QCD at finite temperature

From Heavy-Ion Collisions to Effective Field Theories

Jacopo Ghiglieri, SUBATECH, Nantes
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The weak-coupling picture

(Quasi)particles and fields

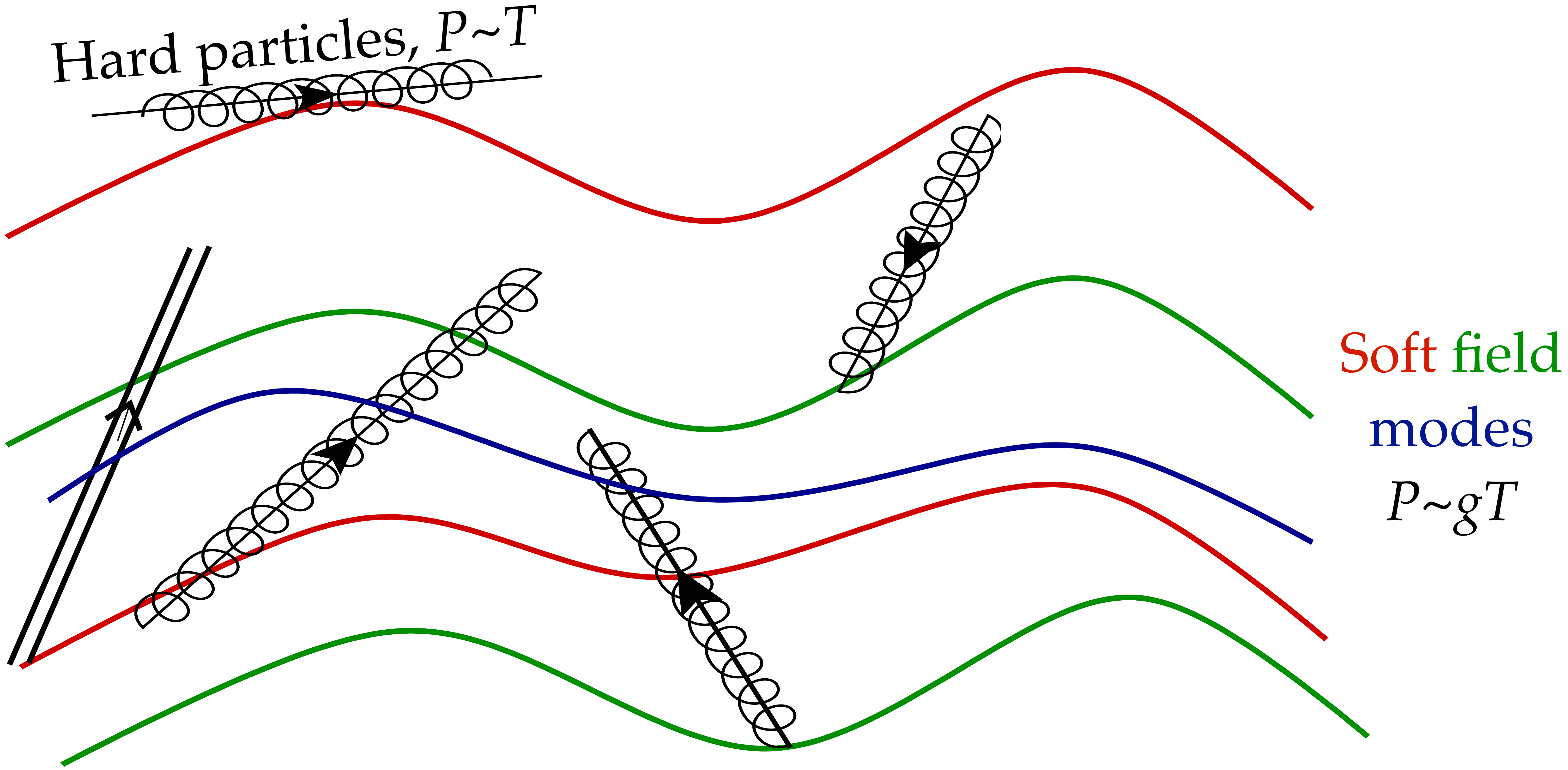


Figure by D. Teaney

Perturbative calculations

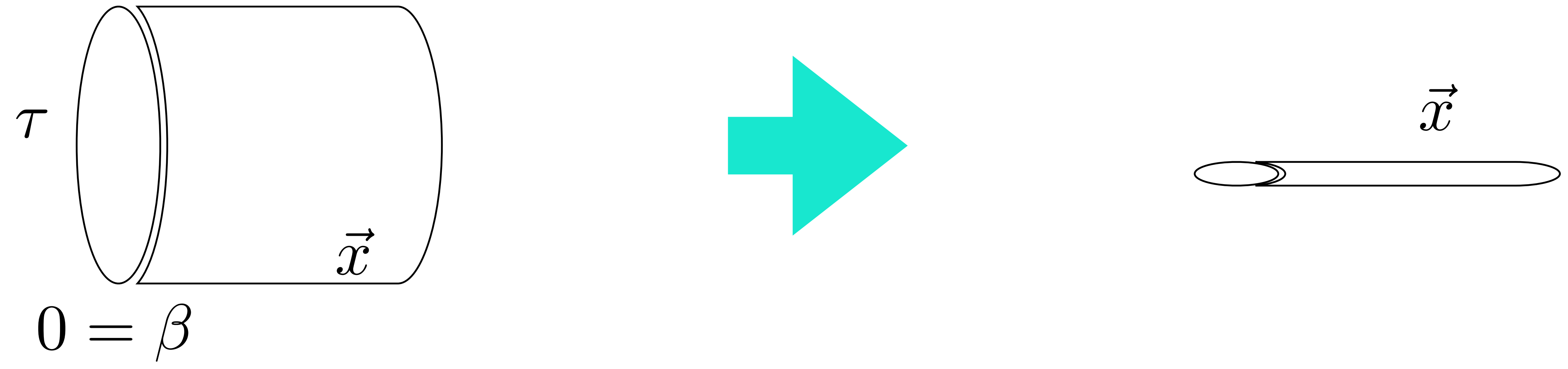
A game of modes

- When computing an observable, one will see the contribution from these modes. For $g \ll 1$ they are well separated and one can resort to EFTs
- Before we do that, let's mention that each mode will start contributing from a given order in perturbation theory, and that the order depends on the observable.
- For the pressure $p \propto \int d^3p \, p \, n_{\rm B}(p)$ we can see that
  - $p \sim T$ starts at order $g^0 T^4$
  - $p \sim gT$ starts at order $g^3 T^4$
  - $p \sim g^2 T$ starts at order $g^6 T^4$

Dealing with the soft modes

Dimensionally-reduced EFTs

- Let us concentrate on the Euclidean case and thermodynamics. Soft modes enter at NNLO (order g^3T^4).
- Exploit scale separation between hard (T) and soft (gT) modes: integrate out the former to obtain an EFT for the latter, valid for $p \ll T$ ($x \gg 1/T$)
- Dimensional reduction, as for $\tau \ll x$ we expand around the zeroth order, $\tau = 0$. We obtain a 3D IR EFT!



Dealing with the soft modes

Dimensional reduction

- Take a massless scalar field, $\mathscr{L}_E = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{\lambda}{4!}\phi^4$ Euclidean action
- See blackboard
- Laine Vuorinen chapter 3 for extra details

Dimensional reduction in scalar field theory

Take a massless scalar field : $S = \int_0^{1/T} d\tau \int d^3x \left[\frac{1}{2}\partial_\mu \varphi \partial_\mu \varphi + \frac{\lambda}{4!}\varphi^4\right]$ Euclidean action

The hard scale is $p \sim T$

What is the soft scale? $S = \frac{1}{2}\int_0^{1/T} d\tau \int d^3x \; \varphi\left[-\partial^2 + 2\frac{\lambda}{4!}\varphi^2\right]\varphi$

$\Rightarrow \; p^2 + \lambda \langle \varphi_T^2 \rangle$ with $\langle \varphi_T^2 \rangle = \int \frac{d^3p}{(2\pi)^3}\frac{n_B(p)}{p} \sim T^2$

$\Rightarrow$ for $p \sim \sqrt{\lambda}\,T$ we have $p^2 \sim \lambda \langle \varphi_T^2 \rangle$, perturbative breakdown of hard-soft interactions

$$\langle \varphi^2_{\sqrt{\lambda}T} \rangle = \int^{\sqrt{\lambda}T} \frac{d^3p}{(2\pi)^3}\frac{n_B(p)}{p} \sim \int^{\sqrt{\lambda}T} dp \; \underbrace{p}_{\sqrt{\lambda}T} \; \underbrace{\frac{T}{p}}_{1/\sqrt{\lambda}} \sim \sqrt{\lambda}\,T^2 \quad \Rightarrow \quad p^2 \gg \lambda \langle \varphi^2_{\sqrt{\lambda}T} \rangle \text{ for } p \sim \sqrt{\lambda}\,T$$

And we shall see that there is no ultrasoft scale

Do a Matsubara decomposition $\varphi(\tau,x) = \sum_{m\in\mathbb{Z}} e^{i\tau\omega_m} \varphi_m(x)$

The kinetic term becomes $S = \sum_{m m'} \underbrace{\int_0^{1/T} d\tau \; e^{i(\omega_m+\omega_{m'})\tau}}_{1/T \, \delta_{m,-m'}} \int d^3x \left[\frac{1}{2} \partial_i\varphi \partial_i\varphi - \frac{1}{2} \omega_m \omega_{m'} \varphi^2 \right]$

$= \beta \sum_m \int d^3x \left[\frac{1}{2} \partial_i\varphi \partial_i\varphi + \frac{1}{2} \omega_m^2 \varphi^2 \right]$ a set of ∞ 3D scalar fields with mass $\omega_m = 2\pi T m$

$\Longrightarrow$ only the $m=0$ zero mode can have a soft momentum, all others have frequencies of the order of the temperature

The only d.o.f. of the EFT will then be the zero mode! $m \neq 0$ modes are integrated out

$\Rightarrow$ the 3D EFT will take the form

$$S_{3D} = \int d^3x \left[\frac{1}{2}\partial_i \phi \partial_i \phi + \frac{1}{2} m^2 \phi^2 + \frac{\tilde{\lambda}}{4!} \phi^4 \right]$$

at first order, we can identify $\phi = \varphi_0 / \sqrt{T}$

We can also match $\tilde{\lambda} = \lambda T$ at first order : same diagram in both theories

How about m? for $p \sim \sqrt{\lambda}\, T$

i.e. $\frac{1}{p^2 + m^2} = \frac{1}{p^2} - \frac{m^2}{p^4} + \dots$

$$-\frac{\lambda}{2} T \sum_{m\neq 0} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\omega_m^2 + q^2} = -\frac{\lambda}{2} T \sum_{m \neq 0} \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} (\omega_m^2)^{d/2-1} = +\frac{\lambda}{2} T \sum_{m\neq 0} \frac{|\omega_m|}{4\pi} = +\frac{\lambda}{2} T^2 \zeta(-1) = -\frac{\lambda T^2}{24}$$

($\frac{1}{2}$ $\hookrightarrow$ symmetry factor)

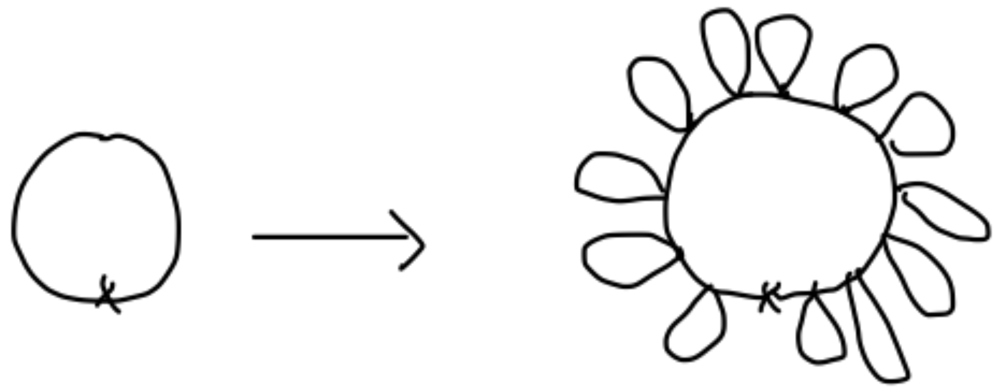
$$\Longrightarrow m^2 = \frac{\lambda T^2}{24}$$

Trivial power counting, $\partial_i \sim p \sim m$

Go back to the pressure. We had $P = \frac{1}{3} T \sum_n \int \frac{d^dp}{(2\pi)^d} \frac{p^2}{p^2 + \omega_n^2}$. The zero-mode contribution vanished in dim. reg

Now we can evaluate it in the 3d EFT

$$P_{3d} = \frac{1}{3} \overset{\text{from field normalisation}}{\overset{\uparrow}{T}} \int \frac{d^dp}{(2\pi)^d} \frac{p^2}{p^2 + m^2} \overset{\text{dim. reg.}}{=} -\frac{T}{3} \int \frac{d^dp}{(2\pi)^d} \frac{m^2}{p^2 + m^2} = \frac{T}{3} \frac{m^3}{4\pi} = \frac{T}{12\pi} \frac{T^3 \lambda \sqrt{\lambda}}{24\sqrt{24}} = \frac{T^4 \lambda \sqrt{\lambda}}{12 \cdot 24 \cdot 2\sqrt{6}\, \pi}$$

We did daisy resummation  and resummed to all orders the non-perturbative

soft-hard interaction. Reminder: soft-soft interactions will give higher-order corrections.

N.B. First correction to the pressure comes at $O(\lambda)$ from hard modes: $P = \frac{\pi^2 T^4}{90} \left[\overset{\text{hard}}{1} - \overset{\text{hard}}{\frac{5}{64\pi^2}\lambda} + \overset{\text{soft}}{\frac{15}{16}\left(\frac{\lambda}{6\pi^2}\right)^{3/2}} + \overset{\text{soft and hard}}{\overset{\uparrow}{O(\lambda^2)}} \right]$

Dimensional reduction gave us a single, massive zero mode. There are thus no ultrasoft modes

Dealing with the soft modes

Electrostatic QCD

- This EFT is called EQCD
  Braaten Nieto 1995, Kajantie Laine Rummukainen Shaposhnikov 1995
- Degrees of freedom
  - ~~Quarks~~ have no zero modes
  - Zero modes of the gauge bosons
- Symmetries: what happens to gauge symmetry?

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Electrostatic QCD

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$$A_\mu(X) \to U(X)A_\mu(X)U^\dagger(X) - \frac{i}{g}(\partial_\mu U(X))U^\dagger(X)$$

There is no time (derivative), so A_0 is no longer a gauge field but an adjoint scalar (adjoint Higgs). What are the consequences?

Electrostatic QCD

The action

- The A_0 field can now have a mass term. Rename it $A_0 \to \Phi\sqrt{T}$
- Sticking to relevant and marginal operators, the action reads

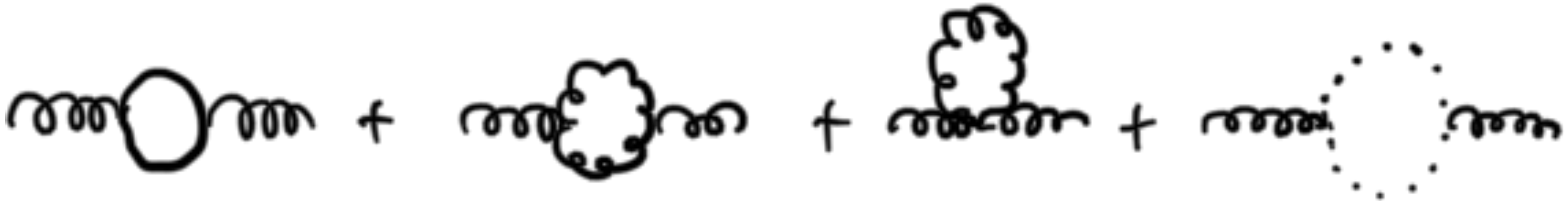
$$S_{\rm EQCD} = \int d^3x \left\{ \frac{1}{2}{\rm Tr}\, F_{ij}F_{ij} + {\rm Tr}\, D_i\Phi D_i\Phi + m_E^2 {\rm Tr}\, \Phi^2 + \lambda_E ({\rm Tr}\, \Phi^2)^2 \right\}$$

- It contains a mass term and a quartic coupling for the adjoint scalar field
- The covariant derivative is $D_i = \partial_i - ig_E A_i$. g_E has mass dimension 1/2 and λ_E mass dimension 1. EQCD super-renormalizable!
- Tree-level matching $g_E^2 = g^2 T(1 + \mathcal{O}(g^2))$

Electrostatic QCD

Matching the mass term

- One-loop QCD matching results in $\Pi_{00}(\omega_n = 0, p \to 0) = m_E^2$

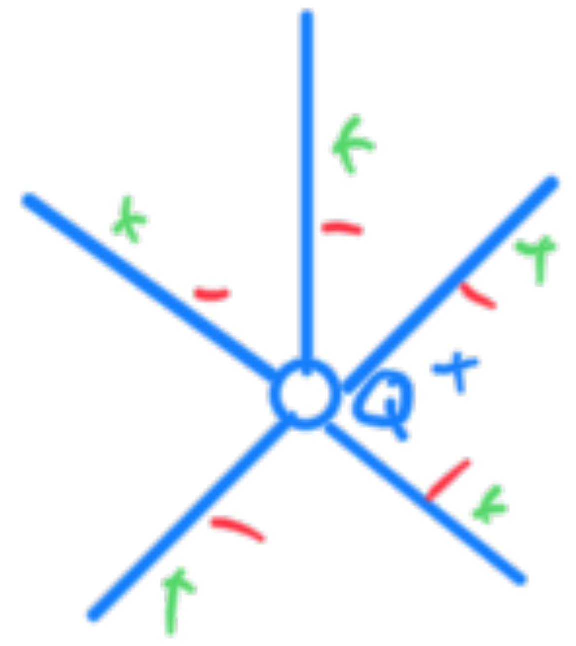


- Diagrammatic evaluation yields $m_E^2 = g^2 T^2 \left(\frac{N_c}{3} + \frac{N_\mathrm{f}}{6} \right)$

- The Φ propagator $1/(p^2 + m_E^2)$ resums this hard-soft interactions to all orders!

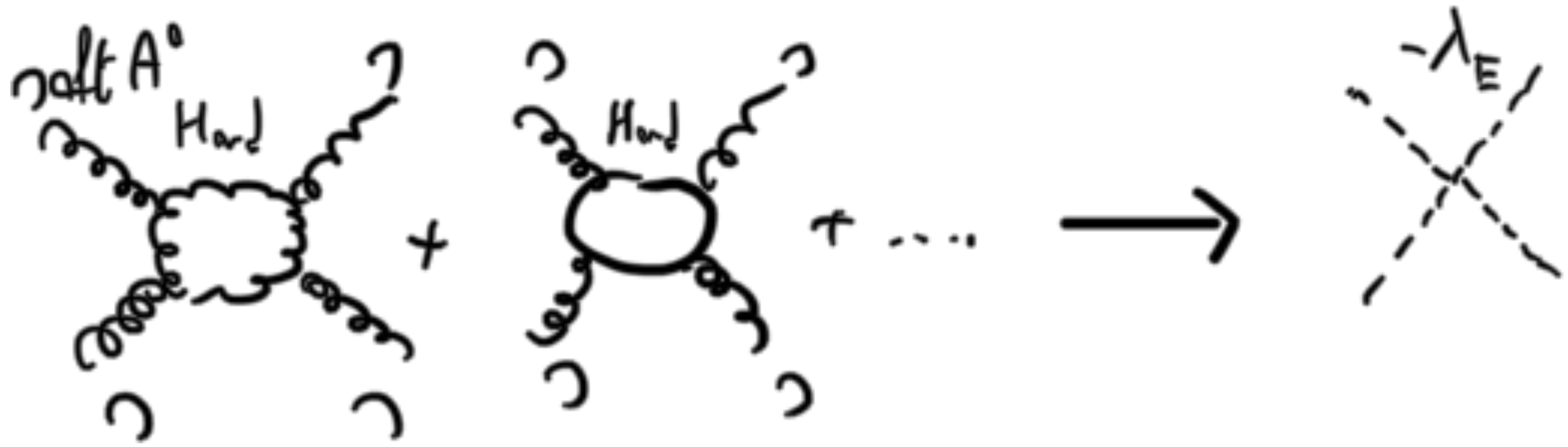
Electrostatic QCD
Consequences of the mass term

- The Φ propagator $1/(p^2 + m_E^2)$ resums these hard-soft interactions to all orders!
- Recall that in QCD the chromoelectric field is $E^i = F^{i0}$. In EQCD it reduces to $E_i = D_i\Phi$.
- As this field has become massive, electrostatic fields will be *Debye screened* ($\propto e^{-m_E r}$) at distances $r \gtrsim 1/m_E$, the so-called Debye radius.
- No long-distance electrostatic interactions



Electrostatic QCD

Matching the quartic term



- One-loop QCD matching gives $\lambda_E = (6 + N_c - N_f)\dfrac{g^4 T}{24\pi^2} + \mathcal{O}(g^6)$
- Even though the EQCD Lagrangian contains no dimensionless coupling, for $g \ll 1$ EQCD has two small parameters (expansion parameters) g_E^2/m_E and λ_E/g_E^2

Towards the ultrasoft scale

Magnetostatic QCD

- EQCD can be used to obtain the soft contribution to the pressure starting from order g^3T^4
- See my review for other applications of EQCD
- But what about the ultrasoft scale and the breakdown of perturbation theory? EQCD has two dynamical scales, the soft and US ones
- If they are separated, we can apply again the EFT paradigm and integrate out the soft scale. The resulting theory is called Magnetostatic QCD (MQCD)

MQCD

Degrees of freedom and action

- The adjoint scalar Φ lives at the soft scale, because of its mass m_E. It will be integrated out
- Only surviving d.o.f.s are US spatial gauge fields
- Sticking again to relevant and marginal operators only

$$S_{\text{MQCD}} = \int d^3x \left\{ \frac{1}{2} \text{Tr}\, F_{ij} F_{ij} \right\}$$

- To first order $g_M^2 = g_E^2 + \ldots$ MQCD has a dimensionful coupling and no other dimensionful or dimensionless parameter. No expansion parameter!

MQCD

Degrees of freedom and action

$$S_{\mathrm{MQCD}} = \int d^3x \left\{ \frac{1}{2} \mathrm{Tr}\, F_{ij} F_{ij} \right\}$$

- MQCD is thus inherently non-perturbative. Though classical, it is still confining!
- Confinement implies that no long-range magnetic forces can exist, they are screened by the lightest glueballs in 3D Yang-Mills
- The $\mathcal{O}(g^6 T^4)$ ultrasoft contribution to the pressure can be obtained by solving MQCD on the lattice
- In cases where the hard and soft scales are well separated, but the soft and US are not, one can solve EQCD on the lattice

The QCD pressure

Perturbative summary

- With $g = g(2\pi T)$ all RG logs disappear and we have

$$p = \frac{\pi^2 T^4}{90}\left(2d_A + \frac{7}{8}4N_c N_f\right)\left[1 + c_2 g^2 + c_3 g^3 + g^4\left(c_{4,1}\ln\frac{T}{m_E} + c_{4,2}\right)\right.$$
$$\left. + c_5 g^5 + g^6\left(c_{6,1}\ln\frac{T}{m_E} + c_{6,2}\ln\frac{m_E}{g^2 T} + c_{6,3}\right) + \ldots\right]$$

- Can we understand the zeroth-order term?

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- Can we understand the zeroth-order term?
- Multiplicity counting: 2 polarisation x d_A colors for bosons (gluons)

  7/8 for the fermionic integral $\int_0^\infty dp p^3 n_{\mathrm{F}}(p)$. 2 spins for $N_c N_f$ q and $\bar{q}$

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- Which scales are responsible for which terms?

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- Which scales are responsible for which terms?

- Hard, soft, ultrasoft

The QCD pressure

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- The hard contribution to $c_{6,3}$ is at present not fully known. The soft and US conversely are known
- Logs signal IR divergences in the naive evaluation at the harder of the two scales

The QCD pressure
Perturbative summary

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$$p = \frac{\pi^2 T^4}{90}\left(2d_A + \frac{7}{8}4N_cN_f\right)\left[1 + c_2 g^2 + c_3 g^3 + g^4\left(c_{4,1}\ln\frac{T}{m_E} + c_{4,2}\right)\right.$$
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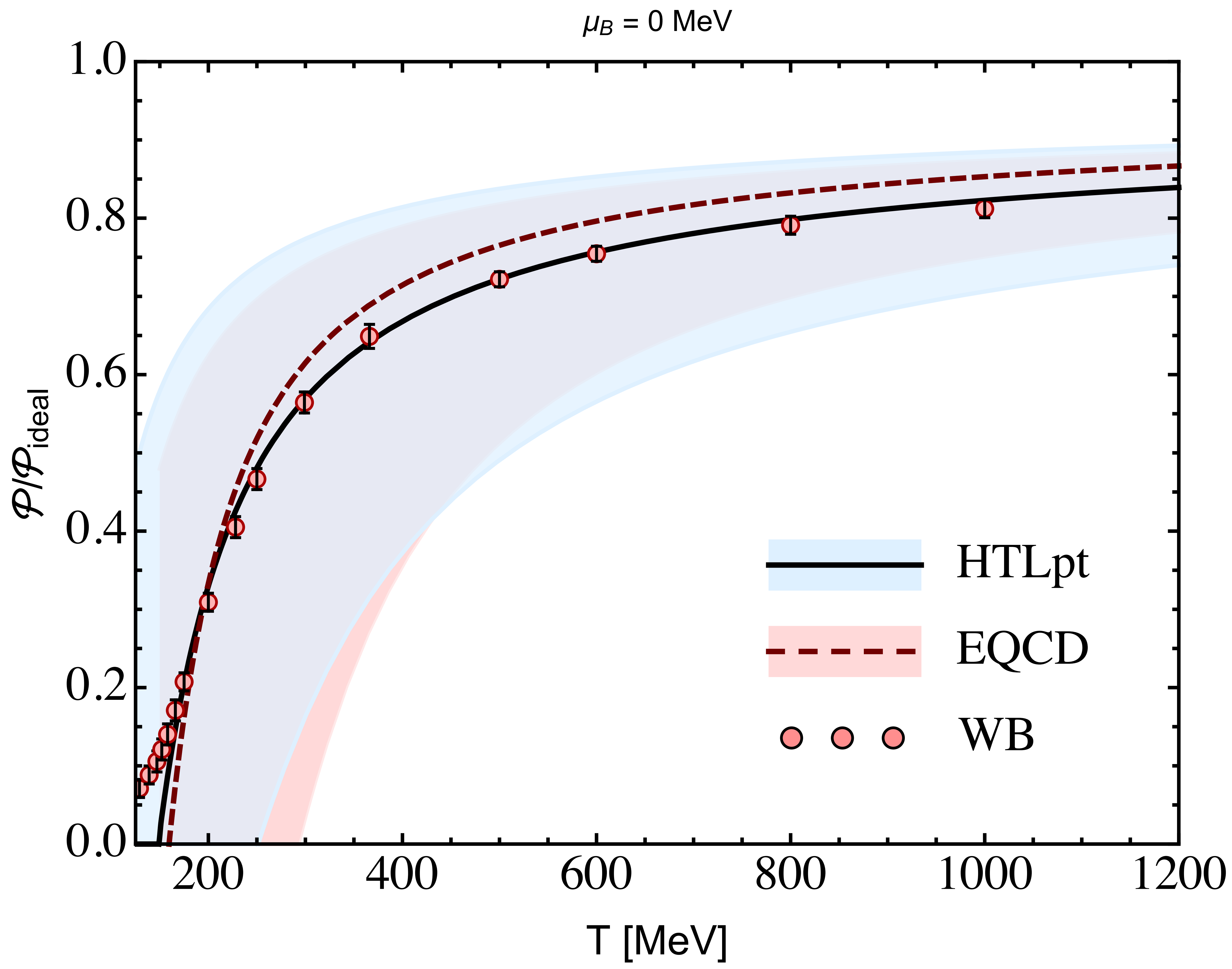
- NB alternative expansion scheme (HTLpt) not shown, see review

The QCD pressure

Perturbative summary

- Large scale-setting uncertainty from varying the renormalisation scale between πT and $4\pi T$
- Midpoint value $2\pi T$ in good agreement with lattice



Intermediate summary

Thermodynamics, the Euclidean formalism and EFTs

- Introduced the Euclidean formalism, best suited for the evaluation of thermodynamics
- Explained how for non-abelian gauge theories, even at arbitrarily small couplings (think about $\mathcal{N}=4$ super Yang-Mills) the pressure contains an inherently non-perturbative contribution
- Showed how a tower of EFTs allows to factor the contribution of the different scales