QCD at finite temperature

From Heavy-Ion Collisions to Effective Field Theories

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Classifying observables

An important difference

- Even if the equilibrium state is time-independent, we can classify these observables by how they are affected by time
- For thermodynamics, $T^{\mu\nu} = \mathrm{diag}(e, p, p, p)$ for an ideal fluid in its rest frame. In QFT $T^{\mu\nu} \to \Theta^{\mu\nu}$, which is a local operator ($\Theta^{\mu\nu}(X)$). Then $p = \langle \Theta^{ii} \rangle / 3$ (with vacuum subtraction)
- Thermodynamics deals with operators which are local in time. As we shall soon see, that is a big simplification

Dealing with local observables

The Matsubara formalism

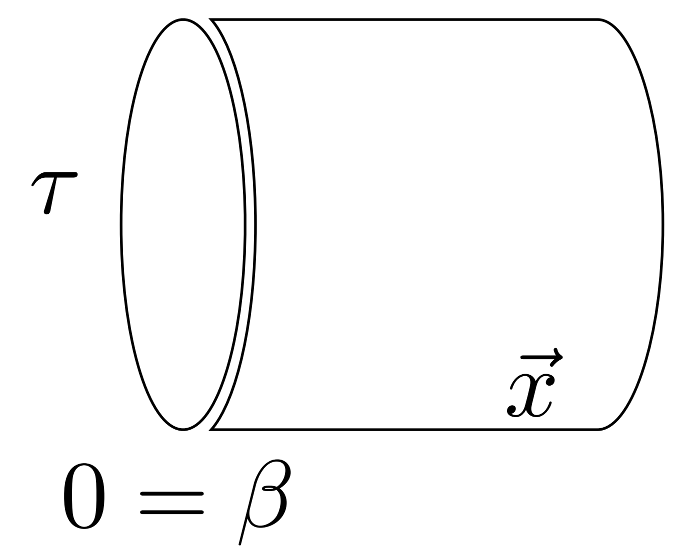
- $\langle \hat{O}(t) \rangle = \mathrm{Tr}[\hat{\rho}(t)\hat{O}(t)] = \sum_i \langle i \,|\, \hat{\rho}(t)\hat{O}(t) \,|\, i \rangle$: used t-invariance of eq. operator
- Now use $\hat{\rho} = e^{-\beta(\hat{H}-\mu_i\hat{N}_i)}/Z$ and identify $e^{-\beta\hat{H}}$ as a time-evolution operator in the imaginary direction τ, i.e $it \leftrightarrow \beta$ ($U(t) = e^{-i\hat{H}t}$)

$$\langle \hat{O} \rangle = \frac{\int \mathcal{D}\phi\, \hat{O}\, e^{-S_E}}{\int \mathcal{D}\phi\, e^{-S_E}} \qquad S_E \equiv \int_0^\beta d\tau L_E$$

- The trace, when transformed into a path integral, implies
  $\phi(0,\vec{x}) = \phi(\beta,\vec{x})$ for bosons (periodicity)
  $\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$ for fermions (antiperiodicity)

Dealing with local observables

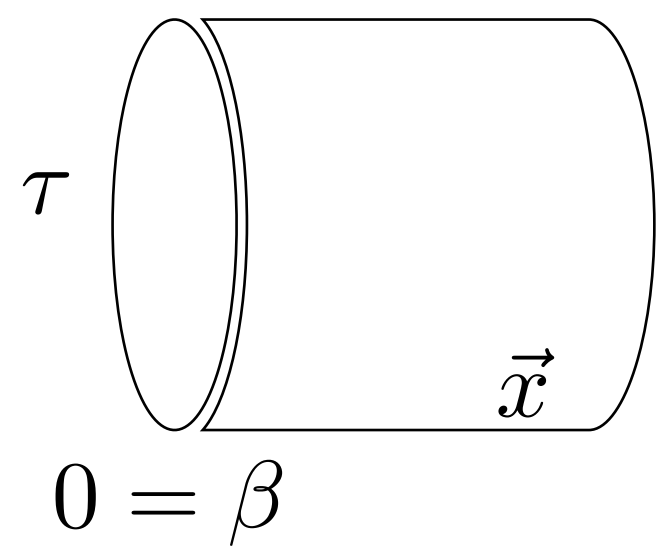
Lattice QCD at finite temperature



- We thus have 3D Euclidean space X compactified Euclidean time
- Ideal (at vanishing chem. pots) for lattice
- Non-perturbatively: compute the discretized Euclidean path integrals numerically with Monte Carlo techniques — see previous plot
- Excellent for vanishing chemical potentials
- Runs into a sign problem at finite μ: the action becomes complex and importance sampling fails (oscillations)
- Runs into another sign problem for non-local observables in t: must resort to non-trivial analytical continuations

Dealing with local observables

The Matsubara formalism



- We thus have 3D Euclidean space X compactified Euclidean time
- Perturbatively: Euclidean field theory with discrete *Matsubara frequencies* $\omega_n = 2\pi T n$ for bosons, $\tilde{\omega}_n = \pi T(2n+1)$ for fermions, $n \in \mathbb{Z}$.
  $$\int d\omega/(2\pi) \to T \sum_n$$
- **Blackboard/exercise**: for a theory of massless, non-interacting real scalars compute the pressure using dim reg. Recall that $\Theta_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{\delta_{\mu\nu}}{2} \partial_\rho \phi \partial_\rho \phi$.
  Solution: $p = \frac{4}{3}\pi^2 T^4 \zeta(-3) = \frac{\pi^2 T^4}{90}$

Pressure of the free boson gas

$$P = \frac{1}{3}\langle \Theta_{ii}\rangle = \frac{1}{3}\left\{ \langle \partial_i \varphi(x)\partial_i \varphi(x)\rangle - \frac{1}{2}\, 3 \,\langle \partial_\ell \varphi \partial_\ell \varphi\rangle \right\} = \frac{1}{3}\, T\sum_m \int \frac{d^d p}{(2\pi)^d} \frac{p^2 - \underbrace{3/2}_{=0 \text{ in dim. reg.}}(\omega_m^2 + p^2)}{\omega_m^2 + p^2} \qquad d = 3 - 2\varepsilon$$

in DR (dim. reg.)

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^2}{p^2 + \omega_m^2} = -\int \frac{d^d p}{(2\pi)^d} \frac{\omega_m^2}{p^2 + \omega_m^2}$$

and

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{\omega_m^2 + p^2} = \frac{1}{(4\pi)^{d/2}} \Gamma(1 - d/2) \left(\omega_m^2\right)^{d/2 - 1} \overset{d=3}{=} -\frac{|\omega_m|}{4\pi}$$

$$\Rightarrow \quad P = \frac{1}{3}\frac{1}{4\pi}\, T \sum_{n\neq 0} (2\pi T)^3 |m|^3 = \frac{1}{3}\frac{8\pi^3 T^4}{4\pi}\, 2\,\zeta(-3) \qquad \left[\sum_{m=1}^{\infty} \frac{1}{m^a} = \zeta(a)\right]$$

$$= \frac{4}{3}\pi^2 T^4 \zeta(-3) = \frac{\pi^2 T^4}{90}.$$

Stefan - Boltzmann pressure for a free bosonic scalar gas.

But what happened? Where is the physics?! Answer later

Pressure of the free boson gas

$$P = \frac{1}{3}\langle \theta_{ii} \rangle = \frac{4}{3}\pi^2 T^4 \zeta(-3) = \frac{\pi^2 T^4}{90}$$

Remark: for a more familiar form, use $\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1}$ for $m \in \mathbb{N}$

$$\zeta(2m) = (-1)^{m+1} \frac{B_{2m} (2\pi)^{2m}}{2(2m)!} \quad \text{for } m \in \mathbb{N}$$

$$\Rightarrow \zeta(-3) = \frac{-B_4}{4}, \quad \zeta(4) = -\frac{B_4 (2\pi)^4}{4 \cdot 24}$$

$$\Rightarrow \zeta(-3) = \frac{24}{16\pi^4}\zeta(4)$$

Redo it with a more transparent physics connection

One can prove that $T\sum_n \frac{1}{\omega_n^2+p^2} = \frac{1+2n_B(p)}{2p}$ with $n_B(p) = \frac{1}{e^{p/T}-1}$ Bose-Einstein distribution

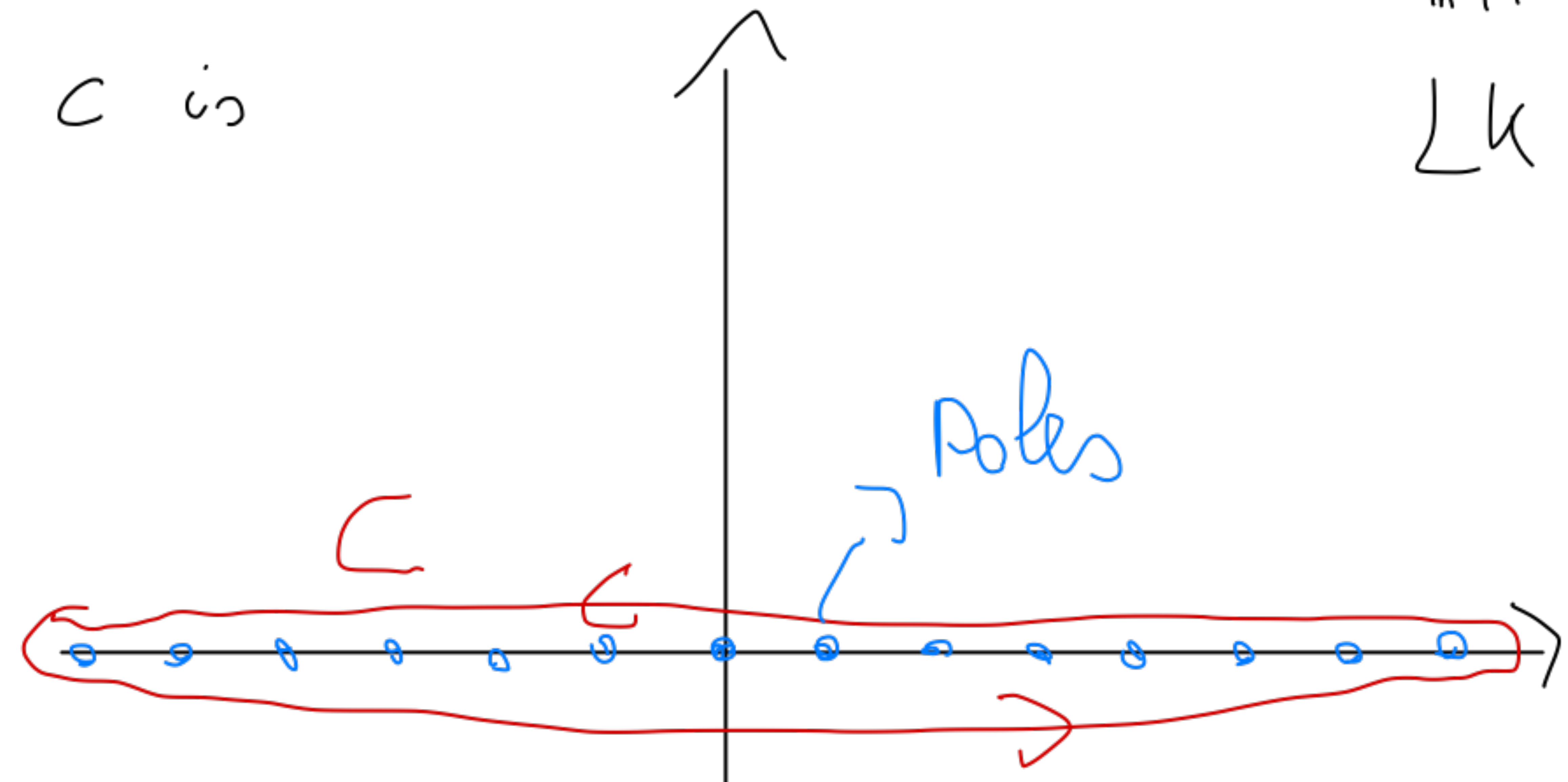
Proof (see Laine Vuorinen 2.2)

Take the function $i n_B(ik) = \frac{i}{e^{ik/T}-1}$. Whenever $e^{ik/T}=1$ there is a pole $\Rightarrow k = 2\pi n T$

The residue is T $\forall$ poles : $i n_B(i(2\pi n T + k)) = \frac{i}{e^{ik/T}-1} \overset{k\to 0}{\approx} \frac{i}{ik/T + O(k^2)} = \frac{T}{k} + O(1)$

$\Rightarrow \oint_C \frac{dk}{2\pi i} \frac{i n_B(ik)}{k^2+p^2}$ the method works not just for $\frac{1}{\omega_m^2+p^2}$ but for any meromorphic function of ω_m

C is



$$\oint_C \frac{dk}{2\pi i} \frac{i n_B(ik)}{k^2+p^2}$$

$$= \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \frac{dk}{2\pi} \frac{n_B(ik)}{k^2+p^2} + \int_{+\infty+i\varepsilon}^{-\infty+i\varepsilon} \frac{dk}{2\pi} \frac{n_B(ik)}{k^2+p^2}$$

$k \to -k$ in the second term. $n_B(-x) = \frac{1}{e^{-x/T}-1} = -\frac{e^{x/T}}{e^{x/T}-1} = -(1+n_B(x))$

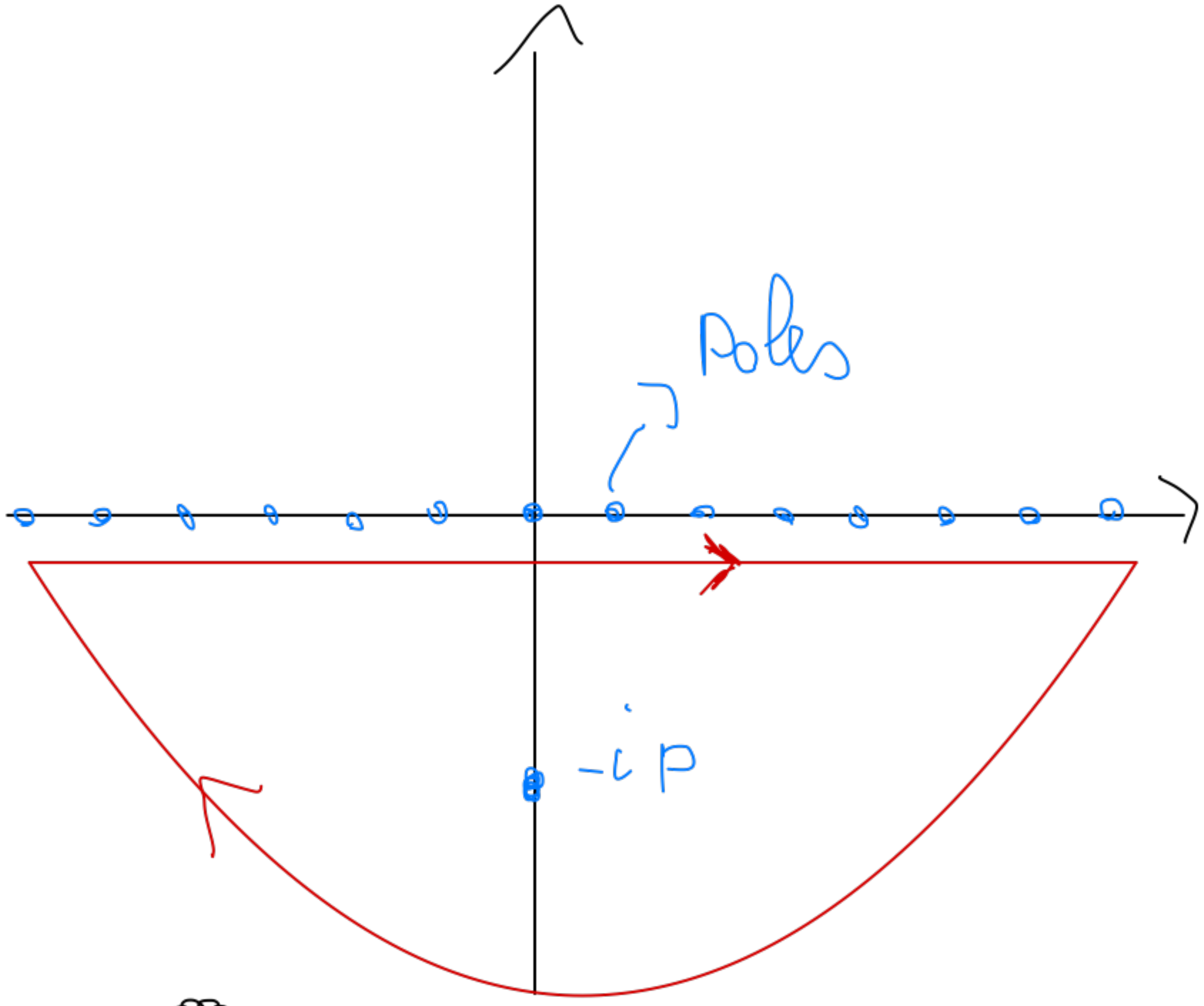
$$\int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \frac{dk}{2\pi} \frac{n_B(ik)}{k^2+p^2} + \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \frac{dk}{2\pi} \frac{1+n_B(+ik)}{k^2+p^2}$$

$$= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{k^2+p^2} + \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \frac{dk}{2\pi} \frac{2n_B(ik)}{k^2+p^2}$$

vacuum thermal

$$= \frac{1}{2p} + 2\int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} \frac{dk}{2\pi}\, \frac{n_B(ik)}{p^2+k^2}$$

We can close below because $n_B(ik)$ falls off fast enough on this arc. We then pick the pole at $k=-ip$



$$= \frac{1}{2p} + 2\,\frac{-2\pi i}{2\pi}\,\frac{n_B(p)}{-2ip} = \frac{1+2n_B(p)}{2p}$$

vacuum part : $\displaystyle\int \frac{d\omega_E}{2\pi}\,\frac{1}{\omega_E^2+p^2} = \frac{1}{2p}$

$$\Rightarrow P = \frac{1}{3}\int \frac{d^dp}{(2\pi)^d}\, p^2\, \frac{1+2n_B(p)}{2p}$$

$\longrightarrow$ thermal part: vanishes at $T=0$ on-shell medium constituents.

Θ_{ii}

The propagator at finite T does not sample vacuum fluctuations only, it also includes statistical fluctuations

The vacuum part is power-law UV divergent. It vanishes in D.R. Physically, we are computing a pressure difference between the thermal and vacuum states

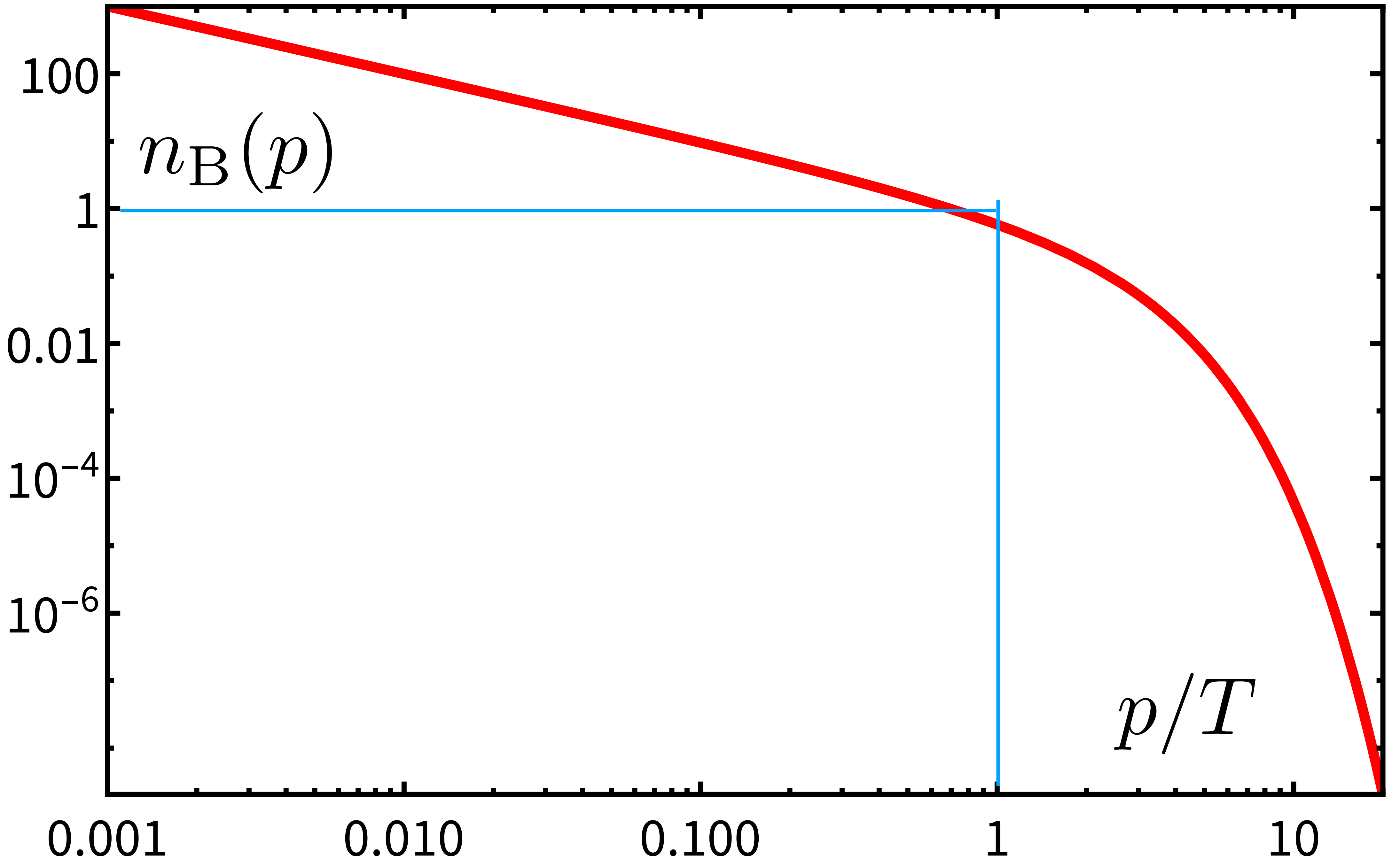
$$\overset{D.R.}{\Longrightarrow} \quad P = \frac{1}{3}\int \frac{d^3p}{(2\pi)^3}\, p\, n_B(p) = \frac{1}{6\pi^2}\int_0^\infty dp\; p^3 n_B(p) = \frac{T^4}{6\pi^2}\int_0^\infty dx\, \frac{x^3}{e^x - 1} = \frac{T^4}{6\pi^2}\,\zeta(4)\,\Gamma(4) \qquad \left\{ \zeta(n) = \frac{1}{\Gamma(n)}\int_0^\infty dx\, \frac{x^{n-1}}{e^x - 1} \right\}$$

$$= \frac{\pi^2 T^4}{90}$$

Emergence of multiple regimes

Quantum and classical modes

- Recall that $p_{\mathrm{SB}} \propto \int d^3p\, p\, n_{\mathrm{B}}(p)$ and $n_{\mathrm{SB}} \propto \int d^3p\, n_{\mathrm{B}}(p)$ the number density
- Since $n \sim T^3$ the typical interparticle separation is of order $\Delta_x \sim 1/T$
- for $p \sim T$ the wavelength $\lambda \sim 1/p \sim 1/T$ is of the order of Δ_x: *quantum regime*
- for $p \gg T$, $\lambda \ll \Delta_x$: *classical particle regime, Maxwell-Boltzmann tail*
- for $p \ll T$, $n_{\mathrm{B}}(p) \approx \dfrac{T}{p} \gg 1$ *classical field (Rayleigh-Jeans) regime* **bosons only**



Interactions

Emergence of multiple scales

- Introduce interactions. When are they a perturbation and when are they not?
- Fluctuations, as before $\langle A^2 \rangle = \int_{\mathbf{p}} \frac{1}{2E_p}(1 \pm 2f_p)$ with $\int_{\mathbf{p}} \equiv \int \frac{d^3p}{(2\pi)^3}$
- Thermal fluctuation $\langle A_T^2 \rangle = \int_{\mathbf{p}} \frac{f_p}{E_p}$. Dominated by $p \sim T$, $\langle A_T^2 \rangle \sim T^2$
- Sidenote: thermal fluctuations are always UV-finite. UV renormalization unaffected by thermal physics. Renormalisation scale typically chosen at multiples of $2\pi T$

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- <u>Def.</u> a mode is perturbative if its interactions (fluctuations) are much smaller than the kinetic term
  <u>Def.</u> $\bar{A} \equiv \langle A^2 \rangle$ and $\partial_\mu - igA_\mu \to p_\mu - ig\bar{A}$

Interactions

Hard modes

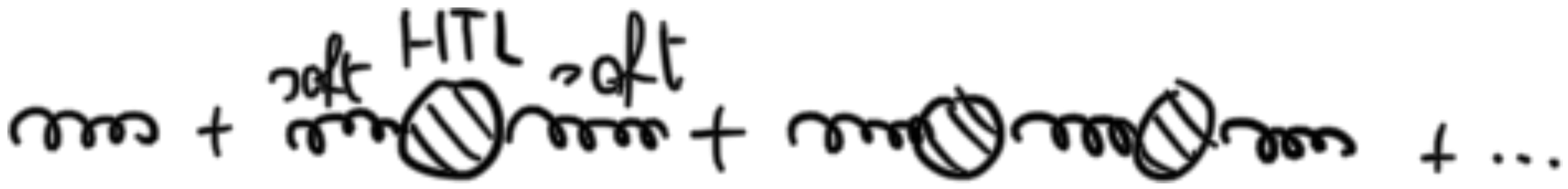
- $p \sim T$, dominate thermodynamics. $p_\mu - ig\bar{A}_T$ perturbative

NB the scale is $p \sim \pi T$, which comes from the Matsubara frequency, or equivalently $p \sim 3T$ the typical thermal momentum.

Interactions

Soft modes

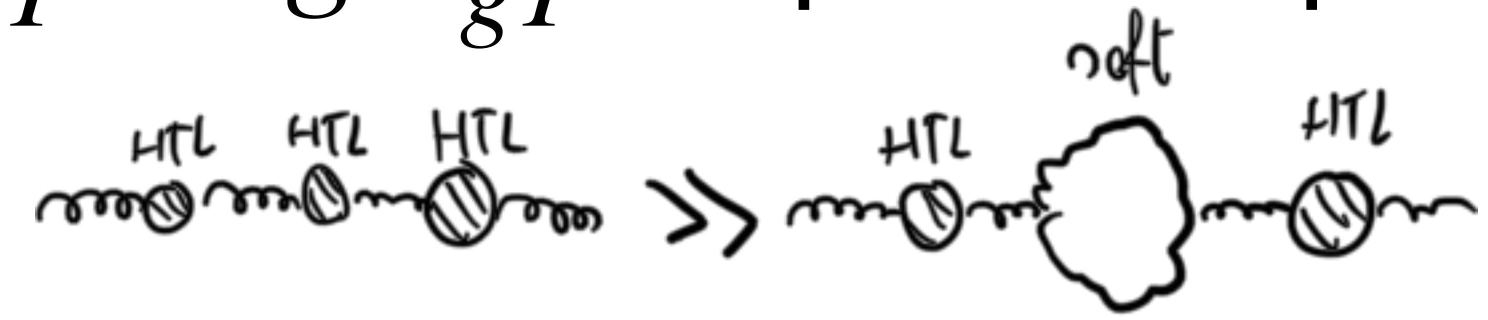
- $p \sim gT$, $\quad p_\mu - ig\bar{A}_T$: both terms have the same size. Soft-hard interactions are non-perturbative.



Resummations are needed, EFTs come to the rescue: Electrostatic QCD (EQCD), Hard Thermal Loop (HTL). More later

- Soft-soft interactions remain perturbative $\langle A^2_{gT} \rangle = \int_{\mathbf{p}}^{gT} \frac{n_{\mathrm{B}}(p)}{E_p} \sim gT^2$

$\bar{A}_{gT} \sim \sqrt{g}T \quad p_\mu - ig\bar{A}_T - ig\bar{A}_{gT}$. Expansion parameter g rather than α_s



Interactions
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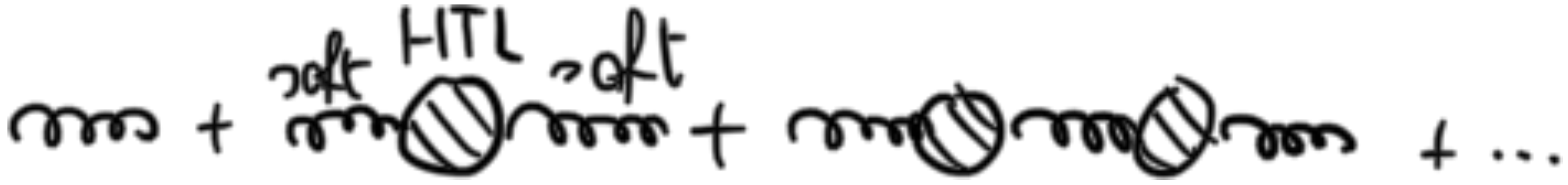
  $\bar{A}_{gT} \sim \sqrt{g}T \quad p_\mu - ig\bar{A}_T - ig\bar{A}_{gT}$. Expansion parameter g rather than α_s
- In other words, at zero temperature (and for massless fields) $\Pi(P) \sim g^2P^2$, so smaller than P^2. Here we can have $\Pi(P) \sim g^2T^2 \sim P^2$ if $P \sim gT$



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  $\bar{A}_{gT} \sim \sqrt{g}T \quad p_\mu - ig\bar{A}_T - ig\bar{A}_{gT}$. Expansion parameter g rather than α_s
- Again, forgetting numerical factors. At zero temperature loop expansion would be α_s/π, here it could be $g/(2\pi)$
- Both gluons and quarks can be soft, but only gluons become classical

Interactions

Ultrasoft modes

- $p \sim g^2 T$, $\langle A^2_{g^2T} \rangle = \int_{\mathbf{p}}^{g^2T} \frac{n_{\mathrm{B}}(p)}{E_p} \sim g^2 T^2$ and $\bar{A}_{g^2T} \sim gT$
- $p_\mu - ig\bar{A}_{g^2T}$ have the same size, ultrasoft self-interactions are non-perturbative
- They can only exist for non-abelian gauge theories and only for transverse gluons, as we shall see

The weak-coupling picture

(Quasi)particles and fields

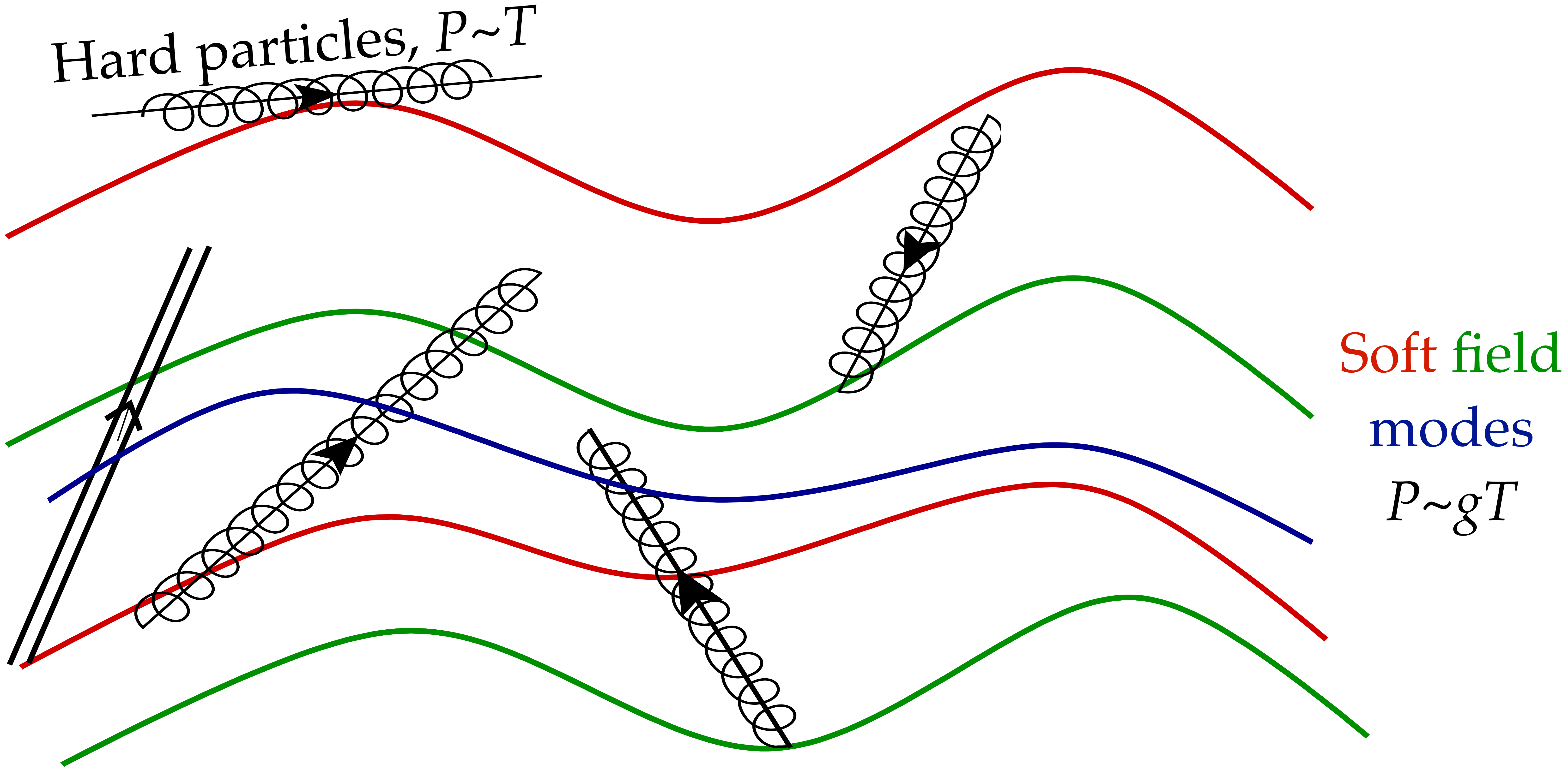


Figure by D. Teaney

Perturbative calculations

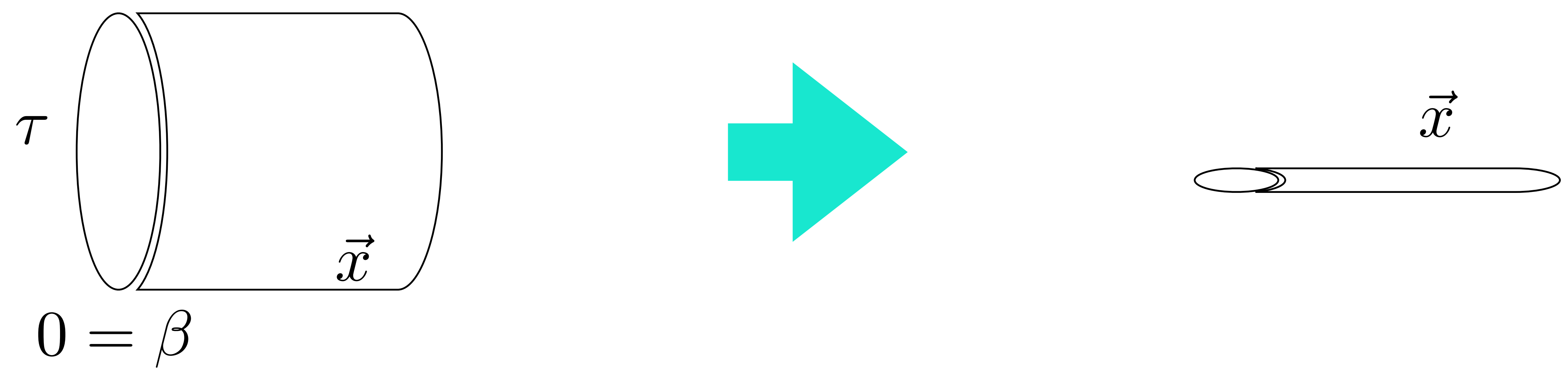
A game of modes

- When computing an observable, one will see the contribution from these modes. For $g \ll 1$ they are well separated and one can resort to EFTs
- Before we do that, let's mention that each mode will start contributing from a given order in perturbation theory, and that the order depends on the observable.
- For the pressure $p \propto \int d^3p\, p\, n_B(p)$ we can see that
  - $p \sim T$ starts at order g^0T^4
  - $p \sim gT$ starts at order g^3T^4
  - $p \sim g^2T$ starts at order g^6T^4

Dealing with the soft modes

Dimensionally-reduced EFTs

- Let us concentrate on the Euclidean case and thermodynamics. Soft modes enter at NNLO (order g^3T^4).
- Exploit scale separation between hard (T) and soft (gT) modes: integrate out the former to obtain an EFT for the latter, valid for $p \ll T$ $(x \gg 1/T)$
- Dimensional reduction, as for $\tau \ll x$ we expand around the zeroth order, $\tau = 0$. We obtain a 3D IR EFT!



Dealing with the soft modes

Dimensional reduction

- Take a massless scalar field, $\mathscr{L}_E = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{\lambda}{4!}\phi^4$ Euclidean action
- See blackboard
- Laine Vuorinen chapter 3 for extra details

Dimensional reduction in scalar field theory

Take a massless scalar field : $S = \int_0^{1/T} d\tau \int d^3x \left[\frac{1}{2} \partial_\mu \varphi \partial_\mu \varphi + \frac{\lambda}{4!} \varphi^4 \right]$ Euclidean action

The hard scale is $p \sim T$

What is the soft scale ? $S = \frac{1}{2} \int_0^{1/T} d\tau \int d^3x \; \varphi \left[-\partial^2 + 2 \frac{\lambda}{4!} \varphi^2 \right] \varphi$

$\Rightarrow \quad p^2 + \lambda \langle \varphi_T^2 \rangle \qquad$ with $\langle \varphi_T^2 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{n_B(p)}{p} \sim T^2$

$\Rightarrow$ for $p \sim \sqrt{\lambda}\, T$ we have $p^2 \sim \lambda \langle \varphi_T^2 \rangle$, perturbative breakdown of hard-soft interactions

$$\langle \varphi^2_{\sqrt{\lambda} T} \rangle = \int^{\sqrt{\lambda} T} \frac{d^3p}{(2\pi)^3} \frac{n_B(p)}{p} \sim \int^{\sqrt{\lambda} T} dp \; \overset{\sqrt{\lambda} T}{p} \; \overset{1/\sqrt{\lambda}}{\frac{T}{p}} \sim \sqrt{\lambda}\, T^2 \quad \Rightarrow \quad p^2 \gg \lambda \langle \varphi^2_{\sqrt{\lambda} T} \rangle \text{ for } p \sim \sqrt{\lambda}\, T$$

And we shall see that there is no ultrasoft scale

Do a Matsubara decomposition $\varphi(\tau, x) = \sum_{m\in\mathbb{Z}} e^{i\tau\omega_m} \varphi_m(x)$

The kinetic term becomes $S = \sum_{m m'} \underbrace{\int_0^{1/T} d\tau \; e^{i(\omega_m + \omega_{m'})\tau}}_{1/T \, \delta_{m,-m'}} \int d^3x \left[\frac{1}{2} \partial_i \varphi \partial_i \varphi - \frac{1}{2} \omega_m \omega_{m'} \varphi^2 \right]$

$= \beta \sum_m \int d^3x \left[\frac{1}{2} \partial_i \varphi \partial_i \varphi + \frac{1}{2} \omega_m^2 \varphi^2 \right]$ a set of ∞ 3D scalar fields with mass $\omega_m = 2\pi T m$

$\Longrightarrow$ only the $m=0$ zero mode can have a soft momentum, all others have frequencies of the order of the temperature

The only d.o.f. of the EFT will then be the zero mode! $m \neq 0$ modes are integrated out

$\Rightarrow$ the 3D EFT will take the form

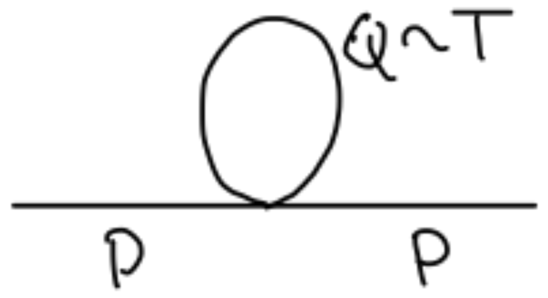
$$S_{3D} = \int d^3x \left[\frac{1}{2} \partial_i \phi \, \partial_i \phi + \frac{1}{2} m^2 \phi^2 + \frac{\tilde{\lambda}}{4!} \phi^4 \right]$$

at first order, we can identify $\phi = \varphi_0 / \sqrt{T}$

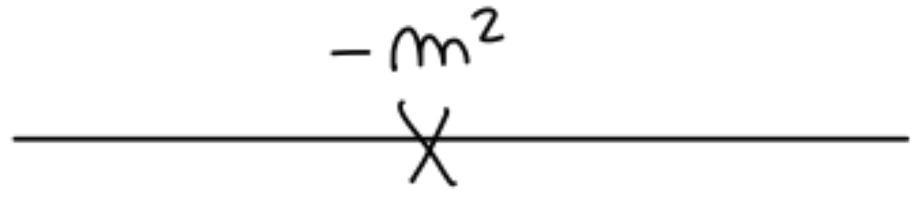
We can also match $\tilde{\lambda} = \lambda T$ at first order : same diagram in both theories

How about m? for $p \sim \sqrt{\lambda}\, T$

4D



3D



i.e. $\dfrac{1}{p^2 + m^2} = \dfrac{1}{p^2} - \dfrac{m^2}{p^4} + \dots$

$$-\frac{\lambda}{2} T \sum_{m \neq 0} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\omega_m^2 + q^2} = -\frac{\lambda}{2} T \sum_{m \neq 0} \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} (\omega_m^2)^{d/2 - 1} = + \frac{\lambda}{2} T \sum_{m \neq 0} \frac{|\omega_m|}{4\pi} = + \frac{\lambda}{2} T^2 \zeta(-1) = -\frac{\lambda T^2}{24}$$

$\hookrightarrow$ symmetry factor

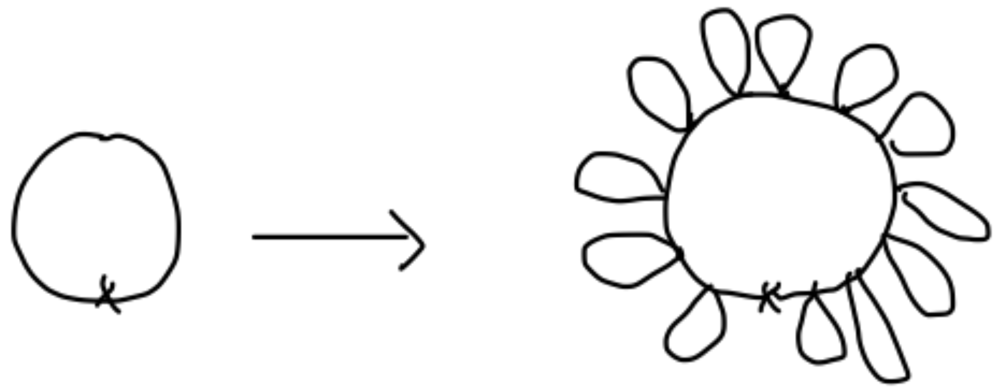
$$\Longrightarrow m^2 = \frac{\lambda T^2}{24}$$

Trivial power counting, $\partial_i \sim p \sim m$

Go back to the measure. We had $P = \frac{1}{3} T \sum_n \int \frac{d^dp}{(2\pi)^d} \frac{p^2}{p^2 + \omega_n^2}$. The zero-mode contribution vanished in dim. reg

Now we can evaluate it in the 3d EFT

$$P_{3d} = \frac{1}{3} \overset{\substack{\text{from field} \\ \text{normalisation} \\ \uparrow}}{T} \int \frac{d^dp}{(2\pi)^d} \frac{p^2}{p^2 + m^2} \overset{\text{dim. reg.}}{=} -\frac{T}{3} \int \frac{d^dp}{(2\pi)^d} \frac{m^2}{p^2 + m^2} = \frac{T}{3} \frac{m^3}{4\pi} = \frac{T}{12\pi} \frac{T^3 \lambda \sqrt{\lambda}}{24\sqrt{24}} = \frac{T^4 \lambda \sqrt{\lambda}}{12 \cdot 24 \cdot 2\sqrt{6}\,\pi}$$

We did daisy resummation  and resummed to all orders the non-perturbative soft-hard interaction. Reminder: soft-soft interactions will give higher-order corrections.

N.B. First correction to the measure comes at $O(\lambda)$ from hard modes: $P = \frac{\pi^2 T^4}{90} \left[\overset{\text{hard}}{1} - \overset{\text{hard}}{\frac{5}{64\pi^2} \lambda} + \overset{\text{soft}}{\frac{15}{16} \left(\frac{\lambda}{6\pi^2} \right)^{3/2}} + \overset{\substack{\text{soft and hard} \\ \uparrow}}{O(\lambda^2)} \right]$

Dimensional reduction gave us a single, massive zero mode. There are thus no ultrasoft modes

Dealing with the soft modes

Electrostatic QCD

- This EFT is called EQCD
  Braaten Nieto 1995, Kajantie Laine Rummukainen Shaposhnikov 1995
- Degrees of freedom
  - ~~Quarks~~ have no zero modes
  - Zero modes of the gauge bosons
- Symmetries: what happens to gauge symmetry?

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$$A_\mu(X) \to U(X)A_\mu(X)U^\dagger(X) - \frac{i}{g}(\partial_\mu U(X))U^\dagger(X)$$

There is no time (derivative), so A_0 is no longer a gauge field but an adjoint scalar (adjoint Higgs). What are the consequences?

Electrostatic QCD
The action

- The A_0 field can now have a mass term. Rename it $A_0 \to \Phi\sqrt{T}$
- Sticking to relevant and marginal operators, the action reads

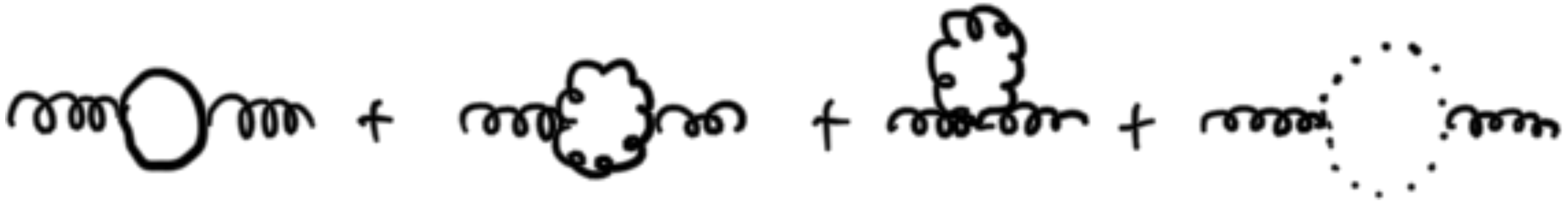
$$S_{\text{EQCD}} = \int d^3x \left\{ \frac{1}{2}\mathrm{Tr}\, F_{ij}F_{ij} + \mathrm{Tr}\, D_i\Phi D_i\Phi + m_E^2 \mathrm{Tr}\,\Phi^2 + \lambda_E (\mathrm{Tr}\,\Phi^2)^2 \right\}$$

- It contains a mass term and a quartic coupling for the adjoint scalar field
- The covariant derivative is $D_i = \partial_i - ig_E A_i$. g_E has mass dimension 1/2 and λ_E mass dimension 1. EQCD super-renormalizable!
- Tree-level matching $g_E^2 = g^2 T(1 + \mathcal{O}(g^2))$

Electrostatic QCD

Matching the mass term

- One-loop QCD matching results in $\Pi_{00}(\omega_n = 0, p \to 0) = m_E^2$



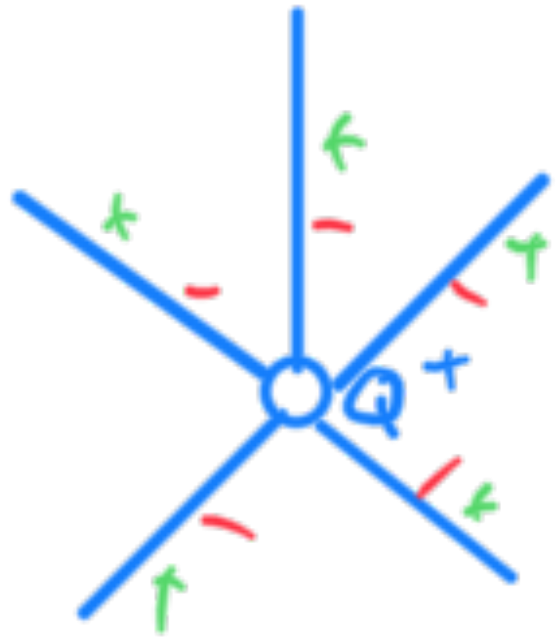
- Diagrammatic evaluation yields $m_E^2 = g^2T^2\left(\frac{N_c}{3} + \frac{N_{\mathrm{f}}}{6}\right)$

- The Φ propagator resums this hard-soft interactions to all orders!

Electrostatic QCD

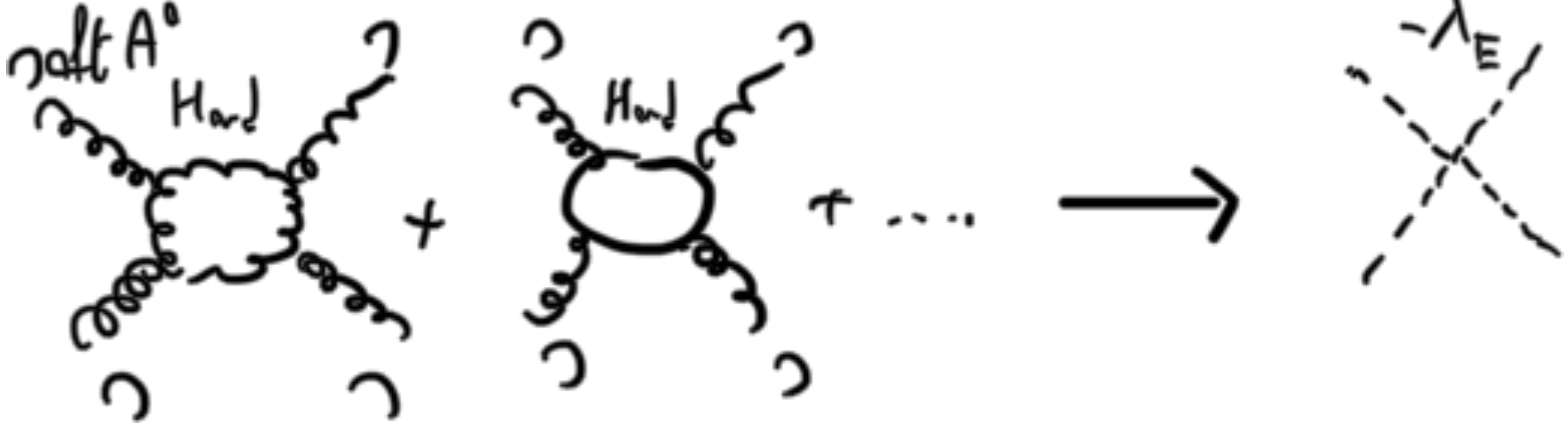
Consequences of the mass term

- The Φ propagator resums these hard-soft interactions to all orders!
- Recall that in QCD the chromoelectric field is $E^i = F^{i0}$. In EQCD it reduces to $E_i = D_i\Phi$.
- As this field has become massive, electrostatic fields will be *Debye screened* ($\propto e^{-m_E r}$) at distances $r \gtrsim 1/m_E$, the so-called Debye radius.
- No long-distance electrostatic interactions



Electrostatic QCD

Matching the quartic term



- One-loop QCD matching gives $\lambda_E = (6 + N_c - N_f)\dfrac{g^4 T}{24\pi^2} + \mathcal{O}(g^6)$

Towards the ultrasoft scale

Magnetostatic QCD

- EQCD can be used to obtain the soft contribution to the pressure starting from order g^3T^4
- See my review for other applications of EQCD
- But what about the ultrasoft scale and the breakdown of perturbation theory? EQCD has two dynamical scales, the soft and US ones
- If they are separated, we can apply again the EFT paradigm and integrate out the soft scale. The resulting theory is called Magnetostatic QCD (MQCD)

MQCD

Degrees of freedom and action

- The adjoint scalar Φ lives at the soft scale, because of its mass m_E. It will be integrated out
- Only surviving d.o.f.s are US spatial gauge fields
- Sticking again to relevant and marginal operators only

$$S_{\mathrm{MQCD}} = \int d^3x \left\{ \frac{1}{2} \mathrm{Tr}\, F_{ij} F_{ij} \right\}$$

- To first order $g_M^2 = g_E^2 + \ldots$ MQCD has a dimensionful coupling and no other dimensionful or dimensionless parameter. No expansion parameter!

MQCD

Degrees of freedom and action

$$S_{\mathrm{MQCD}} = \int d^3x \left\{ \frac{1}{2} \mathrm{Tr}\, F_{ij} F_{ij} \right\}$$

- MQCD is thus inherently non-perturbative. Though classical, it is still confining!
- Confinement implies that no long-range magnetic forces can exist, they are screened by the lightest glueballs in 3D Yang-Mills
- The $\mathcal{O}(g^6 T^4)$ ultrasoft contribution to the pressure can be obtained by solving MQCD on the lattice
- In cases where the hard and soft scales are well separated, but the soft and US are not, one can solve EQCD on the lattice

The QCD pressure
Perturbative summary

- With $g = g(2\pi T)$ all RG logs disappear and we have

$$p = \frac{\pi^2 T^4}{90}\left(2d_A + \frac{7}{8}4N_c N_f\right)\left[1 + c_2 g^2 + c_3 g^3 + g^4\left(c_{4,1}\ln\frac{T}{m_E} + c_{4,2}\right)\right.$$
$$\left. + c_5 g^5 + g^6\left(c_{6,1}\ln\frac{T}{m_E} + c_{6,2}\ln\frac{m_E}{g^2 T} + c_{6,3}\right) + \ldots\right]$$

- Can we understand the zeroth-order term?

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- Can we understand the zeroth-order term?

- Multiplicity counting: 2 polarisation x d_A colors for bosons (gluons)

  7/8 for the fermionic integral $\int_0^\infty dp p^3 n_{\mathrm{F}}(p)$. 2 spins for $N_c N_f$ q and $\bar{q}$

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- Which scales are responsible for which terms?

The QCD pressure

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- Which scales are responsible for which terms?

- Hard, soft, ultrasoft

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- The hard contribution to $c_{6,3}$ is at present not fully known. The soft and US conversely are known
- Logs signal IR divergences in the naive evaluation at the harder of the two scales

The QCD pressure

Perturbative summary

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$$p = \frac{\pi^2 T^4}{90}\left(2d_A + \frac{7}{8}4N_c N_f\right)\left[1 + c_2 g^2 + c_3 g^3 + g^4\left(c_{4,1}\ln\frac{T}{m_E} + c_{4,2}\right)\right.$$
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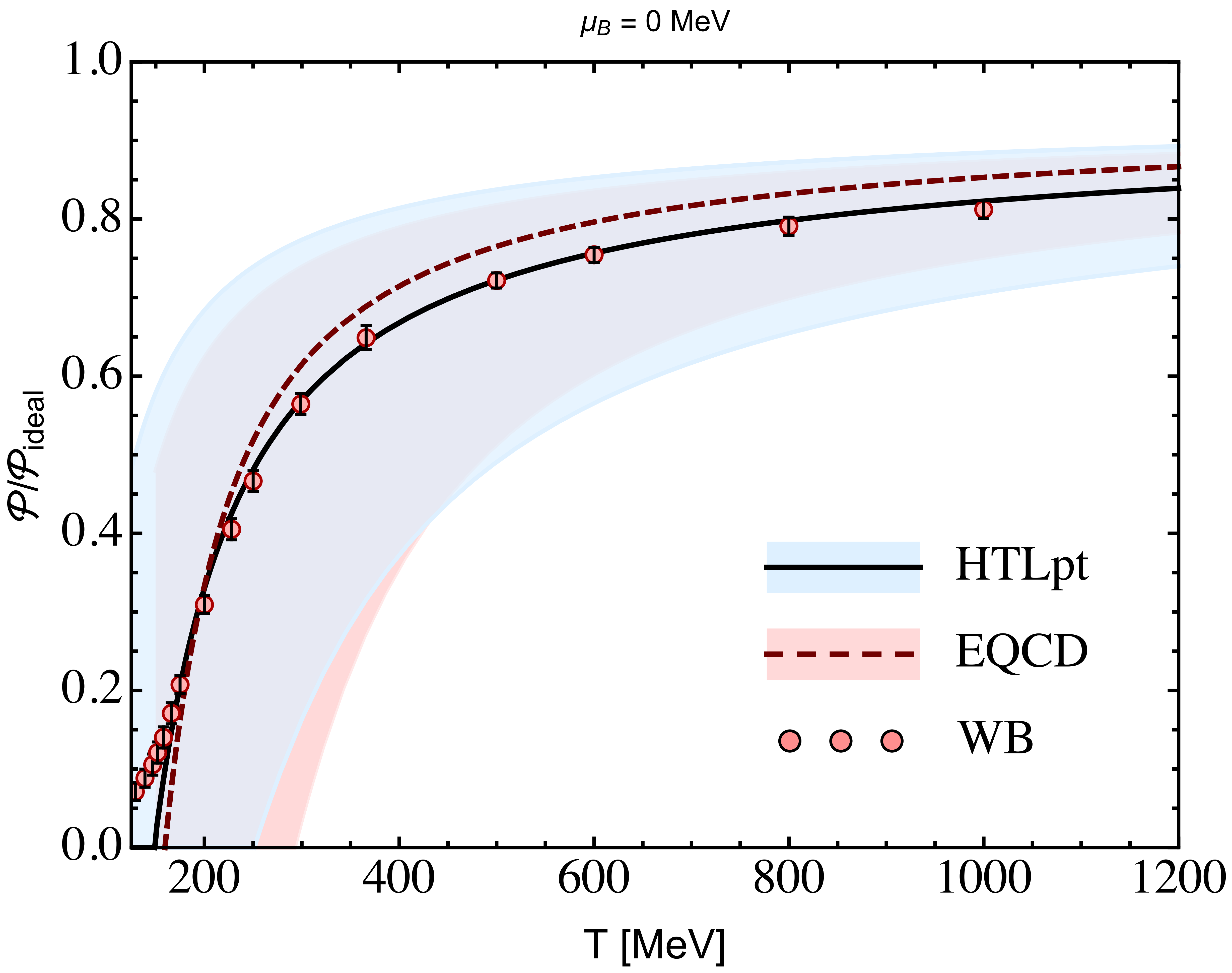
- NB alternative expansion scheme (HTLpt) not shown, see review

The QCD pressure
Perturbative summary

- Large scale-setting uncertainty from varying the renormalisation scale between πT and $4\pi T$
- Midpoint value $2\pi T$ in good agreement with lattice



Intermediate summary

Thermodynamics, the Euclidean formalism and EFTs

- Introduced the Euclidean formalism, best suited for the evaluation of thermodynamics
- Explained how for non-abelian gauge theories, even at arbitrarily small couplings (think about $\mathcal{N} = 4$ super Yang-Mills) the pressure contains an inherently non-perturbative contribution
- Showed how a tower of EFTs allows to factor the contribution of the different scales