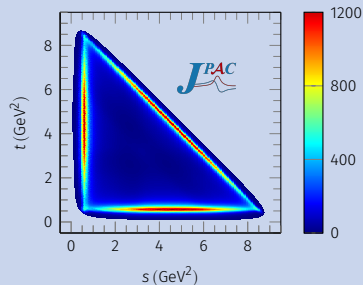


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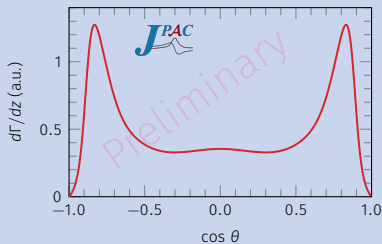
$J/\psi \rightarrow 3\pi$ decays: additional information

[JPAC Collab., 2304.09736]

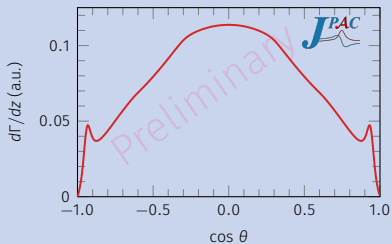
- Dalitz plot distribution similar to exp. one
- More statistics will allow to unveil other effects (resonances, interferences,...)
- Predictions can be done for angular [$z = \cos \theta_s$] distributions, specially restricted to ρ -mass region.



Full \sqrt{s} range



$|\sqrt{s} - m_\rho| \leq 50 \text{ MeV}$



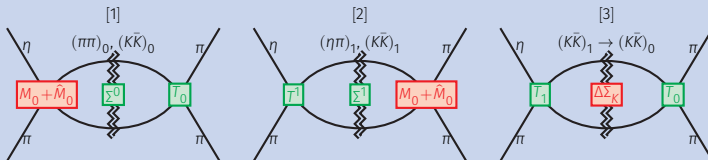
Coupled channels: take into account **intermediate states** other than $(\pi\pi)_I$.

$$\mathbf{M}_0 = \begin{bmatrix} M_0 & G_{10} \\ N_0 & H_{10} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_1 \rightarrow (\pi\pi)_0 & (K\bar{K})_1 \rightarrow (\pi\pi)_0 \\ (\eta\pi)_1 \rightarrow (K\bar{K})_0 & (K\bar{K})_1 \rightarrow (K\bar{K})_0 \end{bmatrix},$$

$$\mathbf{T}_0 = \begin{bmatrix} t_{(\pi\pi)_0 \rightarrow (\pi\pi)_0} & t_{(\pi\pi)_0 \rightarrow (K\bar{K})_0} \\ t_{(\pi\pi)_0 \rightarrow (K\bar{K})_0} & t_{(K\bar{K})_0 \rightarrow (K\bar{K})_0} \end{bmatrix}, \quad \mathbf{T}_1 = \begin{bmatrix} t_{(\eta\pi)_1 \rightarrow (\eta\pi)_1} & t_{(\eta\pi)_1 \rightarrow (K\bar{K})_1} \\ t_{(\eta\pi)_1 \rightarrow (K\bar{K})_1} & t_{(K\bar{K})_1 \rightarrow (K\bar{K})_1} \end{bmatrix}$$

$$\begin{aligned} \text{disc } \mathbf{M}_0(s) &= \mathbf{T}_0^*(s) \Sigma^0(s) [\mathbf{M}_0(s + i\epsilon) + \hat{\mathbf{M}}_0(s)] && \rightarrow [1] \\ &+ [\mathbf{M}_0(s - i\epsilon) + \hat{\mathbf{M}}_0(s)] \Sigma^1(s) \mathbf{T}_1^*(s) && \rightarrow [2] \\ &+ \mathbf{T}_0^*(s) \Delta \Sigma_K(s) \mathbf{T}_1^*(s) && \rightarrow [3] \end{aligned}$$

Schematically:



- Extension of KT equations to coupled-channels

- DP variables X, Y : $x = \frac{\sqrt{3}}{2m_\eta Q_c} (u - t), y = \frac{3}{2m_\eta Q_c} (s_- - s) - 1$

- Charged mode amplitude written as:

$$\frac{|M_c(X, Y)|^2}{|M_c(0, 0)|^2} = \frac{1 + aY + bY^2 + dX^2 + fY^3 + gX^2Y + \dots}{1 + aY + bY^2 + dX^2 + fY^3 + gX^2Y + \dots}$$

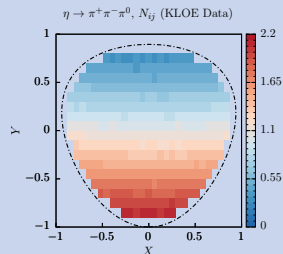
- Neutral decay mode amplitude $[Q_c \rightarrow Q_n]$:

$$\frac{|M_n(X, Y)|^2}{|M_n(0, 0)|^2} = \frac{1 + 2\alpha |Z|^2 + 2\beta \text{Im}(Z^3) + \dots}{1 + 2\alpha |Z|^2 + 2\beta \text{Im}(Z^3) + \dots}$$

	$\mathcal{O}(p^4)$	elastic	coupled	KLOE	BESIII	
charged	a	-1.328	-1.156	-1.142(45)	-1.095(4)	-1.128(15)
	b	0.429	0.200	0.172(16)	0.145(6)	0.153(17)
	d	0.090	0.095	0.097(13)	0.081(7)	0.085(16)
	f	0.017	0.109	0.122(16)	0.141(10)	0.173(28)
	g	-0.081	-0.088	-0.089(10)	-0.044(16)	-
neutral	PDG					
	α	+0.0142	-0.0268	-0.0319(34)	-0.0318(15) [old]	-
	β	-0.0007	-0.0046	-0.0056	-	-

BESIII Collab., Phys. Rev. D92,012014 (2015)

KLOE-2 Collab., JHEP 1605, 019 (2016)



- (Theory) **uncertainty estimation**:

- 1 $\eta\pi$ interaction zero or “large”
- 2 $10^3 L_3^r = -3.82 \rightarrow -2.65$

- General trend: **improve agreement** [$\mathcal{O}(p^4) \rightarrow$ elastic \rightarrow coupled]
- Particularly relevant: α .

Dalitz plot parameters for $\eta \rightarrow 3\pi$

MA, B. Moussallam, EPJ, C77,508('17)

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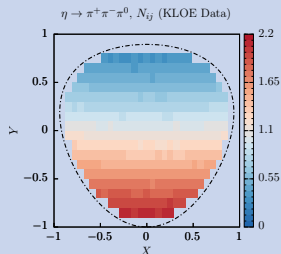
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neutral	PDG					
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BESIII Collab., Phys. Rev. D92,012014 (2015)

KLOE-2 Collab., JHEP 1605, 019 (2016)



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- Particularly relevant: α .

Quark mass ratio from $\eta \rightarrow 3\pi$

MA, B. Moussallam, EPJ, C77,508('17)

From the amplitudes $M_I(s)$ one can compute the width up to the unknown factor Q^2 :

$$\Gamma = \epsilon_L^2 \int_{4m_\pi^2}^{m_\eta^2} ds \int_{t_-(s)}^{t_+(s)} dt |M_0(s) + \dots|^2$$

$$\epsilon_L = Q^{-2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3}f_\pi^2} \frac{m_K^2}{m_\pi^2}, \quad Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - \hat{m}^2}$$

$\Gamma(\eta \rightarrow 3\pi^0)/\Gamma(\eta \rightarrow \pi^+\pi^-\pi^0)$	
PDG (fit)	1.426(26)
PDG (average)	1.48(5)
CLEO	1.496(43)(35)
chiral $\mathcal{O}(p^4)$	1.425
elastic	1.449
coupled	1.451

Decay	Q	
	elastic	coupled
$\Gamma_{(neu.)}^{(exp)} = 299(11)$ eV	21.9(2)	21.7(2)
$\Gamma_{(cha.)}^{(exp)} = 427(15)$ eV	21.8(2)	21.6(2)

- Effect of inelastic channels $\sim 1\%$ (decreasing)
- Theoretical error on Q:
 - ▶ Phase shifts [$s \leq 1 \text{ GeV}^2$]: $\sim 1\%$
 - ▶ $\mathcal{O}(p^4)$ ampl. [L_3]: $\sim 1\%$
 - ▶ NNLO ampl.: $\Delta Q_{th.} = \pm 2.2$

$$Q = 21.6 \pm 0.2 \pm 2.2$$

- Fitted (not matched) polynomial parameters:

$$Q_{fit} = 21.50 \pm 0.67 \pm 0.70$$

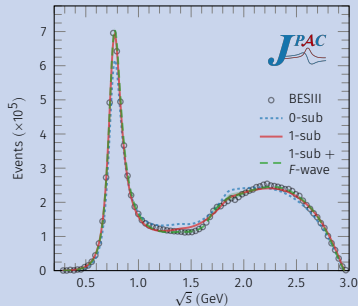
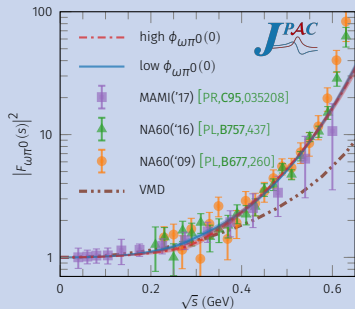
Summary

- KT equations are a powerful tool to study **3-body decays**
- They allow to implement **two-body unitarity** in all the **three channels** (s, t, u).
- For $\omega \rightarrow 3\pi$ decays: JPAC Collab., EPJ,C80,1107('20)
 - ▶ Using once-subtracted DRs, we are able to reproduce the $\omega \rightarrow 3\pi$ DP parameters,
 - ▶ and the $\omega \rightarrow \pi^0\gamma^*$ transition form factor data.
- For $J/\psi \rightarrow 3\pi$ decays: JPAC Collab., 2304.09736
 - ▶ Good agreement with the data is found assuming elastic unitarity (P - and F -waves).
 - ▶ Paves the way for an event-based analysis of J/ψ and ψ' decays.
- For $\eta \rightarrow 3\pi$: MA, B. Moussallam, EPJ,C77,508('17)
 - ▶ A quite general extension of KT equations to coupled channels.
 - ▶ Better determination of the DP parameters, Q ratio determined.

Summary of talk by M. Albaladejo (Mon. 4:15pm)

J/ψ , ω , and η decays to 3π with Khuri-Treiman equations

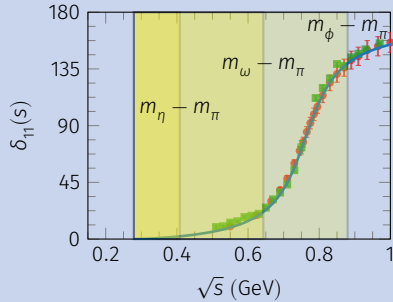
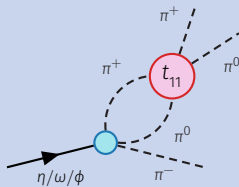
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 - ▶ A quite general extension of KT equations to coupled channels.
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Summary of KT-related works

- KT equations are a powerful tool to study **3-body decays**.
- They allow to implement **two-body unitarity** in all the **three channels** (s, t, u).
- Iterative solution converges fast, linear in subtraction constants.
- For $\eta \rightarrow 3\pi$: MA, B. Moussallam, EPJ,C77,508('17)
 - ▶ Not well described by the perturbative chiral amplitudes.
 - ▶ We have presented an **extension** of this approach to **coupled channels**. The extension is quite **general**.
 - ▶ Effects of $K\bar{K}$ and $\eta\pi$ amplitudes [$f_0(980), a_0(980)$] play some role in the DP parameters, tend to improve.
- For $\pi\pi$ scattering: JPAC Collab., EPJ,C78,574('18)
 - ▶ We have applied KT equations to $\pi\pi$ scattering as benchmark.
 - ▶ Restricted to S - and P -waves, KT equations are equal to Roy equations.
 - ▶ When other waves are included, good comparison is obtained with GPKY equations.
- We have presented a **generalization** of the KT equations for arbitrary quantum numbers of the decaying particle. JPAC Collab., PR,D101,054018('20)
 - ▶ Not trivial, because of spin/crossing.
- For $\omega \rightarrow 3\pi$ decays: JPAC Collab., EPJ,C80,1107('20)
 - ▶ Using once-subtracted DRs, we are able to reproduce the $\omega \rightarrow 3\pi$ DP parameters,
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- For $J/\psi \rightarrow 3\pi$ decays: JPAC Collab., 2304.09736
 - ▶ Good agreement with the data is found assuming elastic unitarity (P - and F -waves).
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KT and phase space



Generalities about $\eta \rightarrow 3\pi$

- In QCD isospin-breaking phenomena are driven by

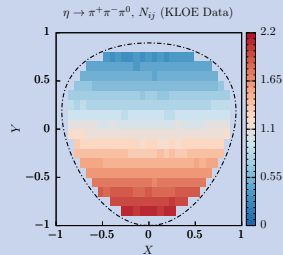
$$H_{IB} = -(m_u - m_d) \bar{\psi} \frac{\lambda_3}{2} \psi$$

- Isospin-breaking induced by EM & strong interactions are **similar** in size, but
- $\eta \rightarrow 3\pi$ is **special**, since EM effects are smaller

- $\Gamma_{\eta \rightarrow 3\pi} \propto Q^4$, with

$$Q^{-2} = \frac{m_d^2 - m_u^2}{m_s^2 - (m_u + m_d)^2/2}$$

- Experimental situation:** Several high-statistics studies; $|T|^2$ well known across the Dalitz plot \Rightarrow stringent tests for the amplitudes (before getting Q!)



$\eta \rightarrow 3\pi^0$

Crys. Ball, PRL**87**,192001('01)
Crys. Ball@MAMI, A2, PRC**79**,035204('09)
Crys. Ball@MAMI, TAPS, A2, EPJA**39**,169('09)
WASA-at-COSY, PLB**677**,24('09)
KLOE, PLB**694**,16('11)

$\eta \rightarrow \pi^+\pi^-\pi^0$

KLOE, JHEP**0805**,006('08)
WASA-at-COSY, PRC**90**,045207('14)
BESIII, PRD**92**,012014('15)
KLOE-2, JHEP**1605**,019('16)

Previous dispersive approaches to $\eta \rightarrow 3\pi$

- Chiral $\mathcal{O}(p^4)$ amplitude fails in describing experiments.

Gasser, Leutwyler, Nucl. Phys. B250, 539 (1985)

- Several attempts to include **unitarity/FSI/rescattering** effects.

Neveu, Scherk, AP57, 39('70); Roiesnel, Truong, NPB187, 293('81); Kambor, Wiesendanger, Wyler, NPB465, 215('96); Anisovich, Leutwyler, PLB375, 335('96); Borasoy, R. Nißler, EPJA26, 383('05); Schneider, Kubis, Ditsche, JHEP1102, 028('11); Kampf, Knecht, Novotný, Zdráhal, PR,D84, 114015('11); Colangelo, Lanz, Leutwyler, Passemar, PRL118, 022001('17); Guo, Danilkin, Fernández-Ramírez, Mathieu, Szczepaniak, PLB771, 497('17).

- Here we reconsider the **KT approach**.

N. Khuri, S. Treiman, Phys. Rev. 119, 1115 (1960)

- $\pi\pi$ scattering **elastic** in the decay region. But **dispersive approaches** require higher energy T -matrix inputs:

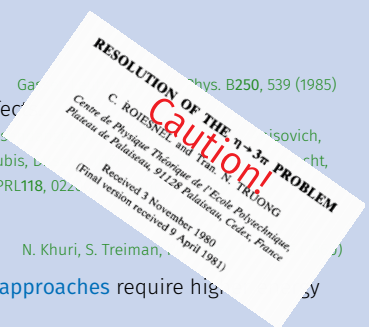
- ▶ $\pi\pi$ near 1 GeV rapid energy variation. $f_0(980)$, $(K\bar{K})_0$
- ▶ Double resonance effect $\eta\pi$ ISI, $a_0(980)$, $(K\bar{K})_1$

Abdel-Rehim, Black, Fariborz, Schechter, PRD67, 054001('03)

We propose a **generalization to coupled channels** [$(K\bar{K})_{0,1}$, $\eta\pi$, $(\pi\pi)_{0,1,2}$] of the KT equations, extending their validity up to the physical $\eta\pi \rightarrow \pi\pi$ region. Allows for the study of the influence of a_0, f_0 into the decay region.

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Isospin amplitudes

- Start with well-defined **isospin amplitudes**:

$$\mathcal{M}^{l,l_z}(s, t, u) = \langle \eta\pi; 1, l_z | \hat{T}_0^{(1)} | \pi\pi; l, l_z \rangle = \langle l, l_z; 1, 0 | 1, l_z \rangle \langle \eta\pi | \hat{T}^{(1)} | \pi\pi; l \rangle$$

- They can be written in terms of a **single amplitude** ($\eta\pi^0 \rightarrow \pi^+\pi^-$), $A(s, t, u)$ (like in $\pi\pi$ scattering):

$$\begin{bmatrix} -\sqrt{3}\mathcal{M}^0(s, t, u) \\ \sqrt{2}\mathcal{M}^1(s, t, u) \\ \sqrt{2}\mathcal{M}^2(s, t, u) \end{bmatrix} = \begin{bmatrix} -\sqrt{3}\mathcal{M}^{0,0}(s, t, u) \\ \sqrt{2}\mathcal{M}^{1,1}(s, t, u) \\ \sqrt{2}\mathcal{M}^{2,1}(s, t, u) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A(s, t, u) \\ A(t, s, u) \\ A(u, t, s) \end{bmatrix}$$

- Reconstruction theorem (for Goldstone bosons):

J. Stern, H. Sazdjian, N. Fuchs, Phys. Rev. D47, 3814 (1993)

$$A(s, t, u) = -\epsilon_L \left[M_0(s) - \frac{2}{3}M_2(s) + M_2(t) + M_2(u) \right. \\ \left. + (s-u)M_1(t) + (s-t)M_1(u) \right] \quad \epsilon_L = \frac{1}{Q^2} \frac{m_K^2 - m_\pi^2}{3\sqrt{3}f_\pi^2} \frac{m_K^2}{m_\pi^2}$$

- Or in general, “the” KT approximation:

Infinite sum of s-channel PW \rightarrow Truncated sums of s-, t-, and u-channels PWs

- Single variable functions:** amenable for dispersion relations.

Partial wave amplitudes

- Summary of previous slide: $\mathcal{M}^l(s, t, u)$ is written in terms of $A(s, t, u)$ (and permutations), and $A(s, t, u)$ is written in terms of $M_l(w)$.
- Now, define **partial waves**: $\mathcal{M}^l(s, t, u) = 16\pi\sqrt{2} \sum_j (2j+1) \mathcal{M}_j^l(s) P_j(z)$

$$\mathcal{M}_0^0(s) = \epsilon_L \frac{\sqrt{6}}{32\pi} [M_0(s) + \hat{M}_0(s)], \quad \mathcal{M}_0^2(s) = \epsilon_L \frac{-1}{32\pi} [M_2(s) + \hat{M}_2(s)],$$

$$\mathcal{M}_1^1(s) = \epsilon_L \frac{\kappa(s)}{32\pi} [M_1(s) + \hat{M}_1(s)],$$

LHC [$\hat{M}_l(s)$]

$\hat{M}_l(s)$ written as angular averages.
Take $M_0(s)$ as an example:

$$\hat{M}_0(s) = \frac{2}{3} \langle M_0 \rangle + \frac{20}{9} \langle M_2 \rangle$$

$$+ 2(s-s_0) \langle M_1 \rangle + \frac{2}{3} \kappa(s) \langle z M_1 \rangle$$

$$\langle z^n M_l \rangle(s) = \frac{1}{2} \int_{-1}^1 dz z^n M_l(t(s, z))$$

$$\kappa(s) = \sqrt{(1 - 4m_\pi^2/s)\lambda(s, m_\eta^2, m_\pi^2)}$$

RHC [$M_l(s)$]

$\hat{M}(s)$ no discontinuity along the RHC:

$$\text{disc } M_l(s) = \text{disc } \mathcal{M}_j^l(s) =$$

$$= \sigma_\pi(s) t^l(s) \mathcal{M}_j^l(s)$$

$$= \sigma_\pi(s) t^l(s) (M_l(s) + \hat{M}_l(s))$$

$$\sigma_\pi(s) = \sqrt{1 - 4m_\pi^2/s}$$

$$\sigma_\pi(s) t^l(s) = \sin \delta_l(s) e^{i\delta_l(s)}$$

Muskhelisvili-Omnès representation

$$\text{disc}M_I(s) = \sigma_\pi(s)t_I^*(s)[M_I(s) + \hat{M}_I(s)]$$

- MO (dispersive) representation of $M_I(s)$:

$$M_0(s) = \Omega_0(s)[\alpha_0 + \beta_0 s + \gamma_0 s^2 + \hat{l}_0(s)s^2] ,$$

$$M_1(s) = \Omega_1(s)[\beta_1 s + \hat{l}_1(s)s] ,$$

$$M_2(s) = \Omega_2(s)[\hat{l}_2(s)s^2] .$$

$$\Omega_I(s) = \exp \left[\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s' - s)} \right] \quad (\text{Omnès function/matrix})$$

$$\hat{l}_I(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\sin \delta_I(s') \hat{M}_I(s')}{|\Omega_I(s')| (s')^{m_I} (s' - s)} , \quad (m_{0,2} = 2, m_1 = 1)$$

- $m_\eta^2 + i\varepsilon$ prescription needed. Integral equations solved iteratively.
- Subtraction constants: Most natural way is to match with **ChPT**:

$$\mathcal{M}(s, t, u) - \overline{\mathcal{M}}_\chi(s, t, u) = \mathcal{O}(p^6)$$

Descotes-Genon, Moussallam, EPJ, C74,2946(2014)

- **Matching conditions:** fix $\alpha_0, \beta_0, \beta_1, \gamma_0$ in terms of ChPT amplitudes (no free parameters).

Coupled channels: MO representations

$$\begin{aligned}
 \text{disc } \mathbf{M}_0(s) &= \mathbf{T}^{0*}(s)\Sigma^0(s) [\mathbf{M}_0(s+i\epsilon) + \widehat{\mathbf{M}}_0(s)] && \rightarrow [1] \\
 &+ [(\mathbf{M}_0(s-i\epsilon) + \widehat{\mathbf{M}}_0(s))\Sigma^1(s) \mathbf{T}^1(s)] && \rightarrow [2] \\
 &+ \mathbf{T}^{0*}(s)\Delta\Sigma_K(s)\mathbf{T}^1(s) && \rightarrow [3]
 \end{aligned}$$

- MO representation for $\mathbf{M}_0(s)$:

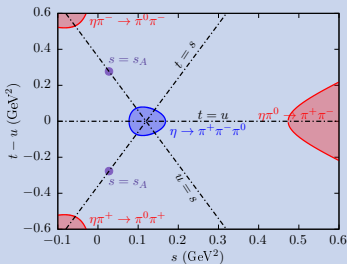
$$\begin{bmatrix} M_0(s) & G_{10}(s) \\ N_0(s) & H_{10}(s) \end{bmatrix} = \mathbf{\Omega}_0(s) [\mathbf{P}_0(s) + s^2 (\widehat{\mathbf{I}}_a(s) + \widehat{\mathbf{I}}_b(s))] {}^t\mathbf{\Omega}_1(s)$$

- $\mathbf{P}_0(s)$ is a matrix of polynomials (subtractions matched to ChPT: no free parameters).
- The $\widehat{\mathbf{I}}(s)$ functions are:

$$\widehat{\mathbf{I}}_{a,b}(s) = \frac{1}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{(s')^2(s'-s)} \Delta\mathbf{X}_{a,b}(s'),$$

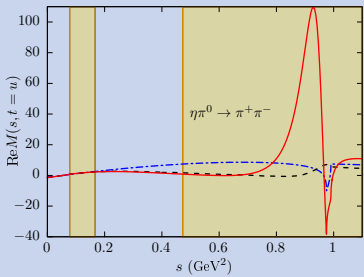
$$\Delta\mathbf{X}_a = \mathbf{\Omega}_0^{-1}(s-i\epsilon) \left[\underbrace{\mathbf{T}^{0*}(s)\Sigma^0(s)\widehat{\mathbf{M}}_0(s)}_{[1]} + \underbrace{\widehat{\mathbf{M}}_0(s)\Sigma^1(s)\mathbf{T}^1(s)}_{[2]} \right] {}^t\mathbf{\Omega}_1^{-1}(s+i\epsilon),$$

$$\Delta\mathbf{X}_b = \underbrace{\mathbf{\Omega}_0^{-1}(s-i\epsilon)\mathbf{T}^{0*}(s)\Delta\Sigma_K(s)\mathbf{T}^1(s)}_{[3]} {}^t\mathbf{\Omega}_1^{-1}(s+i\epsilon)$$

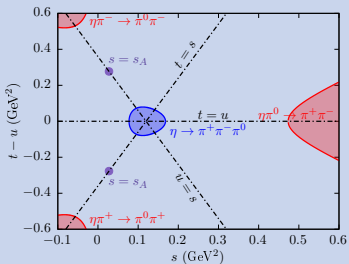


Behaviour in different regions:

- $s \sim 1 \text{ GeV}^2$ Very sharp energy variation,
 - ▶ $a_0(980)$ and $f_0(980)$ interference,
 - ▶ K^+K^- and $K^0\bar{K}^0$ thresholds.
- $0.7 \lesssim s \lesssim 0.97 \text{ GeV}^2$ Coupled channel largely **enhanced** compared with elastic amplitude.
- $s \lesssim 0.7 \text{ GeV}^2$ Effect of coupled channels is to reduce the amplitude.
- $s \lesssim s_{\text{th}}$ Elastic and inelastic amplitudes indistinguishable.

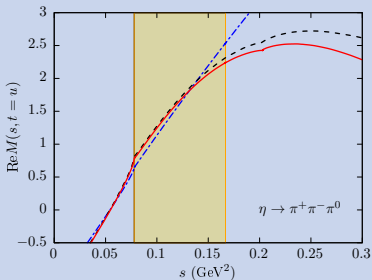


Chiral $\mathcal{O}(p^4)$ --- Elastic --- Coupled ———

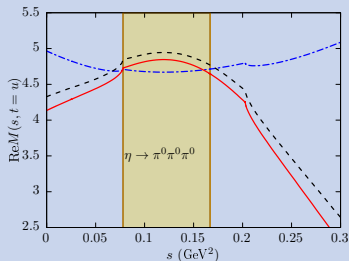
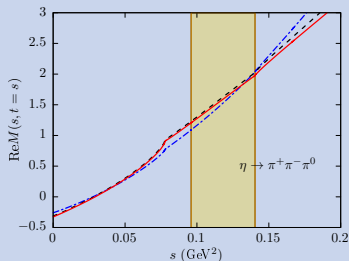


Behaviour in different regions:

- $s \sim 1 \text{ GeV}^2$ Very sharp energy variation,
 - ▶ $a_0(980)$ and $f_0(980)$ interference,
 - ▶ K^+K^- and $K^0\bar{K}^0$ thresholds.
- $0.7 \lesssim s \lesssim 0.97 \text{ GeV}^2$ Coupled channel largely **enhanced** compared with elastic amplitude.
- $s \lesssim 0.7 \text{ GeV}^2$ Effect of coupled channels is to reduce the amplitude.
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Chiral $\mathcal{O}(p^4)$ --- Elastic --- Coupled —

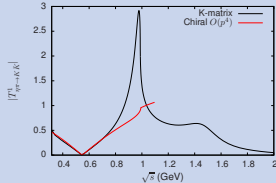
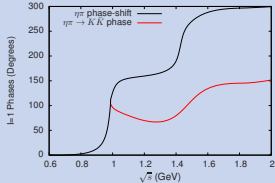
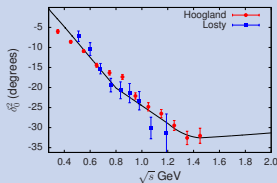
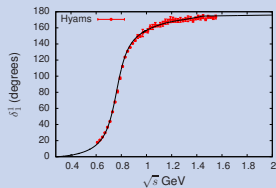
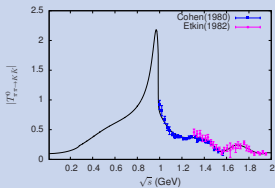
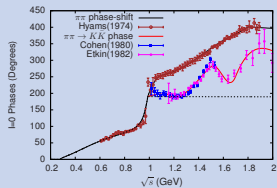


- Subthreshold region: chiral, elastic, and coupled amplitudes **very close**.
- Adler zero ($s_A \approx 0.03 \text{ GeV}^2$):

	NLO	el.	cou.
$s_A/m_{\pi^+}^2 =$	1.42	1.45	1.49

- Substantial influence of coupled channels in the whole region,
- and there is no region in which dispersive and chiral amplitudes agree.
- $T_{\eta \rightarrow 3\pi^0} = M_0(s) + M_0(t) + M_0(u) + \dots$

$\pi\pi$ conserving T -matrices



- B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rept. **353**, 207 (2001);
 R. García-Martín, B. Moussallam, Eur. Phys. J. **C70**, 155 (2010);
 B. Moussallam, Eur. Phys. J. **C71**, 1814 (2011);
 M. Albaladejo, B. Moussallam, Eur. Phys. J. **C75**, 488 (2015);

Amplitudes M_1 and M_2

An analogous analysis can be done with $M_1(s)$ and $M_2(s)$ amplitudes:

$M_1(s)$ [*P*-wave]

$$\mathbf{M}_1(s) = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} = \begin{bmatrix} (\eta\pi)_{1-} \rightarrow (\pi\pi)_{1+} \\ (\eta\pi)_{1-} \rightarrow (K\bar{K})_{1+} \end{bmatrix}$$

$$\mathbf{T}_1^1(s) = \begin{bmatrix} (\pi\pi)_{1-} \rightarrow (\pi\pi)_{1-} & (\pi\pi)_{1-} \rightarrow (K\bar{K})_{1-} \\ (\pi\pi)_{1-} \rightarrow (K\bar{K})_{1-} & (K\bar{K})_{1-} \rightarrow (K\bar{K})_{1-} \end{bmatrix}$$

$$\Delta\mathbf{M}_1(s) = \mathbf{T}_1^{1*}(s)\Sigma^0(s) \\ \times [\mathbf{M}_1(s+i\epsilon) + \hat{\mathbf{M}}_1(s)]$$

$M_2(s)$ [*S*-wave]

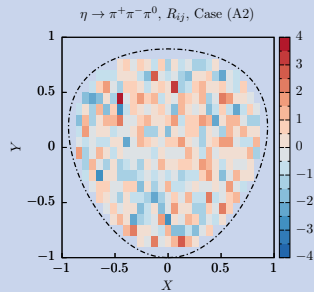
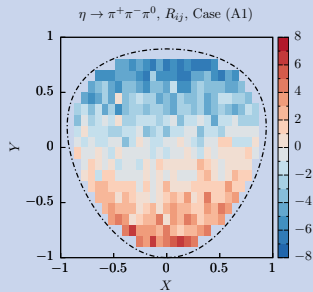
$$\mathbf{M}_2(s) = \begin{bmatrix} M_2 \\ G_{12} \end{bmatrix} = \begin{bmatrix} (\eta\pi)_{1-} \rightarrow (\pi\pi)_{2-} \\ (K\bar{K})_{1-} \rightarrow (\pi\pi)_{2-} \end{bmatrix}$$

$$t_0^2(s) = t_{(\pi\pi)_{2-} \rightarrow (\pi\pi)_{2-}}$$

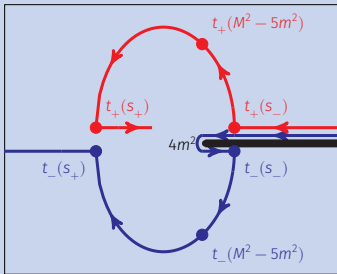
$$\text{disc } \mathbf{M}_2(s) = \mathbf{T}^1(s)\Sigma^1(s) \\ \times (\mathbf{M}_2(s-i\epsilon) + \hat{\mathbf{M}}_2(s)) \\ + \sigma_\pi(s)(t_0^2(s))^*(\mathbf{M}_2(s+i\epsilon) + \hat{\mathbf{M}}_2(s))$$

- **Consistent approximation:** $\hat{N}_0(s)$, $\hat{G}_{10}(s)$, $\hat{H}_{10}(s)$, $\hat{G}_{12}(s)$: we neglect these LHC functions (would require all the related cross channels amplitudes...).
- Further approximation: For $l = j = 1$, we consider elastic $\pi\pi$.

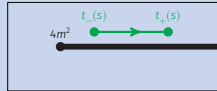
Fitting



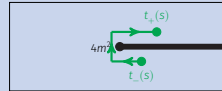
Endpoints



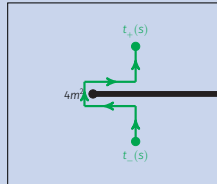
$$4m^2 \ll s \ll \frac{M^2 - m^2}{2}$$



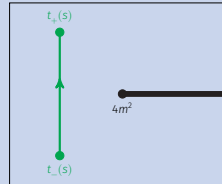
$$\frac{M^2 - m^2}{2} \ll s \ll s_-$$



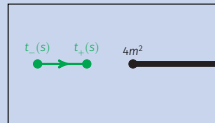
$$s_- \ll s \ll M^2 - 5m^2$$



$$M^2 - 5m^2 \ll s \ll s_+$$



$$s_+ \ll s$$



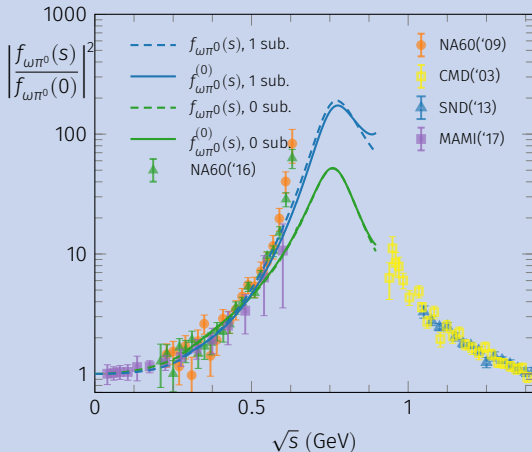
Unitarity and analyticity bounds

- In works by Caprini *et al.* bounds (min and max) of the form factor have been derived.

[EPJ,C74,3209('14); PR,D92,014014('15)]

- $f_{\omega\pi^0}^{(\pm)}(s) = f_{\omega\pi^0}^{(0)}(s) \pm \delta f_{\omega\pi^0}^{(0)}(s)$
- $f_{\omega\pi^0}^{(0)}(s)$ depends on $\Delta f_{\omega\pi^0}(s)$
- $\delta f_{\omega\pi^0}^{(0)}(s) \propto l'$, depends on the value of the TFF for $s \geq (m_\omega + m_\pi)^2$
- High energy data well above our scope...

Tension between low and high energy data?

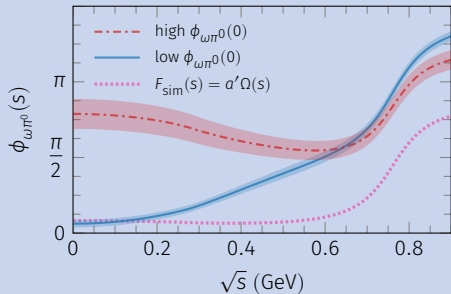
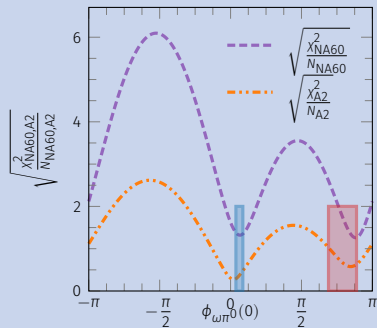


Meaning of the phase?

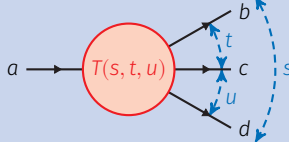
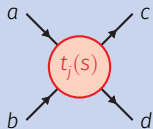
- Original solutions around $\phi_{\omega\pi^0}(0) \sim 0, \pi$
- Global fits remain near the original ones...

If $f_{\omega\pi^0}(0)$, a are considered as part of a microscopic (lagrangian) calculation, they would be real (hermiticity), and their relative phase would be ± 1 .

On the other hand, we find 2σ deviation: almost real, but not exactly...



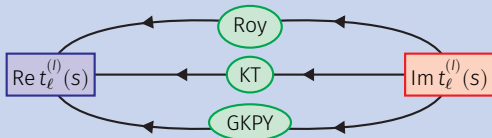
- KT equations for 3-body decays. Crossing: 2-to-2 scattering. Test: $\pi\pi$ scattering.



What happens if you apply KT equations to $\pi\pi$ scattering?

- KT equations for $\pi\pi$ scattering can be written as Roy-like equations:

$$t_\ell^{(l)}(s) = k_\ell^{(l)}(s) + \sum_{\ell', l'} \int_{s_{th}}^{\infty} dt' K_{\ell\ell'}^{ll'}(s, t') \text{Im } t_{\ell'}^{(l')}(t')$$



- Roy equations [PL,36B,353(1971)] and KT equations written as:

$$t_{\ell}^{(l)}(s) = k_{\ell}^{(l)}(s) + \sum_{\ell', l'} \int_{s_{\text{th}}}^{\infty} dt' K_{\ell\ell'}^{ll'}(s, t') \text{Im } t_{\ell'}^{(l')}(t')$$

They differ in the expressions for the polynomial ($k_{\ell}^{(l)}(s)$) and the kernel ($K_{\ell\ell'}^{ll'}(s, t')$).

- Restrict KT to

① S, P -waves ($t_0^{(0)}, t_0^{(2)}, t_1^{(1)}$),

② one subtraction in each channel: only two subtraction constants.

- Difference between KT and Roy equations amplitudes:

$$(t_{\text{KT}})_{\ell}^{(l)}(s) - (t_{\text{Roy}})_{\ell}^{(l)}(s) = \tilde{k}_{\ell}^{(l)}(s) - k_{\ell}^{(l)}(s) + \sum_{\ell', l'} \int_{s_{\text{th}}}^{\infty} dt' \Delta_{\ell\ell'}^{ll'}(4m^2, t') \text{Im } t_{\ell'}^{(l')}(t')$$

- $\Delta_{\ell\ell'}^{ll'}(s, t')$: Difference of kernels is polynomial (logarithmic terms cancel).
- Five conditions that can be fulfilled with the two subtraction constants.

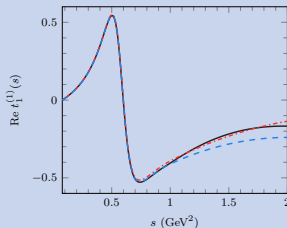
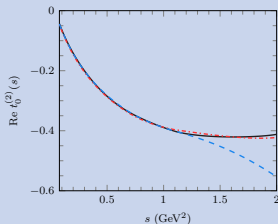
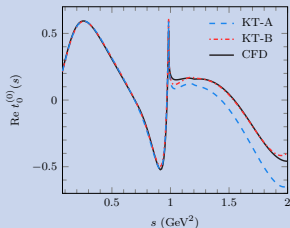
KT equations and Roy equations are equal.

- Take a successful parameterization of the amplitude as **input for $\text{Im}t_\ell^{(l)}(s)$** , and compare the **output $\text{Re}t_\ell^{(l)}(s)$**

Madrid group, PR,D83,074004(2011)

A: one subtraction ($\times 6$), but only 5 free constants. $s_{\max} = 1.0 \text{ GeV}^2$

B: two subtractions ($\times 6$), but only 7 free constants. $s_{\max} = 1.9 \text{ GeV}^2$

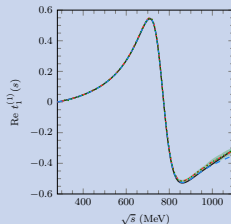
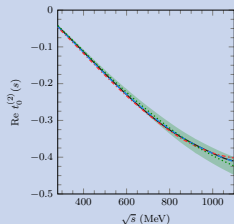
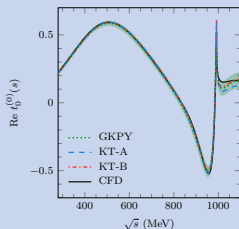


- Take a successful parameterization of the amplitude as **input for $\text{Im}t_\ell^{(l)}(s)$** , and compare the **output $\text{Re}t_\ell^{(l)}(s)$**

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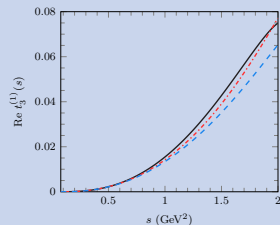
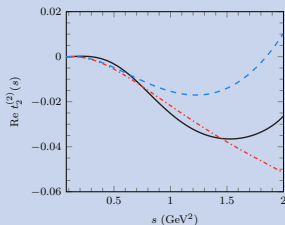
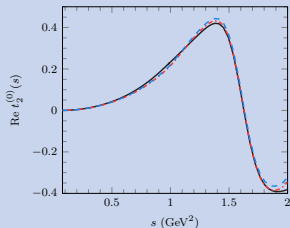


- Take a successful parameterization of the amplitude as **input for $\text{Im}t_\ell^{(l)}(s)$** , and compare the **output $\text{Re}t_\ell^{(l)}(s)$**

Madrid group, PR,D83,074004(2011)

A: one subtraction ($\times 6$), but only 5 free constants. $s_{\max} = 1.0 \text{ GeV}^2$

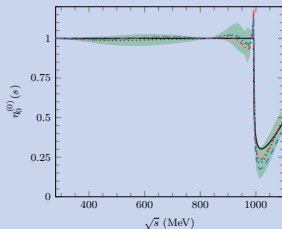
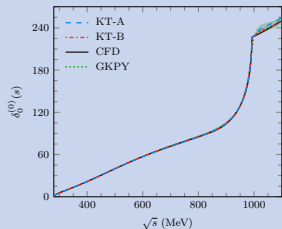
B: two subtractions ($\times 6$), but only 7 free constants. $s_{\max} = 1.9 \text{ GeV}^2$



- Take a successful parameterization of the amplitude as **input for $\text{Im}t_\ell^{(I)}(s)$** , and compare the **output $\text{Re}t_\ell^{(I)}(s)$**

Madrid group, PR,D83,074004(2011)

- A:** one subtraction ($\times 6$), but only 5 free constants. $s_{\text{max}} = 1.0 \text{ GeV}^2$
- B:** two subtractions ($\times 6$), but only 7 free constants. $s_{\text{max}} = 1.9 \text{ GeV}^2$



Results: Comparison with GKPY (II)

- Threshold parameters (right):

$$\frac{m^{2\ell}}{p^{2\ell}(s)} \text{Re } t_{\ell}^{(I)}(s) = a_{\ell}^{(I)} + b_{\ell}^{(I)} \frac{p^2(s)}{m^2} + \dots$$

- Poles and residues (bottom):

$$t_{II}^{-1}(s) = t_I^{-1}(s) + 2i\sigma(s),$$

$$t_{II}(s) \simeq \frac{\tilde{g}_p^2}{s - s_p} + \dots$$

PR,D83,074004('11); PRL,107,072001('11); PL,B749,399('15)

	KT-A	KT-B	GKPY-CFD
$a_0^{(0)}$	0.217	0.213	0.221(9)
$b_0^{(0)}$	0.274	0.275	0.278(7)
$a_0^{(2)}$	-0.044	-0.047	-0.043(8)
$b_0^{(2)}$	-0.078	-0.079	-0.080(9)
$10^3 \cdot a_1^{(1)}$	37.5	37.9	38.5(1.2)
$10^3 \cdot b_1^{(1)}$	5.6	5.7	5.1(3)
$10^4 \cdot a_2^{(0)}$	17.8	17.8	18.8(4)
$10^4 \cdot b_2^{(0)}$	-3.4	-3.4	-4.2(1.0)
$10^4 \cdot a_2^{(2)}$	1.9	1.8	2.8(1.0)
$10^4 \cdot b_2^{(2)}$	-3.2	-3.2	-2.8(8)
$10^5 \cdot a_3^{(1)}$	5.7	5.7	5.1(1.3)
$10^5 \cdot b_3^{(1)}$	-4.0	-4.0	-4.6(2.5)

	KT-A	KT-B	GKPY-CFD
$\sqrt{s_{\sigma}}$ (MeV)	(448, 270)	(448, 269)	(457 $^{+14}_{-13}$, 279 $^{+11}_{-7}$)
$ g_{\sigma} $ GeV	3.36	3.37	3.59 $^{+0.11}_{-0.13}$
$\sqrt{s_{\rho}}$ (MeV)	(762.2, 72.4)	(763.4, 73.5)	(763.7 $^{+1.7}_{-1.5}$, 73.2 $^{+1.0}_{-1.1}$)
$ g_{\rho} $	5.95	6.01	6.01 $^{+0.04}_{-0.07}$
$\sqrt{s_{f_0}}$ (MeV)	(1000, 24)	(995, 26)	(996 \pm 7, 25 $^{+10}_{-6}$)
$ g_{f_0} $ (GeV)	2.4	2.3	2.3 \pm 0.2
$\sqrt{s_{f_2}}$ (MeV)	(1275.1, 89.5)	(1268.9, 86.4)	(1267.3 $^{+0.8}_{-0.9}$, 87 \pm 9)
$ g_{f_2} $ (GeV $^{-1}$)	5.6	5.5	5.0 \pm 0.3

- $\eta \rightarrow 3\pi$, $\pi\pi \rightarrow \pi\pi$: $J = 0$, no spin complications.
- $\omega \rightarrow 3\pi$: single amplitude, $F(s, t, u) = F(s, u, t) = F(t, s, u)$. $J = 1$ particular case.
- For general $J \neq 0$, there are more than a single amplitude, and the t -, u -isobar amplitudes related with s -isobar through **crossing**.

PC	J_{\min}	I	notation (for $I = 0, 1$)
++	1	odd	a_J
+-	1	even	h_J
-+	0	odd	π_J
--	0	even	ω_J/ϕ_J

$$\mathcal{A}^{abcd}(\epsilon(p_X), p_3; p_1, p_2) = \langle \pi^c(p_1) \pi^d(p_2) | \hat{T} | X_J^a(\epsilon(p_X)) \pi^b(p_3) \rangle$$

- Definition of s- and t-channel helicity amplitudes:

$$\mathcal{A}_\lambda^{(s)abcd}(s, t, u) \equiv \mathcal{A}^{abcd}(\epsilon_\lambda^{(s)}(p_X), p_3; p_1, p_2)$$

$$\mathcal{A}_{\lambda'}^{(t)acbd}(t, s, u) \equiv \mathcal{A}^{abcd}(\epsilon_{\lambda'}^{(t)}(p_X'), -p_1', p_2', -p_3')$$

- Crossing, helicity amplitudes: $\mathcal{A}_{\lambda'}^{(t)acbd}(t, s, u) = \sum_\lambda d_{\lambda\lambda'}^J(\omega_t) \mathcal{A}_\lambda^{(s)abcd}(s, t, u)$
Jacob, Wick, Ann.Phys.,7,404('59); Trueman, Wick, Ann.Phys.,26,322('64);
 Hara, PTP,45,584('71); Martin & Spearman ('70);
- Crossing, Isospin: $\mathcal{A}_{\lambda'}^{(t)acbd}(t, s, u) = (-1)^{\lambda'} \mathcal{A}_{\lambda'}^{(s)acbd}(t, s, u)$
- Combining both results:

$$\mathcal{A}_\lambda^{(s)abcd}(s, t, u) = \sum_{\lambda'} (-1)^\lambda d_{\lambda'\lambda}^J(\omega_t) \mathcal{A}_{\lambda'}^{(s)acbd}(t, s, u)$$

Why is this relation so important?

It allows the relation between the same one-variable functions (helicity partial waves or helicity isobars) for s and t.

- Isospin projection:

$$\mathcal{A}_{\lambda_I}(s, t, u) \equiv \frac{1}{(2I+1)} \sum_{a,b,c,d} P_{abcd}^{(I)} \mathcal{A}_{\lambda}^{(s)abcd}(s, t, u)$$

- KT decomposition in terms of isobars:

$$\begin{aligned} \mathcal{A}_{\lambda_I}(s, t, u) &= \sum_{j \geq |\lambda|}^{j_{\max}} (2j+1) d_{\lambda 0}^j(\theta_s) a_{j\lambda_I}(s) \\ &+ \sum_{\lambda' j' l'} (-1)^\lambda (2j'+1) d_{\lambda' \lambda}^{j'}(\omega_t) d_{\lambda' 0}^{j'}(\theta_t) a_{j' \lambda' l'}(t) \frac{1}{2} C_{ll'} \\ &+ \sum_{\lambda' j' l'} (-1)^{\lambda'} (2j'+1) d_{\lambda' \lambda}^{j'}(\omega_u) d_{\lambda' 0}^{j'}(\theta_u) a_{j' \lambda' l'}(u) \frac{1}{2} C_{ll'} (-1)^{l+l'} \end{aligned}$$

- Discontinuity:

$$\Delta a_{j\lambda_I}(s) = \rho(s) t_{jI}^*(s) \left(a_{j\lambda_I}(s) + \bar{a}_{j\lambda_I}(s) \right),$$

- Inhomogeneity:

$$\bar{a}_{j\lambda_I}(s) = (-1)^\lambda \sum_{l' j' \lambda'} \frac{1}{2} C_{ll'} \int d \cos \theta' d_{\lambda 0}^j(\theta') d_{\lambda' \lambda}^{j'}(\omega_{t'}) d_{\lambda' 0}^{j'}(\theta'_t) a_{j' \lambda' l'}(t')$$

- One last point: kinematical singularities and constraints fully taken into account in the paper.

$\pi\pi$ solutions

