Effective field theories: basics

Antonio Vairo

Technische Universität München

Effective field theory: general form

Whenever a system H, described by a fundamental theory, e.g. QCD, is characterized by two energy scales Λ_H and Λ_L such that $\Lambda_H \gg \Lambda_L$, observables may be computed by expanding one scale with respect to the other.

An EFT makes the expansion $\frac{\Lambda_L}{\Lambda_H}$ explicit at the Lagrangian level.

In the following, we characterize the general form of the EFT Lagrangian.

The EFT Lagrangian, $\mathcal{L}$, suitable to describe H at scales lower than Λ_H is defined by

- a cut off $\Lambda_H \gg \mu \gg \Lambda_L$,
- by some degrees of freedom (d.o.f.) that exist at scales lower than μ.

Then $\mathcal{L}$ is made of all operators O_n that may be built out of the effective d.o.f. and are consistent with the symmetries of the fundamental theory

$$\mathcal{L} = \sum_n c_n(\Lambda_H, \mu) \frac{O_n(\mu, \Lambda_L)}{\Lambda_H^{d_n - 4}}$$

where $[O_n] = d_n$ (mass dimension of the operator O_n)

S. Weinberg, Physica A96 (1979) 327

Obs. (i) If $\mu \sim \Lambda_L$ then $\langle O_n(\mu, \Lambda_L)\rangle \sim \Lambda_L^{d_n}$

which is called the <u>power counting</u> of the EFT Lagrangian.

(ii) The EFT Lagrangian is effectively an expansion in $\frac{\Lambda_L}{\Lambda_H}$.

It contains infinite terms, but to a specific precision ε only terms up to

$\left(\frac{\Lambda_L}{\Lambda_H}\right)^N > \varepsilon$ are needed.

The higher the precision, the worse the hierarchy $\Lambda_L \ll \Lambda_H$, the more terms N are needed to achieve the precision ε.

(iii) The coefficients $c_n(\Lambda_H, \mu)$ encode the non-analytic behavior in Λ_H.

They are called Wilson or matching coefficients.

They are computed by imposing that the EFT and the fundamental theory describe the same physics at a given precision ε.

The procedure is called matching.

Sometimes this is also referred to as integrating out the high energy/heavy d.o.f.

(iv) The coefficients c_n may be computed analytically from the fundamental theory if a weak coupling expansion is applicable, elsewhere they are computed numerically or fitted to data.

(v) In general $\mathcal{L}$ is not renormalizable, because it contains couplings with negative mass dimension $\sim \Lambda_H^{-n}$. However it is order by order in $\frac{\Lambda_L}{\Lambda_H}$. Non-renormalizability in the usual sense does not reduce the predictive power of the EFT.

(vi) In the EFT framework exact renormalizability is not so important. What matters more is the realization of the expected power counting:

$$C_n \sim 1 \,, \quad \langle O_n \rangle \sim \Lambda_L^{d_n}$$

If this happens one speaks of naturalness of the EFT.

How to construct an EFT

1. Identify the hierarchy of energy scales and a cut off energy scale μ.

2. Identify the low energy ($<\mu$) degrees of freedom (d.o.f.).

3. Identify the symmetries.

4. Construct the most general Lagrangian out of the d.o.f. consistent with the symmetries: infinite terms and parameters.

5. Determine through power counting the relative importance of the different terms in the Lagrangian. The power counting reflects at the Lagrangian level the hierarchy of energy scales in the system.

6. Chose a desired accuracy $\left(\frac{\Lambda_L}{\Lambda_H}\right)^N$, which allows to truncate the Lagrangian at a finite number of terms.

7. Determine the finite number of parameters, Wilson coefficients, of the Lagrangian by equating amplitudes or Green's functions in the fundamental theory and in the EFT up to order $\left(\frac{\Lambda_L}{\Lambda_H}\right)^N$: matching.

8. Evolve the Wilson coefficients from the matching scale to Λ_L (or a scale close to it) by solving the renormalization group equations.

Relevant, irrelevant and marginal operators

A generic EFT Lagrangian has the form

$$\mathcal{L} = \sum_n c_n \frac{O_n}{\Lambda^{d_n-4}}$$

where Λ is the large scale that has been integrated out and O_n is an operator of mass dimension d_n; c_n are the (dimensionless) matching coefficients.

At energies below Λ, the behavior of the operators O_n is determined by their dimension.

<u>Def.</u>

- if $d_n > 4$ the operator is called <u>irrelevant</u>
- if $d_n < 4$ the operator is called <u>relevant</u>
- if $d_n = 4$ the operator is called <u>marginal</u>

In a 4-dimensional relativistic field theory only the following relevant operators are possible

(ϕ stands for a generic scalar field, ψ for a generic spin $\frac{1}{2}$ field, A_μ for a generic vector field):

i) $d_n = 0$ unit operator

ii) $d_n = 2$ boson mass term : $\phi^+ \phi$

iii) $d_n = 3$ fermion mass term : $\bar{\psi}\psi$; cubic scalar interaction: ϕ^3

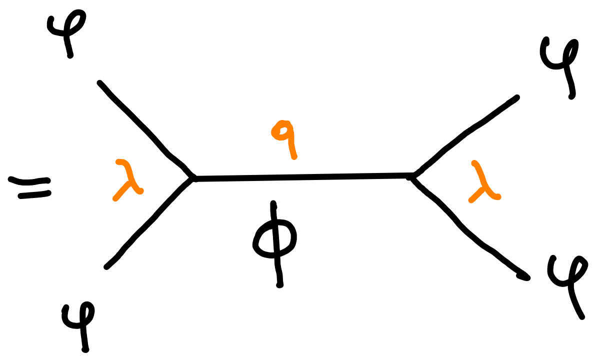
Contributions from relevant operators are large at low energies.

Ex. Consider the Lagrangian (φ and ϕ are two scalar fields)

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}(\partial\phi)^2}_{\text{marginal operators}} \underbrace{- \frac{1}{2} m^2\varphi^2 - \frac{1}{2} M^2\phi^2 - \frac{\lambda}{2}\varphi^2\phi}_{\text{relevant operators}}$$

m is the mass of the field φ, M is the mass of the field ϕ, λ is the coupling ; $[m]=[M]=[\lambda]=1$

We assume $m \ll M$.

amplitude $= \mathcal{A}(\varphi\varphi \to \varphi\varphi) =$ 

$\sim \dfrac{\lambda^2}{q^2 - M^2} \underbrace{\dfrac{1}{E}}$; $q \sim E$

for dimensional reasons to keep $[\mathcal{A}] = -1$

cross section $= \sigma(\varphi\varphi \to \varphi\varphi) \sim |\mathcal{A}(\varphi\varphi\to\varphi\varphi)|^2 \sim \dfrac{1}{E^2} \times \begin{cases} \left(\dfrac{\lambda}{E}\right)^4 & \text{for } E \gg M \\ \left(\dfrac{\lambda}{M}\right)^4 & \text{for } E \ll M \end{cases}$

- at high energies $\sigma \sim \frac{\lambda^4}{E^6} \longrightarrow 0$ for $E \to \infty$

- at low energies $\sigma \sim \frac{1}{E^2} \frac{\lambda^4}{M^4}$

the cross section increases at low energies reflecting the fact tha the operator $\varphi^2\phi$ is relevant.

Examples of marginal operators are

i) ϕ^4

ii) $\bar{\psi}\psi\phi$ (Yukawa interaction)

iii) A^4, ∂A^3, $\bar{\psi}\not{A}\psi$, $\phi^\dagger\phi A^2$ (gauge interaction terms; Lorentz indeces have been suppressed)

iv) $\bar{\psi}\not{\partial}\psi$, $\phi^\dagger\partial^2\phi$ (Kinetic operators)

Obs. (1) Marginal and relevant operators have couplings that are either scaleless constants or have positive mass dimension. Hence marginal and relevant operators are renormalizable.

This explains why renormalizable field theories exist at all:

they appear at some energy scale if there is a large energy gap with the next relevant energy scale.

(2) Marginal operators typically become either relevant or irrelevant once quantum fluctuations are taken into account.

Quantum loops: irrelevant operators

Irrelevant operators multiply coupling constants with negative mass dimension.

This implies that they give rise to non-renormalizable interactions.

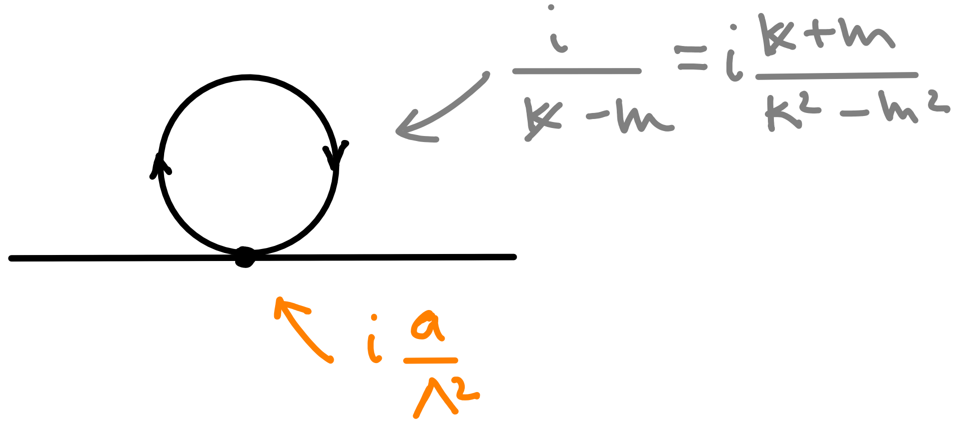
Nevertheless, after renormalization, the power counting of the EFT is preserved.

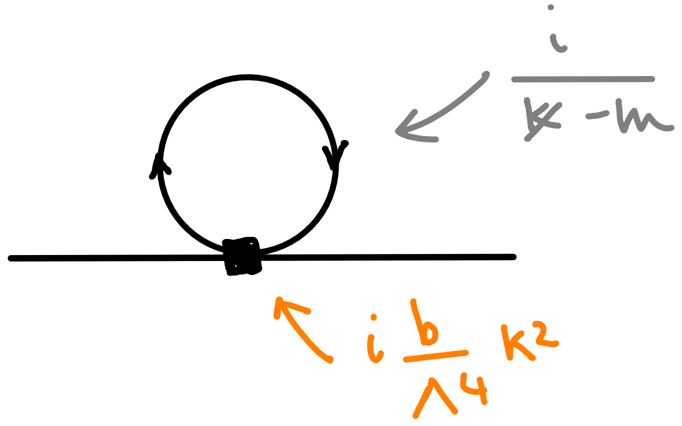
Ex. Consider the EFT Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \underbrace{- \frac{a}{\Lambda^2}(\bar{\psi}\psi)^2 - \frac{b}{\Lambda^4}(\bar{\psi}\partial^2\psi)(\bar{\psi}\psi) + \cdots}_{\text{irrelevant operators (dimension 6 and 8 respectively)}}$$

ψ describes fermions of mass m, and typical momentum and energy $\ll \Lambda$.

We consider the (divergent) contribution to the fermion mass due to dimension 6 and 8 operators in dimensional regularization, which is a mass-independent regularization scheme:





from the dimension 6 operator (in dimensional regularization (DR): $d = 4-2\varepsilon$, $d=4$ for $\varepsilon \to 0$)

Wick rotation: $k^0 \to ik^4$

$$\delta m \sim 2ia\frac{m}{\Lambda^2}\int\frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}}\frac{1}{k^2-m^2+i\varepsilon} = 2\frac{a}{\Lambda^2}m\int\frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}}\frac{1}{k^2+m^2}$$

$$= 2\frac{a}{\Lambda^2}m\frac{m^2}{16\pi^2}\mu^{-2\varepsilon}\left(-\frac{1}{\varepsilon}-1+\gamma_E-\log 4\pi+\log\frac{m^2}{\mu^2}+O(\varepsilon)\right)$$

In the modified MS scheme ($\overline{MS}$) the term proportional to $\left(\frac{1}{\varepsilon} - \gamma_E + \log 4\pi\right)$ is subtracted giving rise to the $\overline{MS}$ renormalized mass correction

$$\delta m^{\overline{MS}} \sim -m \frac{a}{8\pi^2} \frac{m^2}{\Lambda^2}\left(1 - \log\frac{m^2}{\mu^2}\right) \qquad \left(\text{it is "irrelevant" for } \frac{m}{\Lambda} \to 0\right)$$

An analogous result follows for the contribution from the dimension 8 operator:

$$\delta m^{\overline{MS}} \sim m\, b\, \frac{m^4}{\Lambda^4}\left(1 + \ldots \log\frac{m^2}{\mu^2}\right)$$

Obs. (1) After subtraction of the $\frac{1}{\varepsilon}$ poles, the mass corrections satisfy the EFT power counting:

$$m \gg \text{contribution from the dim. 6 operator} \sim m\frac{m^2}{\Lambda^2} \gg \text{contribution from the dim. 8 operator} \sim m\frac{m^4}{\Lambda^4}$$

(2) μ is the renormalization scale of DR. The dependence on μ is fictitious and cancels in physical observables.

Large logarithms $\sim \log\frac{m^2}{\mu^2}$ may be resummed with renormalization group equations.

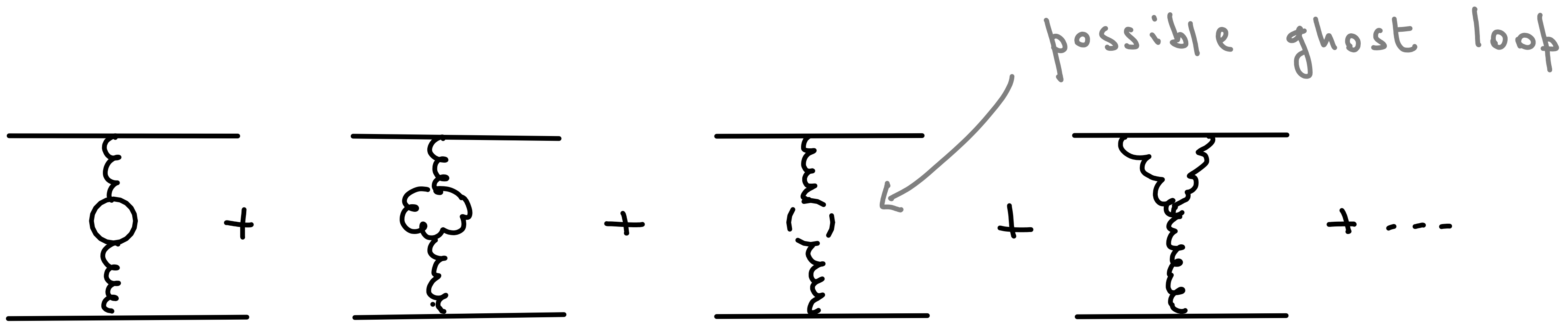
EFTs behave for all practical purposes as renormalizable theories if one works at some fixed order in $\frac{1}{\Lambda_H}$. This is because there is only a finite number of terms in the EFT Lagrangian that contribute at a given order in $\frac{1}{\Lambda_H}$. This can be seen most easily in a mass-independent renormalization scheme, where quantum corrections do not generate power divergences of order Λ_H^n with $n>0$ (at most logarithmic divergences).

Quantum loops: marginal operators

We consider the impact of quantum loops to marginal operators on the example of the renormalization of the coupling constant in QCD.

1-loop strong coupling renormalization in QCD

Several diagrams contribute to the 1 loop renormalization of the strong coupling:



due to the self interaction of the gluons. These non-Abelian contributions are responsible for the antiscreening of the vacuum, which dominates over the screening due to the fermion loop. The beta function in QCD turns out to have the opposite sign of the beta function in QED.

In general for the SU(N_c) case (QCD corresponds to $N_c=3$):

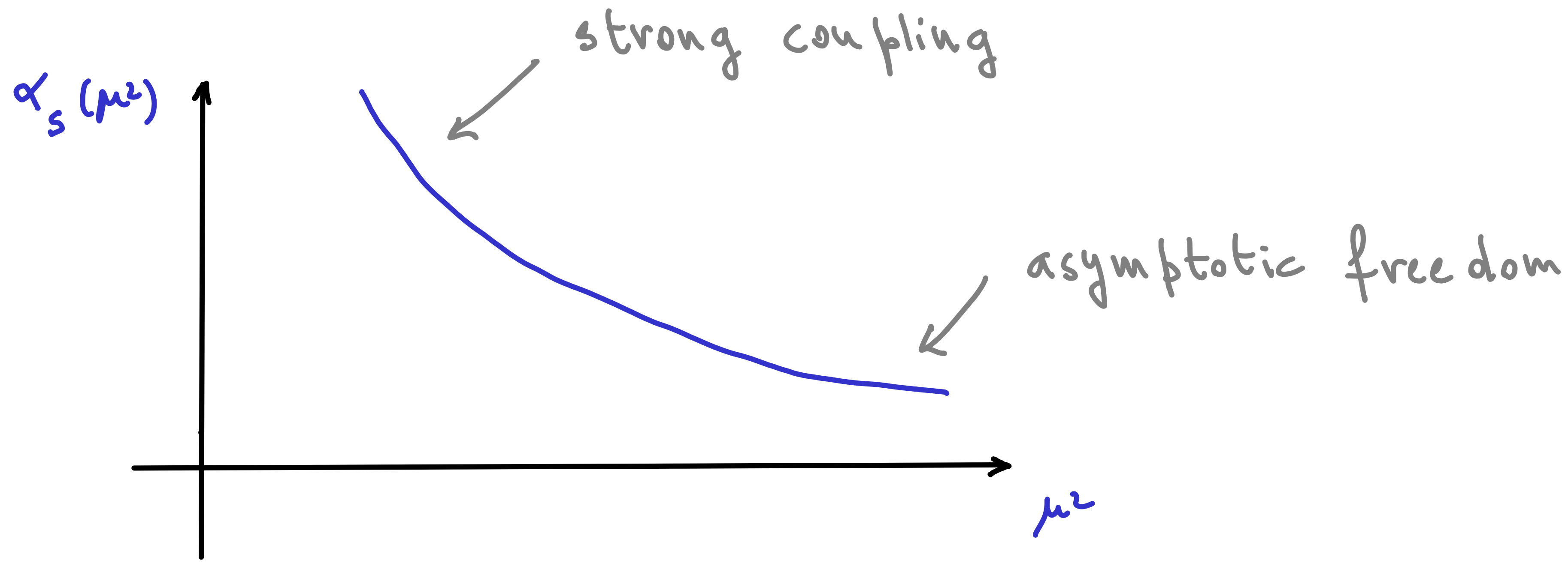
$$\boxed{\frac{d\alpha_s}{d\log\mu} \equiv \alpha_s\,\beta(\alpha_s) = -\frac{\alpha_s^2}{2\pi}\beta_0 + O(\alpha_s^3) \quad \text{with} \quad \beta_0 = \frac{11}{3}N_c - \frac{4}{3}T_F\,n_f}$$

where α_s is the renormalized strong coupling, n_f is the number of massless flavors in the fermion loop, and the normalization T_F is 1 for QED and $\frac{1}{2}$ for SU(N_c).

- The QED case corresponds to $T_F=1$, $N_c=0$ and $n_f=1$, for which we recover $\beta_0=-\frac{4}{3}$.

- The QCD case corresponds to $N_c = 3$.

For $n_f \leq 16$, $\beta_0 > 0$. Hence the running strong coupling constant decreases with μ:



In an EFT language the quark-gluon, gluon-gluon and ghost-gluon operators, which are marginal at tree level, become relevant at low energies due to quantum corrections.

Dimensional transmutation

From solving $\frac{d\alpha}{d\log\mu} = \alpha\,\beta(\alpha) = -\frac{\alpha^2}{2\pi}\beta_0 + O(\alpha^3)$ with α the renormalized coupling, we obtain

$$\int_{\alpha(\mu)}^{\alpha(\Lambda)} \frac{d\alpha}{\alpha^2} = -\int_{\mu}^{\Lambda} \frac{\beta_0}{2\pi}\, d\log\mu$$

If we choose Λ such that $\alpha(\Lambda)=\infty$, then

$$\frac{1}{\alpha(\mu)} = \frac{\beta_0}{4\pi}\log\frac{\mu^2}{\Lambda^2} \qquad \text{or} \qquad \alpha(\mu) = \frac{4\pi}{\beta_0\log\frac{\mu^2}{\Lambda^2}}$$

In this way we generate an <u>intrinsic scale</u> Λ, of mass dimension one, which may be traded for the dimensionless coupling α :

$$\Lambda = \mu \, e^{-\frac{2\pi}{\beta_0}\frac{1}{\alpha(\mu)}}$$

<u>Obs.</u> (i) Λ is independent of μ.

(ii) Λ_{QCD} is non perturbative (it vanishes at any order in α_s).

- In QED: $\Lambda_{QED} = m_e \, e^{\frac{3\pi}{2}\frac{1}{\alpha(m_e)}} \approx 0.5 \text{ MeV } e^{645} \approx 10^{277} \text{ GeV} \gg M_{Planck} \approx 10^{19} \text{ GeV}$

Hence Λ_{QED} is of no physical relevance.

- In QCD: $\Lambda_{QCD} = M_Z \, e^{-\frac{2\pi}{11-3.3}\frac{1}{0.118}} \approx 90 \text{ MeV}$ (≈ 200 MeV at four loop running)

Obs. $n_f = 5$

$\alpha_s(M_Z) \approx 0.118$ with $M_Z \approx 80$ GeV

Λ_{QCD} is the relevant hadronic scale.

From an EFT perspective Λ_{QCD} plays the role of an energy scale under any respect, and has to be included in the study of any physical system described by QCD.

Renormalization group equations: motivation

Renormalization introduces a scale μ that factorizes high energy contributions from low energy ones.

The scale μ is associated with potentially large logarithms of the form $\log \frac{\mu}{\Lambda_H}$.

These are potentially large because they appear in physical observables in the combination:

$$\log \frac{\mu}{\Lambda_H} \quad \langle O_i \rangle (\mu)$$

and the matrix element $\langle O_i \rangle (\mu)$ has natural power counting if $\mu \sim \Lambda_L$.

Hence, we may be in need to resum large $\log \frac{\mu}{\Lambda_H}$.

Obs. This may happen if $c_i \sim \ldots \alpha \left(1 + \ldots \alpha \log \frac{\mu}{\Lambda_H} \ldots\right)$ and $\alpha \log \frac{\mu}{\Lambda_H}$ is comparable to 1.

Logarithms are resummed by requiring that physical observables are independent of renormalization conventions and in particular of μ.

Renormalization group equations in QFT

Let $\Gamma(p_i, \alpha, m, \varepsilon)$ be a Green's function regularized in DR, p^i are the external momenta; let $\Gamma_R(p_i, \alpha_R, m_R, \mu)$ be the renormalized Green's function. Then

$$\Gamma(p_i, \alpha, m, \varepsilon) = Z_\Gamma(\varepsilon, \mu)\, \Gamma_R(p_i, \alpha_R, m_R, \mu)$$

where Z_Γ denotes the appropriate renormalization constant.

Since Γ does not depend on μ: $\quad \mu \dfrac{d}{d\mu} \Gamma = 0$, which implies

$$\mu \frac{d}{d\mu}(Z_\Gamma \Gamma_R) = 0 \qquad \text{or} \qquad \left(\mu \frac{d}{d\mu} + \gamma_\Gamma\right) \Gamma_R(p_i, \alpha_R, m_R, \mu) = 0$$

where

$$\boxed{\gamma_\Gamma \equiv \frac{1}{Z_\Gamma} \mu \frac{d}{d\mu} Z_\Gamma = \frac{d}{d\log\mu} \log Z_\Gamma}$$

If a mass independent renormalization scheme is used then (like in the case of the β function):

$$\gamma_\Gamma = \gamma_\Gamma(\alpha_R)$$

We may also write the renormalization group equation (RG) for Γ_R as

$$\boxed{\left(\mu\frac{\partial}{\partial\mu} + \alpha_R\,\beta(\alpha_R)\frac{\partial}{\partial\alpha_R} - \gamma(\alpha_R)\, m_R\frac{\partial}{\partial m_R} + \gamma_\Gamma(\alpha_R)\right)\Gamma_R(p_i, \alpha_R, m_R, \mu) = 0}$$

where

$$-m_R\,\gamma(\alpha_R) = \mu\frac{\partial}{\partial\mu}m_R(\mu)$$

$$\alpha_R\,\beta(\alpha_R) = \mu\frac{d\alpha_R}{d\mu}$$

The solution of $\left(\mu\frac{d}{d\mu}+\gamma_\Gamma\right)\Gamma_R(p_i,\alpha_R,m_R,\mu)=0$ is

$$\boxed{\Gamma_R(p_i,\alpha_R(\mu),m_R(\mu),\mu)=\Gamma_R(p_i,\alpha_R(\mu_0),m_R(\mu_0),\mu_0)\,e^{-\int_{\alpha_R(\mu_0)}^{\alpha_R(\mu)}\frac{d\alpha}{\alpha}\frac{\gamma_\Gamma(\alpha)}{\beta(\alpha)}}}$$

Obs. We have used $\int_{\mu_0}^{\mu} d\mu'\,\frac{1}{\mu'}\gamma_\Gamma(\alpha(\mu'))=\int_{\alpha_R(\mu_0)}^{\alpha_R(\mu)} d\alpha\,\frac{1}{\alpha\beta(\alpha)}\gamma_\Gamma(\alpha)$ that follows from

$$\alpha=\alpha(\mu)\Rightarrow d\alpha=\frac{d\alpha}{d\mu}d\mu\Rightarrow \mu\, d\alpha=\alpha\beta(\alpha)\,d\mu\Rightarrow\frac{d\mu}{\mu}=\frac{d\alpha}{\alpha\beta(\alpha)}.$$

The solution describes the evolution of Γ_R from the scale μ_0 to μ.

Obs. γ_Γ relates finite (renormalized) quantities, therefore γ_Γ is finite for $\varepsilon\to 0$.

About the meaning of γ_Γ

Let's scale all momenta of Γ_R by ξ ($d_\Gamma \equiv$ mass dimension of Γ):

$$\Gamma_R\left(\xi p_i, \alpha_R(\mu), m_R(\mu), \mu\right) = \xi^{d_\Gamma}\, \Gamma_R\left(p_i, \alpha_R(\mu), \frac{m_R(\mu)}{\xi}, \frac{\mu}{\xi}\right)$$

$$= \xi^{d_\Gamma}\, \Gamma_R\left(p_i, \alpha_R(\mu_0), \frac{m_R(\mu_0)}{\xi}, \frac{\mu_0}{\xi}\right) e^{-\int_{\alpha_R(\mu_0)}^{\alpha_R(\mu)} \frac{d\alpha}{\alpha} \frac{\gamma_\Gamma(\alpha)}{\beta(\alpha)}}$$

We consider now the RG equation for $\Gamma_R\left(\xi p_i, \alpha_R(\mu), m_R(\mu), \mu\right)$.

$$\xi \frac{\partial}{\partial \xi} \Gamma_R(\xi p_i, \alpha_R(\mu), m_R(\mu), \mu) = \xi \frac{\partial}{\partial \xi}\left(\xi^{d_\Gamma} \Gamma_R\left(p_i, \alpha_R(\mu), \frac{m_R(\mu)}{\xi}, \frac{\mu}{\xi}\right)\right)$$

$$= d_\Gamma \xi^{d_\Gamma} \Gamma_R\left(p_i, \alpha_R(\mu), \frac{m_R(\mu)}{\xi}, \frac{\mu}{\xi}\right) - m_R \frac{\partial}{\partial m_R}\left(\xi^{d_\Gamma} \Gamma_R\left(p_i, \alpha_R(\mu), \frac{m_R(\mu)}{\xi}, \frac{\mu}{\xi}\right)\right)$$

$$- \mu \frac{\partial}{\partial \mu}\left(\xi^{d_\Gamma} \Gamma_R\left(p_i, \alpha_R(\mu), \frac{m_R(\mu)}{\xi}, \frac{\mu}{\xi}\right)\right)$$

Obs. $\xi \frac{\partial}{\partial \xi} f\left(\frac{m}{\xi}\right) \underset{x = \frac{m}{\xi}}{=} \xi \left(-\frac{m}{\xi^2}\right) \frac{\partial}{\partial x} f\left(\frac{m}{\xi}\right) = -m \frac{\partial}{\partial m} f\left(\frac{m}{\xi}\right)$ and analogously $\xi \frac{\partial}{\partial \xi} f\left(\frac{\mu}{\xi}\right) = -\mu \frac{\partial}{\partial \mu} f\left(\frac{\mu}{\xi}\right)$.

and finally

$$\xi \frac{\partial}{\partial \xi} \Gamma_R(\xi p_i, \alpha_R(\mu), m_R(\mu), \mu) = \Big(d_\Gamma - m_R \frac{\partial}{\partial m_R} \underbrace{+ \alpha_R \beta(\alpha_R) \frac{\partial}{\partial \alpha_R} - \gamma(\alpha_R)\, m_R \frac{\partial}{\partial m_R} + \gamma_\Gamma}_{= -\mu\, \partial \Gamma_R / \partial \mu \text{ from the RG equation}}\Big) \Gamma_R(\xi p_i, \alpha_R(\mu), m_R(\mu), \mu)$$

Because

$$\left(\xi \frac{\partial}{\partial \xi} - \alpha_R \beta(\alpha_R) \frac{\partial}{\partial \alpha_R} + (1+\gamma(\alpha_R)) m_R \frac{\partial}{\partial m_R} - (d_\Gamma + \gamma_\Gamma(\alpha_R)) \right) \Gamma_R(\xi p_i, \alpha_R(\mu), m_R(\mu), \mu) = 0$$

we can interpret γ_Γ a an <u>anomalous dimension</u> of Γ_R.

Renormalization group equations in EFT

Consider the EFT Lagrangian

$$\mathcal{L} = \sum_i c_i \frac{O_i}{\Lambda^{d_i-4}}$$

Suppose that there is just one single operator for dimension, then the combination $\dfrac{c_i(\mu,\Lambda) \langle O_i(\mu) \rangle_R}{\Lambda^{d_i-4}}$ ($\langle \dots \rangle_R$ is the renormalized Green's function) is an observable: it is the $O(\Lambda^{4-d_i})$ contribution to the spectrum of the theory.

This quantity is therefore independent of μ:

$$\mu \frac{d}{d\mu} c_i(\mu,\Lambda) \langle O_i(\mu) \rangle_R = 0$$

On the other hand, the bare Green's function $\langle O_i \rangle$ is related to the renormilized one $\langle O_i(\mu)\rangle_R$ by

$$\langle O_i \rangle = Z_i(\varepsilon,\mu)\, \langle O_i(\mu)\rangle_R$$

Obs. There is no mixing under renormalization because we have assumed that there is just one operator for dimension (mixing requires a system of RG equations rather than a single one).

The RG equation that follows from $\mu \frac{d}{d\mu}\langle O_i\rangle = 0$ reads

$$\boxed{\left(\mu \frac{d}{d\mu} + \gamma_{O_i}(\alpha_R)\right) \langle O_i(\mu)\rangle_R = 0}$$

$\gamma_{O_i}(\alpha_R) = \frac{\mu}{Z_i}\frac{dZ_i}{d\mu} \equiv \gamma_{O_i}^{(0)} \frac{\alpha_R}{4\pi} + \gamma_{O_i}^{(1)}\left(\frac{\alpha_R}{4\pi}\right)^2 + O(\alpha_R^3)$ is the anomalous dimension of the operator O_i.

Hence

$$0 = \mu \frac{d}{d\mu} c_i(\mu,\Lambda) \langle O_i(\mu)\rangle_R = \left(\mu \frac{d}{d\mu} c_i(\mu,\Lambda)\right) \langle O_i(\mu)\rangle_R + c_i(\mu,\Lambda)\, \mu \frac{d}{d\mu} \langle O_i(\mu)\rangle_R$$

$$= \left(\mu \frac{d}{d\mu} c_i(\mu,\Lambda)\right) \langle O_i(\mu)\rangle_R - c_i(\mu,\Lambda)\, \gamma_{O_i}(\alpha_R) \langle O_i(\mu)\rangle_R$$

which implies the RG equation for the matching coefficient

$$\boxed{\mu \frac{d}{d\mu} c_i(\mu,\Lambda) = \gamma_{O_i}(\alpha_R)\, c_i(\mu,\Lambda)}$$

whose solution is

$$\boxed{c_i(\mu) = c_i(\mu_0)\, e^{\int_{\alpha_R(\mu_0)}^{\alpha_R(\mu)} \frac{d\alpha}{\alpha} \frac{\gamma_{O_i}(\alpha)}{\beta(\alpha)}}}$$

It describes the evolution of $c_i(\mu)$ from the scale μ_0 to the scale μ.

More explicitly

$$c_i(\mu) = c_i(\mu_0)\, e^{\int_{\alpha_R(\mu_0)}^{\alpha_R(\mu)} \frac{d\alpha}{4\pi}\left(\gamma^{(0)} + \frac{\alpha}{4\pi}\gamma^{(1)} + O(\alpha^2)\right)\frac{1}{-\frac{\alpha}{2\pi}\beta_0 - \frac{\alpha^2}{8\pi^2}\beta_1 + O(\alpha^3)}}$$

$$= c_i(\mu_0)\, e^{\int_{\alpha_R(\mu_0)}^{\alpha_R(\mu)} \frac{d\alpha}{\cancel{4\pi}}\left(\gamma^{(0)} + \frac{\alpha}{4\pi}\gamma^{(1)} + O(\alpha^2)\right)\left(-\frac{\cancel{2\pi}}{\alpha\beta_0}\right)\left(1 - \frac{\alpha}{4\pi}\frac{\beta_1}{\beta_0} + O(\alpha^2)\right)}$$

$$= c_i(\mu_0)\, e^{\int_{\alpha_R(\mu_0)}^{\alpha_R(\mu)} d\alpha \left(-\frac{\gamma^{(0)}}{2\alpha\beta_0} - \frac{1}{8\pi\beta_0}\left(\gamma^{(1)} - \frac{\beta_1}{\beta_0}\gamma^{(0)}\right) + O(\alpha)\right)}$$

$$= c_i(\mu_0)\, e^{-\frac{\gamma^{(0)}}{2\beta_0}\log\frac{\alpha_R(\mu)}{\alpha_R(\mu_0)} - \frac{1}{8\pi\beta_0}\left(\gamma^{(1)} - \frac{\beta_1}{\beta_0}\gamma^{(0)}\right)\left(\alpha_R(\mu) - \alpha_R(\mu_0)\right) + O(\alpha_R^3)}$$

$$= c_i(\mu_0)\left(\frac{\alpha_R(\mu)}{\alpha_R(\mu_0)}\right)^{-\frac{\gamma^{(0)}}{2\beta_0}}\left[1 - \left(\frac{\gamma^{(1)}}{2\beta_0} - \frac{\beta_1}{2\beta_0^2}\gamma^{(0)}\right)\frac{\alpha_R(\mu) - \alpha_R(\mu_0)}{4\pi} + O(\alpha_R^3)\right]$$

Obs. Let us choose $\mu_0 = m$, a heavy mass threshold, and $c_i(m) = c_i^{(0)} + \frac{\alpha_R(m)}{4\pi} c_i^{(1)} + O(\alpha_R^2)$ then

$$c_i(\mu) = \underbrace{\underbrace{c_i^{(0)} \left(\frac{\alpha_R(\mu)}{\alpha_R(m)}\right)^{-\frac{\gamma^{(0)}}{2\beta_0}} \Big[\, 1}_{\text{LL}} + \frac{\alpha_R(m)}{4\pi} \frac{c_i^{(1)}}{c_i^{(0)}} - \left(\frac{\gamma^{(1)}}{2\beta_0} - \frac{\beta_1}{2\beta_0^2}\gamma^{(0)}\right) \frac{\alpha_R(\mu) - \alpha_R(m)}{4\pi}}_{\text{NLL}} + O(\alpha_R^3) \Big]$$

Obs. Recall that $\alpha_R(\mu) = \alpha_R(m)\left(1 + \frac{\alpha_R(m)}{\pi} \frac{\beta_0}{2} \log\frac{m}{\mu} + O(\alpha_R^2)\right)$.

Def. Leading logarithmic accuracy (LL) resums all terms of the type $1 + \sum_{n>0} \cdots \left(\alpha \log\frac{\mu}{m}\right)^n$

for it $c_i^{(0)}$ and $\gamma^{(0)}$ are necessary.

Next to leading logarithmic accuracy (NLL) resums all terms of the type $\alpha + \sum_{n>0} \cdots \alpha^{n+1} \log^n \frac{\mu}{m}$

for it also $c_i^{(1)}$ and $\gamma^{(1)}$ are necessary.

How to RG improve the EFT

Consider the EFT Lagrangian

$$\mathcal{L} = \sum_i c_i(\mu, \Lambda_H) \frac{O_i(\mu, \Lambda_L)}{\Lambda_H^{d_i - 4}}$$

The matrix element $\langle O_i(\mu, \Lambda_L)\rangle$ has the natural scaling $\Lambda_L^{d_i}$ if μ is run down from the matching scale Λ_H to Λ_L.

This is performed in 3 steps.

(1) $C_i(\Lambda_H, \Lambda_H)$ is computed at some accuracy by matching at the scale Λ_H Green's functions in the fundamental theory with Green's functions in the EFT.

(2) The anomalous dimension of C_i is computed at the required accuracy (LL accuracy requires the anomalous dimension at 1 loop and $C_i(\Lambda_H, \Lambda_H)$ at leading order (LO), NLL accuracy requires the anomalous dimension at 2 loops and $C_i(\Lambda_H, \Lambda_H)$ at next-to-leading order (NLO)...).

(3) The RG equation for C_i is solved and C_i is run down to Λ_L so that the EFT Lagrangian reads

$$\mathcal{L} = \sum_i C_i(\Lambda_L, \Lambda_H) \frac{O_i(\Lambda_L, \Lambda_L)}{\Lambda_H^{d_i - 4}}$$

Obs. One has to assume that perturbation theory works for both Λ_H and Λ_L, elsewhere one can only run down to $\max\{\Lambda_L, \text{lowest scale where perturbation theory works, e.g., } \Lambda_{QCD}\}$.