

1 The Wigner rotation

The following derivation is taken from Refs [1] and [2]. Let us consider a massive particle at rest. Its 4-momentum $p_0^\mu = (m, 0, 0, 0)$ can be boosted along an arbitrary momentum p_1 , followed by another boost to become p_2 and then returned at rest as follow:

$$p_0 = B(p_0 \leftarrow p_2)B(p_2 \leftarrow p_1)B(p_1 \leftarrow p_0)p_0. \quad (1)$$

The rest momentum p_0 is invariant under (spacial) rotation. The sub-group of Lorentz transformation that leaves a (rest) momentum invariant is called the “little group”. The Lorentz transformation equivalent to the boost chain in Eq. (1) is therefore a rotation, called the Wigner rotation.

This result is a particular case of a more general statement. To see this, let us replace the boost $B(p_2 \leftarrow p_1)$ by a general Lorentz transformation $p' = \Lambda p$ et denote the Wigner rotation by $R(\Lambda, p)$. We have, by definition,

$$R(\Lambda, p) = B(p_0 \leftarrow \Lambda p)L(\Lambda p \leftarrow p)B(p \leftarrow p_0). \quad (2)$$

Now let us apply the relation (27) to a particle in its rest frame with helicity λ :

$$B(p_0 \leftarrow \Lambda p)L(\Lambda p \leftarrow p)B(p \leftarrow p_0)|\mathbf{0}, \lambda\rangle = R(\Lambda, p)|\mathbf{0}, \lambda\rangle. \quad (3)$$

The Wigner rotation is applied to the rest state, and thus mixes the helicities. We then see that the effect any Lorentz transformation on a state is a rotation:

$$\begin{aligned} L(\Lambda p \leftarrow p)|\mathbf{p}, \lambda\rangle &= B(\Lambda p \leftarrow p_0)R(\Lambda, p)|\mathbf{0}, \lambda\rangle \\ &= \sum_{\lambda'} D_{\lambda'\lambda}^J(\Omega_W)|\Lambda p, \lambda'\rangle \end{aligned} \quad (4)$$

where J is the spin of the particle and Ω_W are the polar angles of the Wigner rotation (determined by Λ and p). The action of a Lorentz transformation on a particle is given by the representation of its little group. For a massive particle its little group are the rotations, thus any Lorentz transformation on a massive particle acts as a rotation. These consideration does not applied to massless particles since their little group is different. Massless and massive particles are therefore independent representations of the Poincaré group. This method was used by Wigner to classify the representations of the Poincaré group. More details can be found in Wigner original reference [3] or in chapter 2 in Weinberg [4].

All what we have to do is to determine the Wigner rotation $R(\Lambda, p) \equiv R(\Omega_W)$ for a given momentum p and a given Lorentz transformation Λ . But we first need to specify our convention for the boost $B(p \leftarrow p_0)$. Indeed since p_0 is invariant under rotation, there is an ambiguity in the definition of the boost. A boost that brings p_0 to a momentum p having angle $\Omega = (\theta, \phi)$ and rapidity ξ ($\cosh \xi = E/m$) is

$$B(p \leftarrow p_0) = R(\Omega)B_z(\xi)R(\psi). \quad (5)$$

The convention for helicity state is $\psi(p) = 0$. But for state quantized along the z axis (and for boosting momenta), the convention is $\psi(p) = -\Omega$.

With our general convention Eq. (5), the Wigner rotation becomes

$$R(\Omega_W) = R^{-1}(\psi')B_z^{-1}(\xi')R^{-1}(\Omega')L(\Lambda p \leftarrow p)R(\Omega)B_z(\xi)R(\psi) \quad (6)$$

With the notation $\psi' = \psi(p') = \psi(\Lambda p)$. The angles and rapidity of the boosted momentum p' are Ω' and ξ' . The calculation of the Wigner convention is simplified if, as in many applications, the quantization axis for rest states, the z axis, is in the plane formed by p and $p' = \Lambda p$. In other words, we consider Lorentz transformation in the xz plane. In this case the rotation are performed around the y axis, $R(\Omega_W) = R_y(\omega)$, $R(\Omega) = R_y(\theta)$ and $\psi^{(\prime)} = \theta^{(\prime)} + \pi$ for quantization of the third component of the spin. Our Lorentz transformation

$$L(\Lambda p \leftarrow p) = R_y(\alpha')B_z(\chi)R_y^{-1}(\alpha), \quad (7)$$

now involves only three parameters as it acts only in 2+1 dimensions. The Wigner rotation becomes

$$R_y(\omega + \psi' - \psi) = B_z^{-1}(\xi')R^{-1}(\theta' - \alpha')B_z(\chi)R_y(\theta - \alpha)B_z(\xi). \quad (8)$$

On the seven angles in Eq. (8), only three are independent: $\bar{\omega} \equiv \omega + \psi' - \psi$, $\bar{\theta}' = \theta' - \alpha'$ and $\bar{\theta} = \theta - \alpha$. This is because the knowledge of the Lorentz transformation (α', ξ, α) and the momentum p (θ, ξ) determines the transformed momentum p' (θ', ξ') .

Written in term of the independent angles, the Eq. (8)

$$R_y(\bar{\omega}) = B_z^{-1}(\xi')R^{-1}(\bar{\theta}')B_z(\chi)R_y(\bar{\theta})B_z(\xi) \quad (9)$$

admits a simple geometric interpretation in rapidity space, also called the Poincaré disk. In the Poincaré disk, the distance to the origin represents the rapidity and boost are performed along diameters or circles, cf. Fig 1. For a rigorous treatment of the Poincaré disk, see [5].

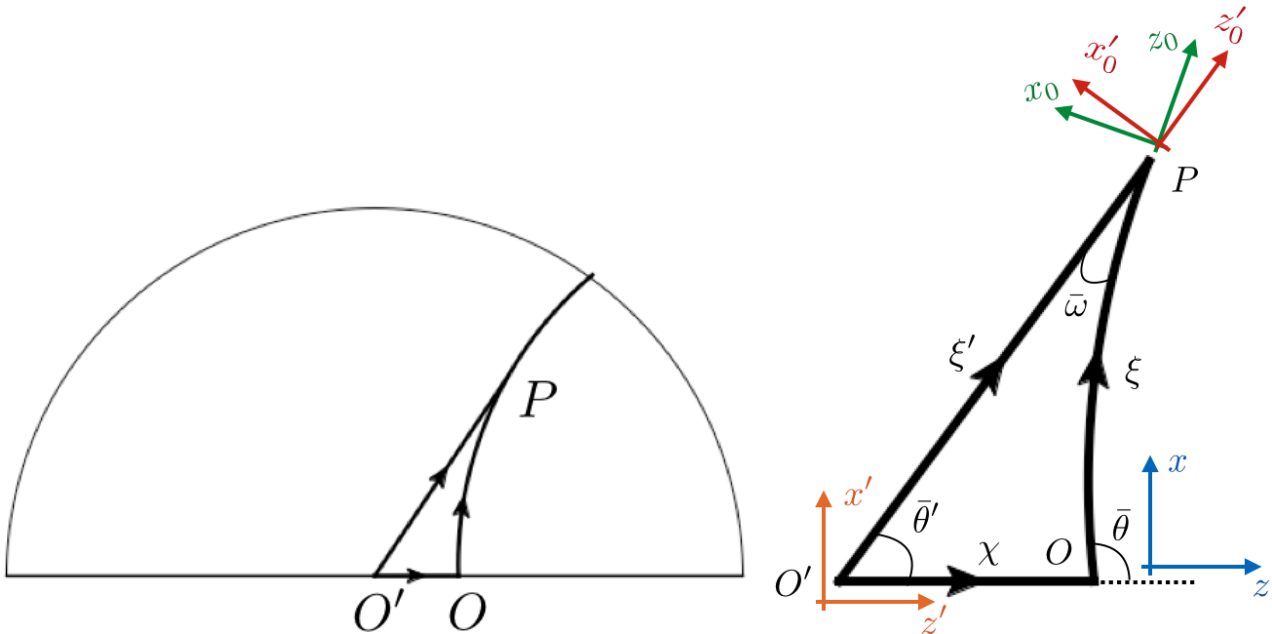


Figure 1: Right: The boost chain in the Poincaré disk. Left: the $OO'P$ hyperbolic triangle and the reference frames.

Since a rotation does not change a rest momentum, to determine the Wigner rotation we should adopt a passive point of view. In this picture the point P , representing the momentum, is fixed and the xz axes undergo the transformations.

Let us consider momentum at rest p_0 that undergoes the chain of transformations in Eq. (9). In the reference system Σ_0 , in green, the momentum is at rest, that is p_0 . The z axis in Σ_0 must be aligned with OP since the boost $B_z(\xi)$ brings the reference system Σ_0 from O to P where after the rotation $R_y(\theta)$ it becomes Σ . In the reference Σ , the blue axes, the point P is seen with angle $\bar{\theta}$ and rapidity ξ , that is $R_y(\bar{\theta})B_z(\xi)p_0$ as we just saw. The system Σ is then boost by χ along the z axis to become Σ' (in orange). In Σ' the point P is seen with an angle $\bar{\theta}'$ and distance (*i.e.* rapidity) ξ' , that is p' (strictly speaking the angle of p' is θ' but we have absorbed the rotation by α' from the general Lorentz transformation to focus only on the boosts). Finally the operation $B_z^{-1}(\xi')R^{-1}(\bar{\theta}')$ brings Σ' in O' to Σ'_0 in P where the momentum is at rest. Now we see that the z axis in Σ'_0 is aligned with $O'P$ and non collinear to the z axis in Σ_0 . Therefore the Wigner angle $\bar{\omega}$ is the angle between OP and $O'P$ in the hyperbolic triangle $OO'P$. We obtain directly from the law of sines in a hyperbolic triangle:

$$\frac{\sin \bar{\omega}}{\sinh \chi} = \frac{\sin \bar{\theta}'}{\sinh \xi} = \frac{\sin \bar{\theta}}{\sinh \xi'} \quad (10)$$

The relation (10) allows us to compute the Wigner angle $\omega = \bar{\omega} - \psi' + \psi$ from a given quantization method (ψ, ψ') , Lorentz transformation (α, α', χ) , $p(\theta, \xi)$ and $p'(\theta', \xi')$. Another equivalent relation we obtain directly from the hyperbolic triangle is

$$\cos \omega = \cos(\theta' - \alpha') \cos(\theta - \alpha) + \cosh \chi \sin(\theta' - \alpha') \sin(\theta - \alpha). \quad (11)$$

It is important to note that the Wigner angle ω is equal to $\bar{\omega}$ for helicity state ($\psi' = \psi = 0$) but is equal to $\omega = \bar{\omega} - \theta + \theta'$ for momenta and state quantized along the z direction. In the case of a pure rotation $\chi = 0$, Eq. (11) proves that the Wigner rotation is the rotation with an angle $\bar{\omega} = (\theta - \alpha) - (\theta' - \alpha')$. The state quantized along the z axis rotates then by $\omega = \alpha' - \alpha$ as expected.

The second hyperbolic law of cosine provides us the relation

$$\cosh \chi = \cosh \xi' \cosh \xi - \cos \bar{\omega} \sinh \xi' \sinh \xi. \quad (12)$$

Note that since this formula does not involve the angles $\bar{\theta}^{(\prime)}$, it applies only for pure Lorentz transformations (no rotation). It can be rewritten with the energies $E = m \cosh \xi$ and $E' = m \cosh \xi'$ of the initial and final momenta

$$\cos \bar{\omega} = \frac{E'E - m^2 \cosh \chi}{[(E^2 - m^2)(E'^2 - m^2)]^{1/2}}. \quad (13)$$

In order to derive a covariant representation of the Wigner angle, we need to represent the pure Lorentz transformation by two momenta a and b . They are such that before the boost, a is in its rest frame, *i.e.* $\mathbf{a} = \mathbf{0}$ and after the boost b is in its rest frame, *i.e.* $\Lambda \mathbf{b} = \mathbf{0}$. We can then express the initial and final energies as $E = a \cdot p$ and $E' = (\Lambda b) \cdot p' = b \cdot p$ (up to irrelevant normalizations that will cancel in $\cos \omega$). We obtain the Wigner angle for boost (no rotation) in term of the vectors (a, b) describing the boost in the following covariant form [2]

$$\cos \bar{\omega} = \frac{(p \cdot a)(p \cdot b) - m^2 a \cdot b}{\{[(p \cdot a)^2 - m^2 a^2][(p \cdot b)^2 - m^2 b^2]\}^{1/2}}. \quad (14)$$

It shows that the Wigner angle corresponds to the angle between the vectors a and b in p rest frame ! In Fig. 1, a and b correspond indeed to O and O' respectively.

The form (14) is very useful to express the Wigner rotation in term of invariant. For instance, in 2-to-2 scattering, the s - and t -channel are related via a (unphysical) boost. The crossing angles are therefore the Wigner angles associated to this boost. More precisely, let

us denote by $A_{\lambda_1\lambda_2\lambda_3\lambda_4}^s(p_i)$, the s -channel amplitude, the center-of-mass of $p_1 + p_2 \rightarrow p_3 + p_4$ and $A_{\mu_1\mu_2\mu_3\mu_4}^t(p_i)$, the t -channel amplitude, the center-of-mass of $p_1 + p_3 \rightarrow p_2 + p_4$. The convention that incoming (outgoing) particles have incoming (outgoing) momenta. With this convention $p_2 = -p_2$ and $p_3 = -p_3$. In this case the s -channel correspond to $a = p_1 + p_2$ and the t -channel correspond to $b = p_1 - p_3$. The crossing hypothesis states that

$$A_{\lambda_1\lambda_2\lambda_3\lambda_4}^s(p_i) = A_{\lambda_1\lambda_2\lambda_3\lambda_4}^t(\bar{p}_i). \quad (15)$$

We use the notation $p_i = \{p_1, p_2, p_3, p_4\}$ and $\bar{p}_i = \{p_1, -p_2, -p_3, p_4\}$ according to our convention. However the helicities λ_i are the s -channel ones since the p_i are in the s -channel. In practice, we need to convert them into t -channel helicities in A^t . We obtain

$$A_{\lambda_1\lambda_2\lambda_3\lambda_4}^t(\bar{p}_i) = \sum_{\mu_i} A_{\mu_1\mu_2\mu_3\mu_4}^t(\Lambda\bar{p}_i) d_{\mu_1\lambda_1}^{s_1}(\bar{\omega}_1) d_{\mu_2\lambda_2}^{s_2}(\bar{\omega}_2) d_{\mu_3\lambda_3}^{s_3}(\bar{\omega}_3) d_{\mu_4\lambda_4}^{s_4}(\bar{\omega}_4). \quad (16)$$

This is the well-known crossing relation, up to a phase. Indeed, we didn't use the Jacob-Wick convention for the second particle.

All the developments we performed assuming that the quantization axis for the rest states was in the (p, p') plane. This is the case for many applications including 2-to-2 scattering. When the axis of quantization of the rest frame, the z axe, is not in the plane determined by \mathbf{p} and \mathbf{p}' , the Wigner rotation $R(\Omega_W)$ is a rotation of angle ω in this plane, *i.e.* around $\mathbf{p}' \times \mathbf{p}$ as we will prove explicitly in the next section. The Wigner rotation for helicity state is recover then given by $R(\psi')R(\Omega_W)R(\psi)$.

Let us consider the decay chain

$$0 \rightarrow A + B \qquad A \rightarrow 1 + 2 \qquad B \rightarrow 3 + 4 \quad (17)$$

Let us write the total amplitude (schematically) as

$$A_{\mu_0\lambda_1\lambda_2\lambda_3\lambda_4} = \sum_{\lambda_A\mu_A\lambda_B\mu_B} \langle \mu_0 | \mathcal{M}_0 | \mu_A \mu_B \rangle_0 \langle \lambda_A | \mathcal{M}_A | \lambda_1 \lambda_2 \rangle_A \langle \lambda_B | \mathcal{M}_B | \lambda_3 \lambda_4 \rangle_B d_{\mu_A\lambda_A}^{s_A}(\omega_A) d_{\mu_B\lambda_B}^{s_B}(\omega_B). \quad (18)$$

The helicities μ_0, μ_A, μ_B are helicities in the 0 RF (rest frame), the helicities $\lambda_A, \lambda_1, \lambda_2$ in the A RF and the helicities $\lambda_B, \lambda_3, \lambda_4$ in the B RF. The Wigner rotations comes from the boost of particle A from the A RF to the 0 RF and for the boost of particle B from B RF to 0 RF. However since A (B) are its RF, the boost is "collinear" $\omega_A = 0$ ($\omega_B = 0$) and the decay amplitude becomes

$$A_{\mu_0\lambda_1\lambda_2\lambda_3\lambda_4} \propto \sum_{\lambda_A\lambda_B} D_{\mu_0, \lambda_B - \lambda_A}^{s_0^*}(\Omega_{B0}) D_{\lambda_A, \lambda_1 - \lambda_2}^{s_A^*}(\Omega_{1A}) D_{\lambda_A, \lambda_3 - \lambda_4}^{s_B^*}(\Omega_{3B}), \quad (19)$$

with Ω_{ij} the angles of i in the j RF. Note that μ_0 and λ_i are helicities in three different frames but since they are not measured, they are summed over. One could express the same amplitude with the helicities other frames. In this case there would be Wigner rotations but in general these rotations are not in the xz plane (defined in the 0 RF). It happens when there is another decay chain

$$0 \rightarrow C + B \qquad C \rightarrow 1 + 3 \qquad D \rightarrow 2 + 4 \quad (20)$$

One should make sure that the two amplitudes involves the same helicities (*i. e.* in the same frame) !

2 An explicit calculation

This calculation is taken from [6]. In order to compute the rotation in term of $p_1^\mu = (E_1, \mathbf{p}_1)$ and $p_2^\mu = (E_2, \mathbf{p}_2)$, we need an explicit form for the Lorentz transformations. A convenient representation involves the Pauli matrices $\sigma^\mu = (\mathbf{I}, \boldsymbol{\sigma})$ and the notation $\underline{p} = p_\mu \sigma^\mu$. We get back the components with $p^\mu = (1/2) \text{Tr}(\sigma^\mu \underline{p})$. With this representation a Lorentz transformation $p' = \Lambda p$ is

$$\underline{p}' = g(\Lambda) \underline{p} g^\dagger(\Lambda). \quad (21)$$

The matrix $r = g(\Lambda)$ for a rotation along the direction \mathbf{n} with angle θ is given by

$$r(\mathbf{n}, \theta) = \mathbf{I} \cos \frac{\theta}{2} + i \boldsymbol{\sigma} \cdot \mathbf{n} \sin \frac{\theta}{2} = \frac{\mathbf{I}(|\mathbf{p}_2||\mathbf{p}_1| + \mathbf{p}_2 \cdot \mathbf{p}_1) + i \boldsymbol{\sigma} \cdot (\mathbf{p}_2 \times \mathbf{p}_1)}{[(|\mathbf{p}_2||\mathbf{p}_1| + \mathbf{p}_2 \cdot \mathbf{p}_1)^2 + (\mathbf{p}_2 \times \mathbf{p}_1)^2]^{1/2}} = r(p_2 \leftarrow p_1) \quad (22)$$

The rotation matrix satisfies $r^\dagger(\mathbf{n}, \theta) = r(\mathbf{n}, -\theta)$. The angle and the direction are given by

$$\mathbf{n} = \frac{\mathbf{p}_2 \times \mathbf{p}_1}{|\mathbf{p}_2 \times \mathbf{p}_1|} \quad \cos \theta = \frac{\mathbf{p}_2 \cdot \mathbf{p}_1}{|\mathbf{p}_2||\mathbf{p}_1|} \quad (23)$$

The matrix $b \equiv g(\Lambda)$ for a boost along the direction \mathbf{n} with rapidity η is given by

$$b(\mathbf{n}, \eta) = \mathbf{I} \cosh \frac{\eta}{2} + \boldsymbol{\sigma} \cdot \mathbf{n} \sinh \frac{\eta}{2} = \frac{\mathbf{I}(E_2 + E_1) + \boldsymbol{\sigma} \cdot (\mathbf{p}_2 - \mathbf{p}_1)}{[(E_2 + E_1)^2 - (\mathbf{p}_2 - \mathbf{p}_1)^2]^{1/2}} = b(p_2 \leftarrow p_1) \quad (24)$$

The boost matrix satisfies $b^\dagger(\mathbf{n}, \eta) = b(\mathbf{n}, \eta)$. The rapidity and the direction are given by

$$\mathbf{n} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{|\mathbf{p}_2 - \mathbf{p}_1|} \quad \cosh \eta = \frac{(E_2 + E_1)^2 + (\mathbf{p}_2 - \mathbf{p}_1)^2}{(E_2 + E_1)^2 - (\mathbf{p}_2 - \mathbf{p}_1)^2} \quad (25)$$

It is worth mentioning that this convention for the boost correspond to the choice $\psi = -\Omega$, *i.e.* the quantization method using a fixed axis. It can be checked that indeed $b(\mathbf{n}, \eta) = r(\hat{\mathbf{n}} \leftarrow \hat{\mathbf{z}}) b(\hat{\mathbf{z}}, \eta) r(\hat{\mathbf{z}} \leftarrow \hat{\mathbf{n}})$.

We are in position to compute the equivalent rotation of the boost chain in Eq. (1):

$$\begin{aligned} b(p_0 \leftarrow p_2) b(p_2 \leftarrow p) b(p \leftarrow p_0) &= \frac{\mathbf{I}[(E_2 + m)(E_1 + m) + \mathbf{p}_2 \cdot \mathbf{p}_1] + i \boldsymbol{\sigma} \cdot (\mathbf{p}_2 \times \mathbf{p}_1)}{[2(E_2 + m)(E_1 + m)(m^2 + E_2 E_1 + \mathbf{p}_2 \cdot \mathbf{p}_1)]^{1/2}} \\ &\equiv r(\mathbf{n}_W, \omega) \end{aligned} \quad (26)$$

This is indeed a rotation with direction and angle:

$$\mathbf{n}_W = \frac{\mathbf{p}_2 \times \mathbf{p}_1}{|\mathbf{p}_2 \times \mathbf{p}_1|} \quad \sin \omega = \frac{|\mathbf{p}_2 \times \mathbf{p}_1|}{m^2 + E_1 E_2 + \mathbf{p}_2 \cdot \mathbf{p}_1} \left(1 + \frac{\mathbf{p}_2 \cdot \mathbf{p}_1}{(E_2 + m)(E_1 + m)} \right) \quad (27)$$

This is the Wigner rotation for momenta and states with the third component of their spin quantized. As already discussed this is not the Wigner rotation for helicity states.

All formulas can be recast in a more traditional form with the Lorentz factors γ_i with the relations

$$\gamma_1 = E_1/m = \cosh \xi_1 \quad \gamma_1 \boldsymbol{\beta}_1 = \mathbf{p}_1/m = \mathbf{n}_1 \sinh \xi_1 \quad (28a)$$

$$\gamma_2 = E_2/m = \cosh \xi_2 \quad \gamma_2 \boldsymbol{\beta}_2 = \mathbf{p}_2/m = \mathbf{n}_2 \sinh \xi_2 \quad (28b)$$

with $\mathbf{n}_{1,2}$ the normalized vectors in the direction of $\mathbf{p}_{1,2}$. The composition of boost gives a Lorentz factor γ and the angle of the Wigner rotation becomes

$$\gamma = \gamma_1 \gamma_2 (1 + \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2) \quad \sin \omega = |\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2| \frac{\gamma_1 \gamma_2 (1 + \gamma_2 + \gamma_2 + \gamma)}{(1 + \gamma_1)(1 + \gamma_2)(1 + \gamma)} \quad (29)$$

References

- [1] G. C. Wick, *Annals Phys.* **18**, 65 (1962). doi:10.1016/0003-4916(62)90059-3
- [2] J. Gomatam, L. S. O' Raifeartaigh and D. H. Tchrakian, *Teor. Mat. Fiz.* **19**, 201 (1974). doi:10.1007/BF01035945
- [3] E. P. Wigner, *Annals Math.* **40**, 149 (1939) [*Nucl. Phys. Proc. Suppl.* **6**, 9 (1989)]. doi:10.2307/1968551
- [4] S. Weinberg,
- [5] J. A. Rhodes and M. D. Semon, *Am. J. Phys.* **72**, 943 (2004) doi:10.1119/1.1652040 [gr-qc/0501070].
- [6] L. A. Kondratyuk and M. V. Terentev, *Sov. J. Nucl. Phys.* **31**, 561 (1980) [*Yad. Fiz.* **31**, 1087 (1980)].