Testing entanglement and Bell inequalities in $H \to ZZ$

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Main goals

1. Reconstruct the spin density matrix, ρ_{ZZ}, from angular observables in the decay $H \to ZZ^* \to \ell_1^+ \ell_1^- \ell_2^+ \ell_2^-$ (Quantum Tomography).
2. Develop a necessary and sufficient condition for entanglement in ρ_{ZZ} by taking into account symmetries of the system.
3. Give a novel sufficient condition for the violation of Bell inequalities of ρ_{ZZ}.

Introduction to Quantum Information Theory

- A general quantum system is described by a density matrix:

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| , \quad p_i \geq 0, \quad \sum_i p_i = 1.$$

For bipartite systems, $|\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, with $dim\, \mathcal{H}_{A(B)} = d_{A(B)}$.

- Expectation values of an observable $\mathcal{O}$ are computed via:

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\{\rho\, \mathcal{O}\}.$$

- For bipartite systems, a density matrix is separable when

$$\rho_{\text{sep}} = \sum_i p_i\, \rho_i^A \otimes \rho_i^B \ ,$$

and entangled otherwise.

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- **Bell inequalities**

  Given ρ for a bipartite system, and A_1, A_2 and B_1, B_2 observables (acting A_i on $\mathcal{H}_A$ and B_j on $\mathcal{H}_B$), any local realistic theory must fulfilled

$$I(P(A_i = k, B_j = l)) = \mathrm{Tr}\left\{\rho\, \mathcal{O}_{Bell}(A_i,\, B_j)\right\} \leq 2.$$

- **Locality**: no physical influences (information) traveling faster than the speed of light.
- **Realism**: physical properties independent of the measurement.
- We use the CGLMP inequality, I_{CGLMP} (generalization of I_{CHSH} optimal for $d_A, d_B \geq 3$).

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Quantum Tomography

- **Spin density matrix ρ_{ZZ}?**

  In our case, $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}_{Spin}$ $(d_A = d_B = 3)$.

  We use the irreducible tensor operators $\{T^{L_1}_{M_1} \otimes T^{L_2}_{M_2}\}$ (transforming under rotations as the spherical harmonics) as a basis:

$$T^{L_1}_{M_1}, T^{L_2}_{M_2} \in \left\{\mathbb{I}_3; T^1_1, T^1_0, T^1_{-1}; T^2_2, T^2_1, T^2_0, T^2_{-1}, T^2_{-2}\right\} .$$

Hence:

$$\rho_{ZZ} = \frac{1}{9}\sum C_{L_1,M_1,L_2,M_2}\, T^{L_1}_{M_1} \otimes T^{L_2}_{M_2},$$

where $C_{L_1M_1L_2M_2} = (-1)^{M_1+M_2}(C_{L_1,-M_1,L_2,-M_2})^*$, deduced from $\rho_{ZZ} = \rho^{\dagger}_{ZZ}$ and $\left(T^L_M\right)^{\dagger} = (-1)^M\, T^L_{-M}$.

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Symmetries of the system

- Momentum J_z conserved. In particular $J_z = 0$.
- CP conservation.

For a particular event, using the symmetries and the Lorentz structure of the HZZ SM vertex:

$$\rho_\beta = |\psi_\beta\rangle \langle\psi_\beta| \,, \quad |\psi_\beta\rangle = \frac{1}{\sqrt{2+\beta^2}} \left(|+-\rangle - \beta\,|00\rangle + |-+\rangle\right),$$

where $\beta = 1 + \dfrac{m_H^2 - (m_{Z_1} + m_{Z_2})^2}{2 m_{Z_1} m_{Z_2}}$.

For $\beta = 1$, $|\psi_{\beta=1}\rangle = |\psi_s\rangle = \dfrac{1}{\sqrt{3}} \left(|+-\rangle - |00\rangle + |-+\rangle\right)$ and $\rho_{\beta=1} = \rho_s$.

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In practice, we have a probability distribution $\mathcal{P}(\beta)$ for the parameter β:

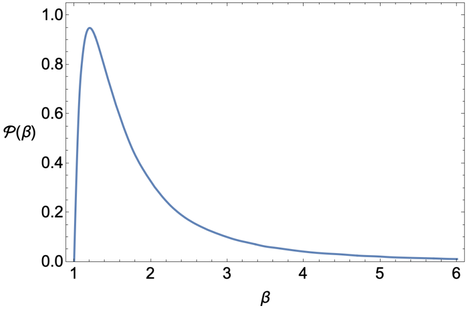

Figure 1: Probability distribution of β.

Thus, the final density matrix is

$$\rho_{ZZ} = \int d\beta \; \mathcal{P}(\beta)\rho_\beta = \rho_{ZZ}\left(C_{2,0,0,0}, C_{2,1,2,-1}, C_{2,2,2,-2}\right).$$

Furthermore, $\rho_{ZZ} \backsim \rho_\beta \backsim \rho_s$.

In order to extract the coefficients, we take into account that for the cross section of $ZZ^* \to \ell_1^+ \ell_1^- \ell_2^+ \ell_2^-$:

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega_1 d\Omega_2} = \left(\frac{3}{4\pi}\right)^2 \mathrm{Tr}\left\{\rho_{ZZ}\, \left(\Gamma_1 \otimes \Gamma_2\right)^T\right\},$$

where $\Gamma_j(\theta_j, \varphi_j)$ are the decay density matrices of each Z boson.

By noticing that $\mathrm{Tr}\left\{T_M^L\, \Gamma^T\right\} = B_L\, Y_L^M(\theta, \varphi)$, with B_L a constant:

$$\int \frac{1}{\sigma}\frac{d\sigma}{d\Omega_1 d\Omega_2} Y_{L_1}^{M_1}(\Omega_1)^* Y_{L_2}^{M_2}(\Omega_2)^* d\Omega_1 d\Omega_2 = \frac{B_{L_1} B_{L_2}}{(4\pi)^2} C_{L_1 M_1 L_2 M_2}$$

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- **Peres-Horodecki criterion**

Given a density matrix ρ describing a bipartite system, if the matrix $\rho' = (\mathbb{I} \otimes T_B)\,\rho$ has at least one negative eigenvalue, then ρ is entangled.

This criterion is only a sufficient condition for:

$$dim\,\mathcal{H}_A = dim\,\mathcal{H}_B = d \geq 3 \text{ (our case)}$$

However, due to the symmetries mentioned and the structure of ρ_{ZZ}:

$$\rho_{ZZ} \text{ entangled } \iff C_{2,1,2,-1} \neq 0 \text{ or } C_{2,2,2,-2} \neq 0.$$

Testing Entanglement

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Testing Bell inequalities

We recall that $\rho_s = |\psi_s\rangle\langle\psi_s|$, with $|\psi_s\rangle = \frac{1}{\sqrt{3}}(|+-\rangle - |00\rangle + |-+\rangle)$.
This state has a U(3) symmetry in the sense:

$$\mathrm{Tr}\left\{\rho_s\,(U\otimes U^*)^\dagger \mathcal{O}^s_{Bell}(U\otimes U^*)\right\} = \mathrm{Tr}\left\{\rho_s\,\mathcal{O}^s_{Bell}\right\} \text{ for } U\in U(3)$$

with $\mathcal{O}^s_{Bell}$ the **known** optimal Bell operator for the singlet.

For $\rho_\beta = |\psi_\beta\rangle\langle\psi_\beta|$, where $|\psi_\beta\rangle = \frac{1}{\sqrt{2+\beta^2}}(|+-\rangle - \beta|00\rangle + |-+\rangle)$, the symmetry is broken to $U(2)\otimes U(1)$:

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$\mathcal{O}_{Bell}^{\beta}$ is obtained maximizing the violation of I_3 in the broken symmetry:

$$\mathrm{Tr}\left\{\rho_\beta\, \mathcal{O}_{Bell}^{\beta}\right\} = \max_{U\in U(3)/(U(2)\otimes U(1))} \mathrm{Tr}\left\{\rho_\beta\, (U\otimes U^*)^\dagger \mathcal{O}_{Bell}^{s}(U\otimes U^*)\right\}.$$

For $\rho_{ZZ} \Rightarrow I_3 = I_3\left(C_{2,0,0,0}, C_{2,1,2,-1}, C_{2,2,2,-2}\right).$

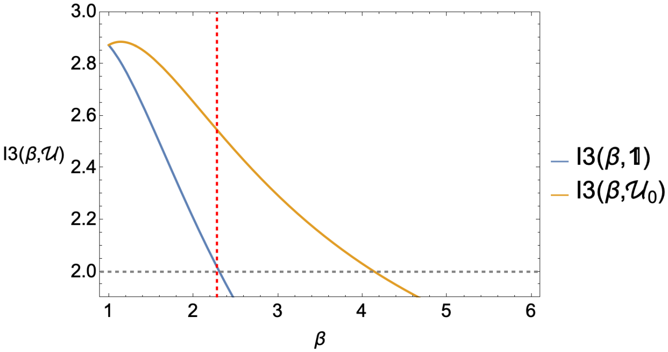

Figure 2: Functions $(I_3(\beta, \mathbb{I}_3), I_3(\beta, U_0))$, local-realistic upper bound (gray line) and mean value of β with respect to $\mathcal{P}(\beta)$ (red line).

Numerical Resuts

- **LHC Run 2+3**

	min m_{Z_2}			
	0	10 GeV	20 GeV	30 GeV
N	450	418	312	129
$C_{2,1,2,-1}$	-0.98 ± 0.31	-0.97 ± 0.33	-1.05 ± 0.38	-1.06 ± 0.61
$C_{2,2,2,-2}$	0.60 ± 0.37	0.64 ± 0.38	0.74 ± 0.43	0.82 ± 0.63
I_3	2.66 ± 0.46	2.67 ± 0.49	2.82 ± 0.57	2.88 ± 0.89

Table 1: Values $C_{2,1,2,-1}$, $C_{2,2,2,-2}$ and I_3 obtained from 1000 pseudo experiments with $L = 300$ fb^{-1}.

- **HL-LHC**

	min m_{Z_2}			
	0	10 GeV	20 GeV	30 GeV
N	4500	4180	3120	1290
$C_{2,1,2,-1}$	-0.95 ± 0.10	-1.00 ± 0.10	-1.04 ± 0.12	-1.04 ± 0.19
$C_{2,2,2,-2}$	0.60 ± 0.12	0.64 ± 0.12	0.74 ± 0.14	0.83 ± 0.20
I_3	2.63 ± 0.15	2.71 ± 0.16	2.81 ± 0.18	2.84 ± 0.28

Table 2: Same as Table 1, for $L = 3$ ab^{-1}.

Conclusions

- The decay channel $H \to ZZ^* \to \ell_1^+ \ell_1^- \ell_2^+ \ell_2^-$ is an excellent way to probe the quantum nature of the Universe:
  - Run 2+3: ρ_{ZZ} entangled in more than 2σ and $I_3 > 2$ in more than 1σ.
  - HL-LHC: ρ_{ZZ} entangled in more than 5σ and $I_3 > 2$ in more than 3σ.
- The quantum tomography formalism developed is practical and generalizable for other kinds of processes.
- Entanglement criteria as well as optimal violations of Bell inequalities can be extracted and implemented taking into account the symmetries of the system.

Thank you for listening!