$\hat{q} = (\sin\theta\sin\phi,\ \sin\theta\cos\phi,\ \cos\theta)$ $\qquad \hat{k} = (\sin\theta'\sin\phi', \sin\theta'\cos\phi', \cos\theta')$

- We want to check that $\sum_M Y_\ell^M(\hat{k})^* \int d\Omega_{\hat{q}}\ \hat{q}\ Y_\ell^M(\hat{q}) = \hat{k}\,\delta_{\ell,1}$

- We write: $\hat{q} = \left(\frac{1}{2}\sqrt{\frac{3}{\pi}}\right)^{-1} \left\{ -\frac{Y_1^1 + Y_1^{-1}}{\sqrt{2}i},\ \frac{Y_1^1 - Y_1^{-1}}{\sqrt{2}},\ Y_0 \right\}$

- Which can be seen directly because: $Y_1^{\pm 1}(\theta,\phi) = \mp \frac{1}{2} e^{\pm i\phi} \sin\theta \sqrt{\frac{3}{2\pi}}$

$Y_1^0(\theta,\phi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta$

- More compactly, $\hat{q}_i = \sum_{M=-1}^{1} c_{iM} Y_1^M(\hat{q})$ [$\hat{q}_i$ here is each of the components]

- We also know that $\int d\Omega_{\hat{q}}\ Y_\ell^M(\Omega_{\hat{q}}) Y_{\ell'}^{M'}(\Omega_{\hat{q}}) = \delta_{\ell\ell'}\delta_{MM'}$

Then $\sum_{M} Y_\ell^M(\hat{k}) \int d\Omega_{\hat{q}}\, \hat{q}_i\, Y_\ell^M(\hat{q})^* =$

$$= \sum_{M} Y_\ell^M(\hat{k}) \int d\Omega_{\hat{q}} \sum_{M'} c_{iM'}\, Y_1^{M'}(\hat{q})\, Y_\ell^M(\hat{q})^*$$

$$= \sum_{M,M'} c_{iM'}\, Y_\ell^M(\hat{k}) \underbrace{\int d\Omega_{\hat{q}}\, Y_1^{M'}(\hat{q})\, Y_\ell^M(\hat{q})^*}_{\delta_{\ell,1}\delta_{MM'}} =$$

$$= \sum_{M} c_{iM}\, Y_\ell^M(\hat{k}) = \hat{k} \qquad \blacksquare$$

On the other hand, the other result was:

$$(2\ell+1) P_\ell(\hat{k}\cdot\hat{q}) = 4\pi \sum_{M=-\ell}^{+\ell} Y_\ell^M(\hat{k})\, Y_\ell^M(\hat{q})^*,$$

which is known as Addition Theorem for spherical harmonics.

From it, we can derive the "beautiful formula":

$$\frac{1}{4\pi}\int d\Omega_{\hat{q}}\, P_\ell(\hat{p}\cdot\hat{k}) P_{\ell'}(\hat{p}'\cdot\hat{k})(2\ell+1)(2\ell'+1) =$$

$$= \frac{1}{4\pi}(4\pi)^2 \int d\Omega_{\hat{q}} \sum_{M,M'} Y_\ell^M(\hat{p})\, Y_\ell^M(\hat{q})^*\, Y_{\ell'}^{M'}(\hat{p}')^*\, Y_{\ell'}^{M'}(\hat{q}) =$$

$$= 4\pi \sum_{M,M'} Y_\ell^M(\hat{p})\, Y_{\ell'}^{M'*}(\hat{p}')\, \delta_{\ell\ell'}\delta_{MM'} = 4\pi\, \delta_{\ell\ell'} \sum_M Y_\ell^M(\hat{p})\, Y_\ell^M(\hat{p}')^* =$$

$$= (2\ell+1) P_\ell(\hat{p}\cdot\hat{p}')\, \delta_{\ell\ell'}$$