

1 Lorentz Invariants

The reaction is $a + b \rightarrow 1 + 2 + 3$. We define 10 invariants as illustrated on the figure.

$$s = (p_a + p_b)^2 \quad s_{ij} = (p_i + p_j)^2 \quad t_i = (p_a - p_i)^2 \quad u_i = (p_b - p_i)^2 \quad (1)$$

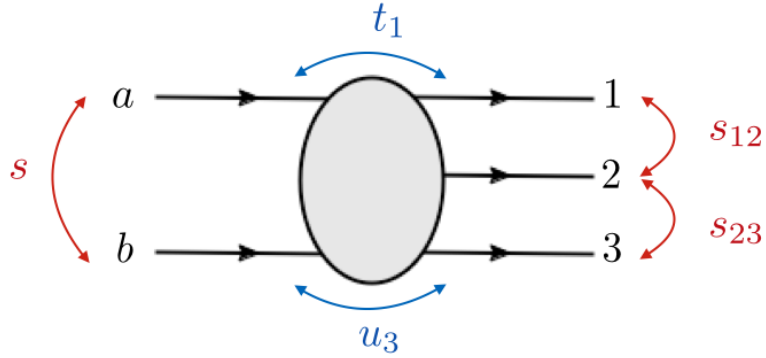


Figure 1: Definition of the invariants.

Only 5 of them are independent. They are subject to the constraints

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ s_{12} \\ s_{23} \\ t_1 \\ u_3 \end{pmatrix} = \begin{pmatrix} m_a^2 + m_b^2 + m_3^2 - t_3 \\ m_a^2 + m_b^2 + m_1^2 - u_1 \\ m_a^2 + m_1^2 + m_2^2 - t_2 \\ m_b^2 + m_2^2 + m_3^2 - u_2 \\ m_1^2 + m_2^2 + m_3^2 - s_{13} \end{pmatrix} \quad (2)$$

The 10 scalar products are, in terms of five independent $(s, t_1, u_3, s_{12}, s_{23})$,

$$2p_a \cdot p_b = s - m_a^2 - m_b^2 \quad 2p_b \cdot p_2 = s_{23} + u_3 - t_1 - m_3^2 \quad (3a)$$

$$2p_a \cdot p_1 = m_a^2 + m_1^2 - t_1 \quad 2p_b \cdot p_3 = m_b^2 + m_3^2 - u_3 \quad (3b)$$

$$2p_a \cdot p_2 = s_{12} + t_1 - u_3 - m_1^2 \quad 2p_1 \cdot p_2 = s_{12} - m_1^2 - m_2^2 \quad (3c)$$

$$2p_a \cdot p_3 = s - s_{12} + u_3 - m_b^2 \quad 2p_1 \cdot p_3 = s - s_{12} - s_{23} + m_2^2 \quad (3d)$$

$$2p_b \cdot p_1 = s - s_{23} + t_1 - m_a^2 \quad 2p_2 \cdot p_3 = s_{23} - m_2^2 - m_3^2 \quad (3e)$$

2 The Gram determinant

The Gram determinant of order 4 is

$$\Delta_4 = -\frac{1}{16} \begin{vmatrix} 0 & m_2^2 & s_{23} & t_1 & m_1^2 & 1 \\ m_2^2 & 0 & m_3^2 & u_3 & s_{12} & 1 \\ s_{23} & m_3^2 & 0 & m_b^2 & s & 1 \\ t_1 & u_3 & m_b^2 & 0 & m_a^2 & 1 \\ m_1^2 & s_{12} & s & m_a^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} \quad (4)$$

In the massless case we obtain

$$16\Delta_4 = (st_1 - t_1s_{12} + s_{12}s_{23} - s_{23}u_3 + u_3s)^2 - 4st_1u_3(s - s_{12} - s_{23}) \quad (5)$$

The Gram determinant of order 4 is at most quadratic in the invariant.

$$16\Delta_4 = -\frac{1}{4t_1^2} \lambda(u_3, t_1, m_2^2) \lambda(t_1, m_1^2, m_a^2) \lambda(t_1, m_b^2, s_{23}) \sin^2 \chi \sin^2 \theta_2 \sin^2 \omega \quad (6a)$$

$$= -\frac{1}{4t_1u_3} \lambda(u_3, t_1, m_2^2) \lambda(t_1, m_1^2, m_a^2) \lambda(u_3, m_b^2, m_3^2) \sin^2 \theta_1 \sin^2 \theta_2 \sin^2 \omega \quad (6b)$$

$$= -\frac{1}{4s_{12}^2} \lambda(s_{12}, m_1^2, m_2^2) \lambda(s, s_{12}, m_3^2) \lambda(s_{12}, u_3, m_a^2) \sin^2 \epsilon \sin^2 \theta \sin^2 \phi \quad (6c)$$

$$= \frac{1}{4s_{12}u_3} \lambda(s_{12}, m_1^2, m_2^2) \lambda(u_3, m_b^2, m_3^2) \lambda(s_{12}, u_3, m_a^2) \sin^2 \bar{\theta}_1 \sin^2 \theta \sin^2 \phi \quad (6d)$$

$$= -\frac{1}{4s_{23}^2} \lambda(s_{23}, m_2^2, m_3^2) \lambda(s, s_{23}, m_1^2) \lambda(s_{23}, t_1, m_b^2) \sin^2 \bar{\epsilon} \sin^2 \bar{\theta} \sin^2 \bar{\phi} \quad (6e)$$

The angles are computed in the next sections. ϵ, θ, ϕ are defined in the (12) Gottfried-Jackson frame ; $\bar{\epsilon}, \bar{\theta}, \bar{\phi}$ are defined in the (23) Gottfried-Jackson frame ; $\theta_1, \theta_2, \omega$ are defined in the double Regge frames.

The triangle function is $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ca)$.

3 Cross section

The cross section is

$$\sigma = \frac{1}{F} R_3(s) \overline{\sum} |M|^2 \quad R_3(s) = \int \frac{d^3\mathbf{p}_1}{2E_1} \frac{d^3\mathbf{p}_2}{2E_2} \frac{d^3\mathbf{p}_3}{2E_3} \delta^4(p_a + p_b - p_1 - p_2 - p_3) \quad (7)$$

with $\overline{\sum}$ the sum over helicities, average over the initial ones and with the flux being

$$F = 2\lambda^{1/2}(s, m_a^2, m_b^2) (2\pi)^{3 \times 3 - 4} \quad (8)$$

In the GJ frames we obtain, with Φ , the polarization angle,

$$\sigma = \frac{1}{(2\pi)^3} \frac{1}{\lambda(s, m_a^2, m_b^2)} \int_0^{2\pi} \frac{d\Phi}{2\pi} \int_{u_3^-}^{u_3^+} \int_{s_{12}^-}^{s_{12}^+} \frac{\lambda^{1/2}(s_{12}, m_1^2, m_2^2)}{32s_{12}} du_3 ds_{12} \int \frac{d\Omega}{4\pi} \overline{\sum} |M|^2 \quad (9a)$$

$$= \frac{1}{(2\pi)^3} \frac{1}{\lambda(s, m_a^2, m_b^2)} \int_0^{2\pi} \frac{d\Phi}{2\pi} \int_{t_1^-}^{t_1^+} \int_{s_{23}^-}^{s_{23}^+} \frac{\lambda^{1/2}(s_{23}, m_2^2, m_3^2)}{32s_{23}} dt_1 ds_{23} \int \frac{d\Omega}{4\pi} \overline{\sum} |M|^2 \quad (9b)$$

4 (12) Gottfried-Jackson frame

We use the invariant variables s, t_1, s_{12} and the two angles $\Omega = (\theta, \phi)$ of particle 1 in the rest frame of (12). The phase space can be rewritten into a product of phase space for $a+b \rightarrow (12)+3$ and $(12) \rightarrow 1+2$:

$$\sigma = \frac{1}{(2\pi)^3} \frac{1}{\lambda(s, m_a^2, m_b^2)} \int_0^{2\pi} \frac{d\Phi}{2\pi} \int_{u_3^-}^{u_3^+} \int_{s_{12}^-}^{s_{12}^+} \frac{\lambda^{1/2}(s_{12}, m_1^2, m_2^2)}{32s_{12}} du_3 ds_{12} \int \frac{d\Omega}{4\pi} \overline{\sum} |M|^2 \quad (10)$$

The integration over the angle Φ between the two decay planes can be done if the target and the beam are not polarized. The boundaries are given by

$$u_3^\pm = m_b^2 + m_3^2 - \frac{1}{2s} [(s + m_b^2 - m_a^2)(s - s_{12} + m_3^2) \mp \lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(s, s_{12}, m_3^2)] \quad (11a)$$

$$s_{12}^\pm = s + m_3^2 - \frac{1}{2m_b^2} [(s + m_b^2 - m_a^2)(m_b^2 + m_3^2 - u_3) \mp \lambda^{1/2}(s, m_b^2, m_3^2)\lambda^{1/2}(u_3, m_b^2, m_3^2)] \quad (11b)$$

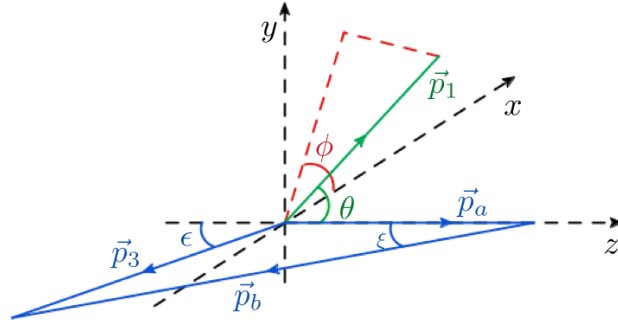


Figure 2: Momenta in the (12) GF frame. The momenta of particle 2 is not indicated, $\mathbf{p}_2 = -\mathbf{p}_1$.

To define the angle (θ, ϕ) , we place the z axis along the direction of the beam and choose them as the angle of particle 1. In that case the momenta are (their norm are $|\mathbf{p}_i|^2 = E_i^2 - m_i^2$)

$$\mathbf{p}_a = |\mathbf{p}_a|(0, 0, 1) \quad E_a = (s_{12} + m_a^2 - u_3)/2\sqrt{s_{12}} \quad (12a)$$

$$\mathbf{p}_b = |\mathbf{p}_b|(-\sin \xi, 0, -\cos \xi) \quad E_b = (s_{12} + m_b^2 - t_3)/2\sqrt{s_{12}} \quad (12b)$$

$$\mathbf{p}_3 = |\mathbf{p}_3|(-\sin \epsilon, 0, -\cos \epsilon) \quad E_3 = -(s_{12} + m_3^2 - s)/2\sqrt{s_{12}} \quad (12c)$$

$$\mathbf{p}_1 = |\mathbf{p}_1|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad E_1 = (s_{12} + m_1^2 - m_2^2)/2\sqrt{s_{12}} \quad (12d)$$

$$\mathbf{p}_2 = -|\mathbf{p}_1|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad E_2 = (s_{12} + m_2^2 - m_1^2)/2\sqrt{s_{12}} \quad (12e)$$

We get $E_b = (s + u_3 - m_a^2 - m_3^2)/2\sqrt{s_{12}}$. The angles are

$$2|\mathbf{p}_a||\mathbf{p}_b| \cos \xi = s - 2E_a E_b - m_a^2 - m_b^2 \quad (13a)$$

$$2|\mathbf{p}_a||\mathbf{p}_3| \cos \epsilon = s - 2E_a E_3 - m_b^2 - s_{12} + u_3 \quad (13b)$$

All energies and the angles ξ and ϵ depend only on s, s_{12} and u_3 . The angles θ and ϕ must depend on the remaining invariants s_{23} and t_1 . Indeed we have

$$t_1 = m_a^2 + m_1^2 - 2E_a E_1 + 2|\mathbf{p}_a||\mathbf{p}_1| \cos \theta \quad (14a)$$

$$s_{23} = m_2^2 + m_3^2 + 2E_2 E_3 - 2|\mathbf{p}_1||\mathbf{p}_3| (\sin \epsilon \sin \theta \cos \phi + \cos \epsilon \cos \theta) \quad (14b)$$

5 (23) Gottfried-Jackson frame

We use the invariant variables s, t_1, s_{23} and the two angles $\bar{\Omega} = (\bar{\theta}, \bar{\phi})$ of particle 3 in the rest frame of (23). The phase space can be rewritten into a product of phase space for $a+b \rightarrow 1+(23)$ and $(23) \rightarrow 2+3$:

$$\sigma = \frac{1}{(2\pi)^3} \frac{1}{\lambda(s, m_a^2, m_b^2)} \int_0^{2\pi} \frac{d\Phi}{2\pi} \int_{t_1^-}^{t_1^+} \int_{s_{23}^-}^{s_{23}^+} \frac{\lambda^{1/2}(s_{23}, m_2^2, m_3^2)}{32s_{23}} dt_1 ds_{23} \int \frac{d\Omega}{4\pi} \sum |M|^2 \quad (15)$$

The integration over the angle Φ between the decay planes can be done if the target and the beam are not polarized. The boundaries are given by

$$t_1^\pm = m_a^2 + m_1^2 - \frac{1}{2s} [(s + m_a^2 - m_b^2)(s - s_{23} + m_1^2) \mp \lambda^{1/2}(s, m_a^2, m_1^2)\lambda^{1/2}(s, s_{23}, m_1^2)] \quad (16a)$$

$$s_{23}^\pm = s + m_1^2 - \frac{1}{2m_1^2} [(s + m_a^2 - m_b^2)(m_a^2 + m_1^2 - t_1) \mp \lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(t_1, m_a^2, m_1^2)] \quad (16b)$$

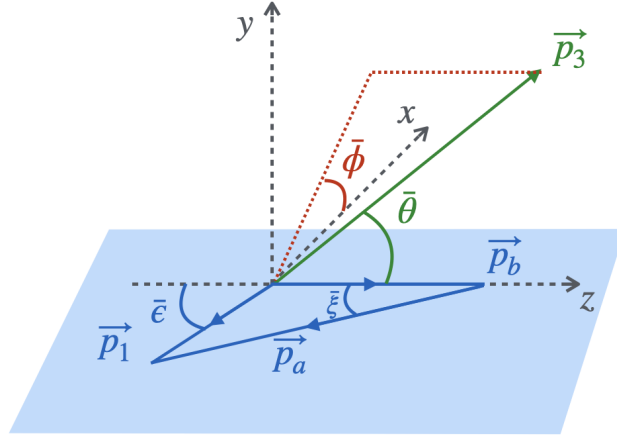


Figure 3: Momenta in the (23) RF. The momenta of particle 2 and 3 are outside the xz plane.

To define the angle $(\bar{\theta}, \bar{\phi})$, we place the z axis along the direction of the target and choose them as the angle of particle 3. In that case the momenta are (their norm are $|\mathbf{p}_i|^2 = E_i - m_i^2$)

$$\mathbf{p}_b = |\mathbf{p}_b|(0, 0, 1) \quad E_b = (s_{23} + m_3^2 - t_1)/2\sqrt{s_{23}} \quad (17a)$$

$$\mathbf{p}_a = |\mathbf{p}_a|(-\sin \bar{\xi}, 0, -\cos \bar{\xi}) \quad E_a = (s_{23} + m_a^2 - u_1)/2\sqrt{s_{23}} \quad (17b)$$

$$\mathbf{p}_1 = |\mathbf{p}_1|(-\sin \bar{\epsilon}, 0, -\cos \bar{\epsilon}) \quad E_1 = (s - s_{23} - m_1^2)/2\sqrt{s_{23}} \quad (17c)$$

$$\mathbf{p}_2 = -|\mathbf{p}_2|(\sin \bar{\theta} \cos \bar{\phi}, \sin \bar{\theta} \sin \bar{\phi}, \cos \bar{\theta}) \quad E_2 = (s_{23} + m_2^2 - m_3^2)/2\sqrt{s_{23}} \quad (17d)$$

$$\mathbf{p}_3 = |\mathbf{p}_3|(\sin \bar{\theta} \cos \bar{\phi}, \sin \bar{\theta} \sin \bar{\phi}, \cos \bar{\theta}) \quad E_3 = (s_{23} + m_3^2 - m_2^2)/2\sqrt{s_{23}} \quad (17e)$$

We obtain $E_a = (s + t_1 - m_b^2 - m_1^2)/2\sqrt{s_{23}}$. The angle are

$$2|\mathbf{p}_a||\mathbf{p}_b| \cos \bar{\xi} = s - 2E_a E_b - m_a^2 - m_b^2 \quad (18a)$$

$$2|\mathbf{p}_1||\mathbf{p}_b| \cos \bar{\epsilon} = s - 2E_1 E_b - m_a^2 - s_{23} + t_1 \quad (18b)$$

All energies and the angles $\bar{\chi}$ and $\bar{\epsilon}$ depend only on s, s_{23} and t_1 . The angles $\bar{\theta}$ and $\bar{\phi}$ must depend on the remaining invariants s_{12} and u_3 . Indeed we have

$$u_3 = m_b^2 + m_3^2 - 2E_3 E_b + 2|\mathbf{p}_2||\mathbf{p}_b| \cos \bar{\theta} \quad (19a)$$

$$s_{12} = m_1^2 + m_2^2 + 2E_1 E_2 - 2|\mathbf{p}_2||\mathbf{p}_1| (\sin \bar{\epsilon} \sin \bar{\theta} \cos \bar{\phi} + \cos \bar{\epsilon} \cos \bar{\theta}) \quad (19b)$$

6 (13) Gottfried-Jackson frame

We use the invariant variables s, t_2, s_{13} and the two angles $\Omega' = (\theta', \phi')$ of particle 3 in the rest frame of (13). The phase space can be rewritten into a product of phase space for $a+b \rightarrow 2+(13)$ and $(13) \rightarrow 1+3$:

$$\sigma = \frac{1}{(2\pi)^3} \frac{1}{\lambda(s, m_a^2, m_b^2)} \int_0^{2\pi} \frac{d\Phi}{2\pi} \int_{t_2^-}^{t_2^+} \int_{s_{13}^-}^{s_{13}^+} \frac{\lambda^{1/2}(s_{13}, m_1^2, m_3^2)}{32s_{13}} dt_2 ds_{13} \int \frac{d\Omega'}{4\pi} \sum |M|^2 \quad (20)$$

Compared to (V.5.6) in Particle Kinematics, the integration over the angle between the decay planes has been done since the target and the beam are not polarized. The boundaries are given by

$$t_2^\pm = m_a^2 + m_2^2 - \frac{1}{2s} [(s + m_a^2 - m_b^2)(s - s_{13} + m_2^2) \mp \lambda^{1/2}(s, m_a^2, m_2^2)\lambda^{1/2}(s, s_{13}, m_2^2)] \quad (21a)$$

$$s_{13}^\pm = s + m_2^2 - \frac{1}{2m_2^2} [(s + m_a^2 - m_b^2)(m_a^2 + m_2^2 - t_2) \mp \lambda^{1/2}(s, m_a^2, m_b^2)\lambda^{1/2}(t_2, m_a^2, m_2^2)] \quad (21b)$$

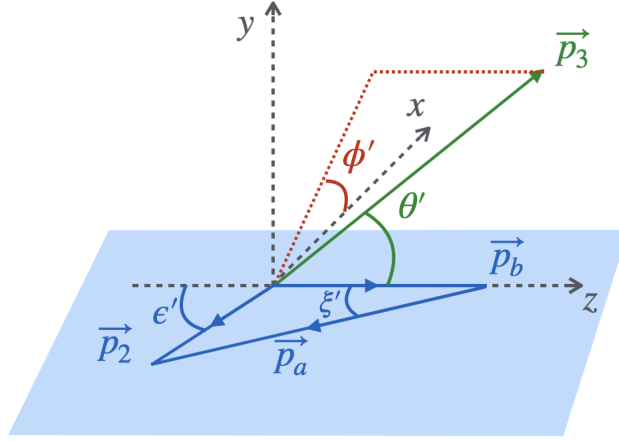


Figure 4: Momenta in the (13) RF. The momenta of particle 1 and 3 are outside the xz plane.

To define the angle (θ', ϕ') , we place the z axis along the direction of the target and choose them as the angle of particle 3. In that case the momenta are (their norm are $|\mathbf{p}_i|^2 = E_i - m_i^2$)

$$\mathbf{p}_b = |\mathbf{p}_b|(0, 0, 1) \quad E_b = (s_{13} + m_3^2 - t_2)/2\sqrt{s_{13}} \quad (22a)$$

$$\mathbf{p}_a = |\mathbf{p}_a|(-\sin \xi', 0, -\cos \xi') \quad E_a = (s_{13} + m_a^2 - u_2)/2\sqrt{s_{13}} \quad (22b)$$

$$\mathbf{p}_2 = |\mathbf{p}_2|(-\sin \epsilon', 0, -\cos \epsilon') \quad E_2 = (s - s_{13} - m_2^2)/2\sqrt{s_{13}} \quad (22c)$$

$$\mathbf{p}_1 = -|\mathbf{p}_3|(\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta') \quad E_1 = (s_{13} + m_1^2 - m_3^2)/2\sqrt{s_{13}} \quad (22d)$$

$$\mathbf{p}_3 = |\mathbf{p}_3|(\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta') \quad E_3 = (s_{13} + m_3^2 - m_1^2)/2\sqrt{s_{13}} \quad (22e)$$

We obtain $E_a = (s + t_2 - m_b^2 - m_2^2)/2\sqrt{s_{13}}$. The angle are

$$2|\mathbf{p}_a||\mathbf{p}_b| \cos \xi' = s - 2E_a E_b - m_a^2 - m_b^2 \quad (23a)$$

$$2|\mathbf{p}_2||\mathbf{p}_b| \cos \epsilon' = s - 2E_2 E_b - m_a^2 - s_{13} + t_2 \quad (23b)$$

All energies and the angles χ' and ϵ' depend only on s, s_{13} and t_2 . The angles θ' and ϕ' must depend on the remaining invariants s_{12} and u_3 . Indeed we have

$$u_3 = m_b^2 + m_3^2 - 2E_3 E_b + 2|\mathbf{p}_2||\mathbf{p}_b| \cos \bar{\theta} \quad (24a)$$

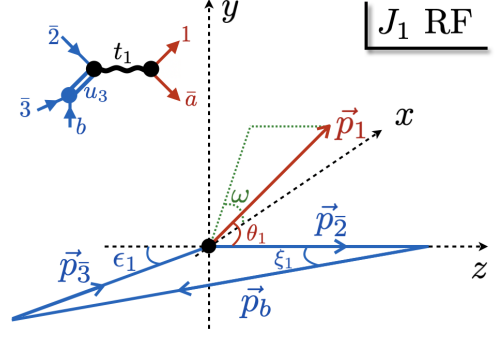
$$s_{12} = m_1^2 + m_2^2 + 2E_1 E_2 - 2|\mathbf{p}_2||\mathbf{p}_1| (\sin \bar{\epsilon} \sin \bar{\theta} \cos \bar{\phi} + \cos \bar{\epsilon} \cos \bar{\theta}) \quad (24b)$$

7 Double Regge Kinematics

In the double Regge limit with the central particle 2, the system is described by two invariants t_1 and u_3 , the invariant mass squared of the Reggeons, and three angles. In the rest frame of $b\bar{3} \rightarrow 2(\bar{a}1)$, θ_2 is the angle between 2 and b . In the rest frame of $(b\bar{3})\bar{2} \rightarrow \bar{a}1$, θ_1 and ω are the polar and azimuthal angle of 1 when $\bar{2}$ is along the z axis with b and $\bar{3}$ in the xz plane.

For crossed particles, the momentum is flipped $p_x = -p_x$. In the J_1 rest frame, the z axis in the \mathbf{p}_2 direction. We choose $\mathbf{y} = \mathbf{p}_b \times \mathbf{p}_2$ and $\mathbf{x} = \mathbf{y} \times \mathbf{z}$.

The momenta are



$$\mathbf{p}_2 = |\mathbf{p}_2|(0, 0, 1) \quad |\mathbf{p}_2| = \frac{\lambda^{1/2}(t_1, u_3, m_2^2)}{2\sqrt{t_1}} \quad E_2 = \frac{t_1 + m_2^2 - u_3}{2\sqrt{t_1}} \quad (25a)$$

$$\mathbf{p}_b = |\mathbf{p}_b|(-\sin \xi_1, 0, -\cos \xi_1) \quad |\mathbf{p}_b| = \frac{\lambda^{1/2}(t_1, s_{23}, m_b^2)}{2\sqrt{t_1}} \quad E_b = \frac{t_1 + m_b^2 - s_{23}}{2\sqrt{t_1}} \quad (25b)$$

$$\mathbf{p}_{\bar{3}} = |\mathbf{p}_{\bar{3}}|(\sin \epsilon_1, 0, \cos \epsilon_1) \quad |\mathbf{p}_{\bar{3}}| = \frac{\lambda^{1/2}(t_1, u_2, m_3^2)}{2\sqrt{t_1}} \quad E_{\bar{3}} = \frac{t_1 + m_3^2 - u_2}{2\sqrt{t_1}} \quad (25c)$$

$$\mathbf{p}_1 = |\mathbf{p}_1|(\sin \theta_1 \cos \omega, \sin \theta_1 \sin \omega, \cos \theta_1) \quad |\mathbf{p}_1| = \frac{\lambda^{1/2}(t_1, m_a^2, m_1^2)}{2\sqrt{t_1}} \quad E_1 = \frac{t_1 + m_1^2 - m_a^2}{2\sqrt{t_1}} \quad (25d)$$

$$\mathbf{p}_{\bar{a}} = -|\mathbf{p}_1|(\sin \theta_1 \cos \omega, \sin \theta_1 \sin \omega, \cos \theta_1) \quad E_{\bar{a}} = \frac{t_1 + m_a^2 - m_1^2}{2\sqrt{t_1}} \quad (25e)$$

One can check that $E_2 + E_b + E_{\bar{3}} = \sqrt{t_1}$.

The scattering angle θ_1 is related to $s_{12} = (p_1 + p_2)^2 = (p_1 - p_2)^2$. The angle ξ_1 , the angle between \mathbf{p}_b and $\mathbf{p}_2 = -\mathbf{p}_{\bar{2}} = \mathbf{p}_b + \mathbf{p}_{\bar{3}}$ is obtained by evaluating $m_3^2 = [(p_{\bar{3}} + p_b) - p_b]^2$. The angle ϵ_1 of $\mathbf{p}_{\bar{3}}$ is obtained similarly. We obtain

$$\cos \theta_1 = \frac{2t_1(s_{12} - m_1^2 - m_2^2) + (t_1 + m_1^2 - m_a^2)(t_1 + m_2^2 - u_3)}{\lambda^{1/2}(t_1, m_1^2, m_a^2)\lambda^{1/2}(u_3, t_1, m_2^2)} \quad (26a)$$

$$\cos \xi_1 = \frac{2t_1(m_3^2 - m_b^2 - u_3) + (t_1 + u_3 - m_2^2)(t_1 + m_b^2 - s_{23})}{\lambda^{1/2}(t_1, m_b^2, s_{23})\lambda^{1/2}(u_3, t_1, m_2^2)} \quad (26b)$$

$$\cos \epsilon_1 = \frac{2t_1(u_3 + m_3^2 - m_b^2) - (t_1 + u_3 - m_2^2)(t_1 + m_3^2 - u_2)}{\lambda^{1/2}(t_1, m_3^2, u_2)\lambda^{1/2}(u_3, t_1, m_2^2)} \quad (26c)$$

They satisfy the relation $|\mathbf{p}_2| = |\mathbf{p}_3| \cos \epsilon_1 - |\mathbf{p}_b| \cos \xi_1$. The azimuthal angle, or Toller angle, is obtained by expressing $s = (p_a + p_b)^2 = (p_{\bar{a}} - p_b)^2$.

$$s = m_a^2 + m_b^2 - \frac{1}{2t_1}(t_1 + m_a^2 - m_1^2)(t_1 + m_b^2 - s_{23}) + \frac{1}{2t_1}\lambda^{1/2}(t_1, m_a^2, m_1^2)\lambda^{1/2}(t_1, m_b^2, s_{23})(\sin \xi_1 \sin \theta_1 \cos \omega + \cos \xi_1 \cos \theta_1) \quad (27)$$

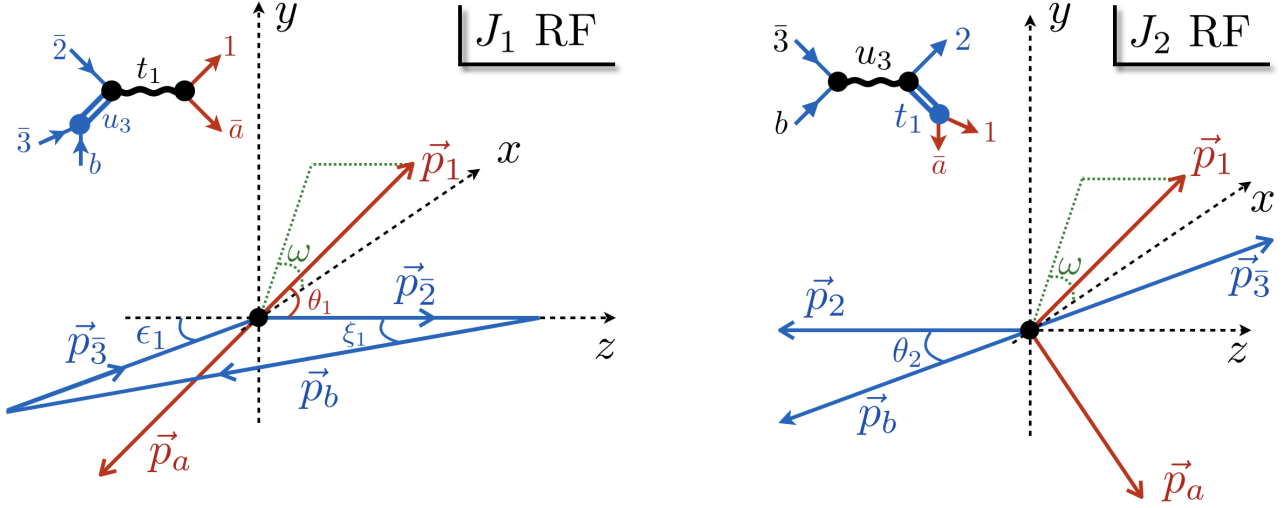


Figure 5: The vectors in blue are in the xz plane. The plane formed by \vec{p}_2 , \vec{p}_1 and \vec{p}_a has an angle ω with the xz plane.

We now boost to bring $(b\bar{3})$ to its rest frame, the J_2 RF. The boost is along the z direction:

$$\begin{pmatrix} E' \\ p'_z \end{pmatrix} = \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix} \begin{pmatrix} E \\ p_z \end{pmatrix} \quad \begin{aligned} \cosh \chi &= (t_1 + u_3 - m_2^2)/(2\sqrt{t_1 u_3}) \\ \sinh \chi &= \lambda^{1/2}(t_1, u_3, m_2^2)/(2\sqrt{t_1 u_3}) \end{aligned} \quad (28)$$

After the boost, the momenta become (particle 2 has been crossed $p_2 = -p_2$)

$$p_2 = \left(\frac{u_3 + m_2^2 - t_1}{2\sqrt{u_3}}, 0, 0, -\frac{\lambda^{1/2}(t_1, u_3, m_2^2)}{2\sqrt{u_3}} \right) \quad (29a)$$

$$\begin{aligned} p_3 &= \left(\frac{u_3 + m_3^2 - m_b^2}{2\sqrt{u_3}}, |\mathbf{p}_3| \sin \epsilon_1, 0, E_3 \sinh \chi + \cosh \chi |\mathbf{p}_3| \cos \epsilon_1 \right) \\ &= \left(\frac{u_3 + m_3^2 - m_b^2}{2\sqrt{u_3}}, |\mathbf{p}'_3| \sin \theta_2, 0, |\mathbf{p}'_3| \cos \theta_2 \right) \end{aligned} \quad (29b)$$

The norm of the momentum in the J_2 rest frame is $|\mathbf{p}'_3| = |\mathbf{p}_b| = \lambda^{1/2}(u_3, m_3^2, m_b^2)/2\sqrt{u_3}$. The scattering angle in the J_2 rest frame, θ_2 , is related to $s_{23} = (p_2 + p_3)^2 = (p_2 - p_3)^2$:

$$\cos \theta_2 = -\frac{2u_3(s_{23} - m_2^2 - m_3^2) + (u_3 + m_3^2 - m_b^2)(u_3 + m_2^2 - t_1)}{\lambda^{1/2}(u_3, m_b^2, m_3^2)\lambda^{1/2}(u_3, t_1, m_2^2)} \quad (30a)$$

The component perpendicular to the $(b23)$ plane is invariant $|\mathbf{p}'_3| \sin \theta_2 = |\mathbf{p}_3| \sin \epsilon_1 = |\mathbf{p}_b| \sin \xi_1$. We thus obtain

$$\sin \xi_1 = \sin \theta_2 \sqrt{\frac{t_1}{u_3} \frac{\lambda^{1/2}(u_3, m_b^2, m_3^2)}{\lambda^{1/2}(t_1, m_b^2, s_{23})}} \quad \text{and} \quad \sin \epsilon_1 = \sin \theta_2 \sqrt{\frac{t_1}{u_3} \frac{\lambda^{1/2}(u_3, m_b^2, m_3^2)}{\lambda^{1/2}(t_1, m_3^2, u_2)}}, \quad (31)$$

which proves that $\cos \omega \sin \theta_1 \sin \theta_2$ is free from kinematic singularities in s , s_{12} and s_{23} . The singularities in s_{23} cancel in Eq. (27). There are still singularities in t_1 and u_3 in $\cos \omega \sin \theta_1 \sin \theta_2$.

The kinematical factor $K = \epsilon_{\alpha\beta\mu\nu} p_a^\alpha p_b^\beta p_1^\mu p_2^\nu$ is

$$K = \frac{1}{8t_1} \lambda^{1/2}(u_3, t_1, m_2^2) \lambda^{1/2}(t_1, m_1^2, m_a^2) \lambda^{1/2}(t_1, m_b^2, s_{23}) \sin \xi_1 \sin \theta_1 \sin \omega \quad (32a)$$

$$= \frac{1}{8\sqrt{t_1 u_3}} \lambda^{1/2}(u_3, t_1, m_2^2) \lambda^{1/2}(t_1, m_1^2, m_a^2) \lambda^{1/2}(u_3, m_b^2, m_3^2) \sin \theta_1 \sin \theta_2 \sin \omega \quad (32b)$$

$$= \sqrt{-\Delta_4(p_a, p_b, p_1, p_2)} \quad (32c)$$

In the double Regge limit, $s, s_{23}, s_{12} \rightarrow \infty$ with t_1, u_3 and $\eta = s/(s_{12}s_{23}) > 0$ fixed we obtain

$$\sin \theta_1 \xrightarrow{DRL} \frac{2s_{12}\sqrt{-t_1^2}}{\lambda^{1/2}(t_1, m_1^2, m_a^2)\lambda^{1/2}(u_3, t_1, m_2^2)} \quad \sin \xi_1 \xrightarrow{DRL} \frac{2\sqrt{-t_1 u_3}}{\lambda^{1/2}(u_3, t_1, m_2^2)} \quad (33a)$$

$$\sin \theta_2 \xrightarrow{DRL} \frac{2s_{23}\sqrt{-u_3^2}}{\lambda^{1/2}(u_3, m_3^2, m_b^2)\lambda^{1/2}(u_3, t_1, m_2^2)} \quad \sin \xi_1 \xrightarrow{DRL} \frac{2\sqrt{-t_1 u_3}}{\lambda^{1/2}(u_3, t_1, m_2^2)} \quad (33b)$$

which yields

$$\frac{s}{s_{12}s_{23}} \xrightarrow{DRL} \frac{2\sqrt{t_1 u_3} \cos \omega + m_2^2 - t_1 - u_3}{\lambda(u_3, t_1, m_2^2)} \quad (34)$$

Rewriting $\lambda(u_3, t_1, m_2^2) = (m_2^2 - t_1 - u_3)^2 - 4t_1 u_3$, we obtain the condition, in the high energy limit,

$$[m_2^2 - t_1 - u_3 + 2\sqrt{t_1 u_3}]^{-1} \leq \frac{s}{s_{12}s_{23}} \leq [m_2^2 - t_1 - u_3 - 2\sqrt{t_1 u_3}]^{-1} \quad (35)$$

in agreement with Humble Eq. (2.2.5) p.17.