Solid matter with zero shear modulus in flat universe

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Contents

- general relativistic solid matter
  - kinematics
  - dynamics
- zero shear modulus, Lamé coefficient $\mu = 0$

  perfect fluid behaviour up to first perturbative order
- a natural restriction

  $$\boxed{\widetilde{w} \equiv \frac{T_i{}^i/3}{-T_0{}^0} = \text{const.}}$$

  $$!!! \quad \widetilde{w} \overset{!}{\neq} w \equiv \frac{p}{\rho} \quad !!!$$
- radiation-like solid

  dark energy-like solid
- equations for perturbations

Solid matter in general relativity

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• M. Karlovini, L. Samuelsson: *Elastic Stars in General Relativity: I. Foundations and Equilibrium Models*, Class. Quant. Grav. **20**, 3613 (2003), [arXiv:gr-qc/0211026]

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• S. Endlich, A. Nicolis, J. Wang: *Solid Inflation*, JCAP **1310**, 011 (2013), [arXiv:1210.0569 [hep-th]]

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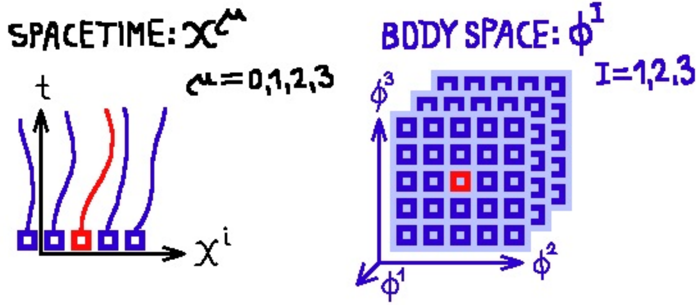
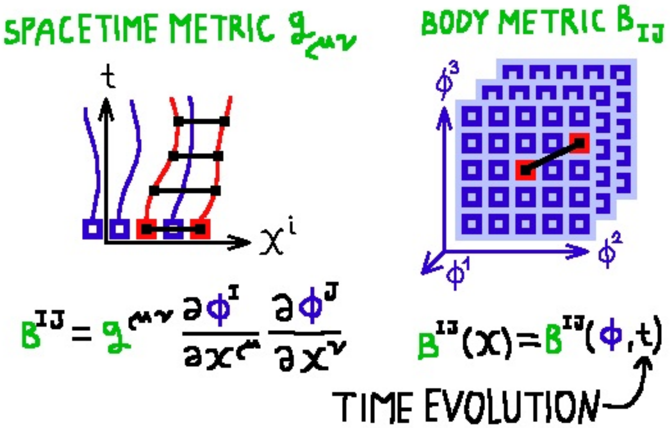
Solid matter kinematics



CONTINUUM KINEMATICS: $\phi^I = \phi^I(x)$

Solid matter kinematics



Homogeneity and isotropy

body metric $B^{IJ} = g^{\mu\nu}\phi^{I}{}_{,\mu}\phi^{J}{}_{,\nu}$

invariance with respect to internal (body) global translations and rotations $\phi^I \mapsto \phi^I + T^I + R^I{}_J\phi^J, \quad T^I \in \mathbb{R}, \quad R^I{}_J \in SO(3)$

three independent invariants

$$[\delta B\ ] = \bar{B}_{IJ}\ \delta B^{IJ}, \qquad [\delta B^2] = \bar{B}_{IJ}\ \delta B^{JK}\ \bar{B}_{KL}\ \delta B^{LI},$$

$$[\delta B^3] = \bar{B}_{IJ}\ \delta B^{JK}\ \bar{B}_{KL}\ \delta B^{LM}\ \bar{B}_{MN}\ \delta B^{NI}$$

$\bar{B}^{IJ}$ = background, $\delta B^{IJ} = B^{IJ} - \bar{B}^{IJ}$ = perturbation

note: $\det A = (\mathrm{Tr} A)^3/6 - \left(\mathrm{Tr} A^2\right)\mathrm{Tr} A/2 + \mathrm{Tr} A^3/3$

$\Longrightarrow$ equation of state

$$\rho = \rho\left(B^{IJ}\right) = \left[\begin{array}{l}\text{homogeneity}\\ \text{\& isotropy}\end{array}\right] = \rho\left([\delta B\ ], [\delta B^2], [\delta B^3]\right)$$

Solid matter dynamics

homogeneous and isotropic equation of state

$$\rho = \bar{\rho} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} C_{abc} \left[\delta B\ \right]^a \left[\delta B^2\right]^b \left[\delta B^3\right]^c$$

dynamics: $S = \int \sqrt{-g}\, d^4x \left(\frac{R}{16\pi\kappa} - \rho\right) \quad \longrightarrow \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi\kappa\, T_{\mu\nu}$

stress-energy tensor

$$T_{\mu\nu} = 2\frac{\partial\rho}{\partial B^{IJ}} \phi^I{}_{,\mu}\phi^J{}_{,\nu} - \rho g_{\mu\nu} = \bar{\rho} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} C_{abc} \left[\delta B\ \right]^a \cdot$$
$$\cdot \left[\delta B^2\right]^b \left[\delta B^3\right]^c \left[2\left(a\frac{\Psi^{(0)}_{\mu\nu}}{[\delta B]} + 2b\frac{\Psi^{(1)}_{\mu\nu}}{[\delta B^2]} + 3c\frac{\Psi^{(2)}_{\mu\nu}}{[\delta B^3]}\right) - g_{\mu\nu}\right]$$

where $\Psi^{(0)}_{\mu\nu} = \overline{B}_{IJ}\phi^I{}_{,\mu}\phi^J{}_{,\nu}, \quad \Psi^{(1)}_{\mu\nu} = \overline{B}_{IK}\delta B^{KL}\overline{B}_{LJ}\phi^I{}_{,\mu}\phi^J{}_{,\nu},$

$$\Psi^{(2)}_{\mu\nu} = \overline{B}_{IK}\delta B^{KL}\overline{B}_{LM}\delta B^{MN}\overline{B}_{NJ}\phi^I{}_{,\mu}\phi^J{}_{,\nu}$$

Lamé coefficients

equation of state

$$\rho = \bar{\rho} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} C_{abc} \left[\delta B\ \right]^a \left[\delta B^2\right]^b \left[\delta B^3\right]^c$$

parametrization

order $= a + 2b + 3c$

order 0	background	$C_{000} = 1$
order 1	$w = \frac{p}{\rho}$	$C_{100} = \frac{w+1}{2}$
order 2	Lamé coefficients	$C_{200} = \frac{\lambda + w + 1}{8}, \quad C_{010} = \frac{\mu - w - 1}{4}$
order 3	beyond Lamé	$C_{110} = \frac{1}{4}\nu_1, \quad C_{001} = \frac{1}{6}\nu_2, \quad C_{300} = \frac{1}{12}\nu_3$

shear modulus set to zero: $\mu = 0$ $\Longrightarrow$

$\Longrightarrow$ perfect fluid behaviour up to first perturbative order

Radiation & Dark Energy

$$\begin{cases} T_{\mu\nu} \stackrel{\text{rad.}}{=} F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} & \longrightarrow \quad T_{\mu}{}^{\mu} \equiv T_{0}{}^{0} + T_{i}{}^{i} = 0 \\ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\kappa T_{\mu\nu} & \longrightarrow \quad T_{\mu}{}^{\nu} \stackrel{\text{d.e.}}{=} -\frac{\Lambda}{8\pi\kappa}\delta_{\mu}^{\nu} \end{cases}$$

$$\Longrightarrow \quad \text{for both} \quad \left.\begin{matrix}\text{rad.}\\ \text{d.e.}\end{matrix}\right\} \Longrightarrow \frac{T_{i}{}^{i}}{T_{0}{}^{0}} = \begin{cases} -1 \\ 3 \end{cases} = \text{const.}$$

introduce $\widetilde{w} \equiv \frac{T_{i}{}^{i}/3}{-T_{0}{}^{0}} = \begin{cases} \frac{1}{3} \\ -1 \end{cases} \stackrel{\text{here}}{=} w \equiv \frac{p}{\rho}$

!!! in general !!! $\boxed{\widetilde{w} \neq w}$!!!

Flat universe

metric $ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$ Friedmann–Lemaître–Robertson–Walker

scalar perturbations: $ds^2 = a^2 \left[-(1+2\phi) d\tau^2 + (1-2\psi)\delta_{ij} dx^i dx^j \right]$

longitudinal gauge $\phi^I = \delta^I_i (x^i + \pi^i) = \delta^I_i (x^i + \sigma_{,i})$

$$B^{IJ} = g^{\mu\nu} \phi^I{}_{,\mu} \phi^J{}_{,\nu} \longrightarrow \delta B^{IJ} \longrightarrow$$

$$\frac{\rho}{\bar{\rho}} = 1 + \frac{w+1}{2}[\delta B\ ] + \frac{\lambda + w + 1}{8}[\delta B\ ]^2 - \frac{w+1}{4}[\delta B^2] + \frac{1}{4}\nu_1 [\delta B\ ][\delta B^2] + \frac{1}{6}\nu_2 [\delta B^3] + \frac{1}{12}\nu_3 [\delta B\ ]^3$$

equation of state $\longrightarrow$ stress-energy tensor $T_{\mu\nu}$ $\longrightarrow$ $\widetilde{w} = \ldots$

up to the second perturbative order

$$\frac{{T_0}^0}{\bar{\rho}} =$$

$$= -1 - 3(w + 1)\psi^{(1)} - (w + 1)\triangle\sigma^{(1)} -$$

$$-\frac{3}{2}\left(5w + 5 + 3\lambda\right)\psi^{(1)2} - 3(w + 1)\psi^{(2)} -$$

$$-3\left(w + 1 + \lambda\right)\psi^{(1)}\triangle\sigma^{(1)} +$$

$$+(w + 1)\left(\frac{1}{2}\sigma^{(1)}_{,ij}\sigma^{(1)}_{,ij} - \frac{1}{2}\left(\triangle\sigma^{(1)}\right)^2 - \frac{1}{2}\sigma^{(1)\prime}_{,i}\sigma^{(1)\prime}_{,i} - \triangle\sigma^{(2)}\right) -$$

$$-\frac{1}{2}\lambda\left(\triangle\sigma^{(1)}\right)^2$$

up to the second perturbative order

$$
\frac{T_i{}^i}{\bar{\rho}} =
$$

$$
= 3\, w + 9\, \lambda\, \psi^{(1)} + 3\, \lambda \triangle\sigma^{(1)} +
$$

$$
+ \frac{3}{2}\left(15\, \lambda + w + 1 + 36\, \nu_1 + 8\, \nu_2 + 36\, \nu_3\right) \psi^{(1)2} + 9\, \lambda\, \psi^{(2)} +
$$

$$
+ \left(w + 1 + 9\, \lambda + 36\, \nu_1 + 8\, \nu_2 + 36\, \nu_3 \right) \psi^{(1)} \triangle\sigma^{(1)} +
$$

$$
+ (w + 1) \left(-\sigma^{(1)}_{,ij}\sigma^{(1)}_{,ij} + \frac{1}{2}\left(\triangle\sigma^{(1)}\right)^2 + \sigma^{(1)\prime}_{,i}\sigma^{(1)\prime}_{,i} \right) +
$$

$$
+ \lambda \left(\frac{3}{2}\sigma^{(1)}_{,ij}\sigma^{(1)}_{,ij} + \frac{1}{2}\left(\triangle\sigma^{(1)}\right)^2 - \frac{3}{2}\sigma^{(1)\prime}_{,i}\sigma^{(1)\prime}_{,i} + 3\triangle\sigma^{(2)} \right) +
$$

$$
+ \nu_1 \left(6\sigma^{(1)}_{,ij}\sigma^{(1)}_{,ij} + 4\left(\triangle\sigma^{(1)}\right)^2 \right) + 4\, \nu_2\, \sigma^{(1)}_{,ij}\sigma^{(1)}_{,ij} + 6\, \nu_3 \left(\triangle\sigma^{(1)}\right)^2
$$

Constraint on parameter space

$$\boxed{\tilde{w} \equiv \frac{T_i{}^i/3}{-T_0{}^0} = \text{const.}} \quad \Longrightarrow \quad \begin{array}{l}\text{constraints on parameters} \\ w,\ \lambda,\ \nu_1,\ \nu_2,\ \nu_3\end{array}$$

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first perturbative order

$$\Longrightarrow \qquad \lambda = w(w+1)$$

second perturbative order

$$\Longrightarrow \qquad (3w-1)(w+1) = 0, \quad 3\nu_1 + 2\nu_2 = 0, \quad 2\nu_1 + 3\nu_3 = 0$$

only independent coefficients are

$$w = \begin{cases} \frac{1}{3} & \text{radiation-like solid} \\ -1 & \text{dark energy-like solid} \end{cases}, \qquad \nu \equiv \nu_1 \text{ arbitrary}$$

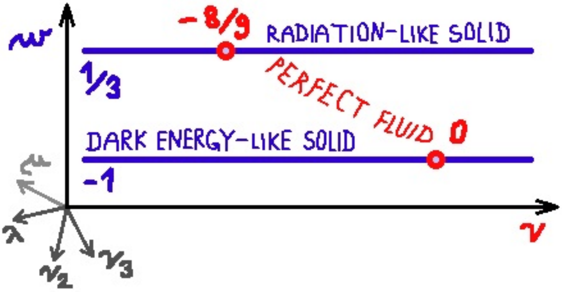
---

perfect fluid

$$\boxed{\rho = F\left(\det B^{IJ}\right)} \quad \Longrightarrow \quad \nu = \begin{cases} -\frac{8}{9} & \text{rad.-like solid} \\ 0 & \text{d.e.-like solid} \end{cases}$$

Constrained equation of state

$$
\begin{aligned}
\frac{\rho}{\rho} &= 1 + \frac{1}{2}(w + 1)[\delta B\ ] + \\
&+ \frac{1}{8}(w+1)^2 [\delta B\ ]^2 - \frac{1}{4}(w+1)[\delta B^2] + \\
&+ \nu \left(\frac{1}{4}[\delta B\ ][\delta B^2] - \frac{1}{4}[\delta B^3] - \frac{1}{18}[\delta B\ ]^3 \right)
\end{aligned}
$$



Notes

- **the same results** also for
  - relaxed longitudinal gauge for scalar perturbations
  - vector and tensor perturbations

- **dark energy-like solid** without pure scalar perturbations up to the second order
  nontrivial because for $\nu \neq 0$ no longer $T_{\mu\nu} \propto g_{\mu\nu}$
  (for $\nu = 0 \quad \rightarrow \quad$ perfect fluid with $T_{\mu\nu} \propto g_{\mu\nu}$)
  - first order tensor perturbations should source equations for second order scalar perturbations - so far omitted

- **radiation-like solid** with modified second order perturbations

Outlook

- investigating effect of linear order tensor perturbations, which are formed also in the dark energy-like case, on evolution of second order perturbations;
- relaxing the validity of condition $\widetilde{w} = \text{const.}$ in arbitrary coordinates, which have led to only two values of pressure to energy density ratio (i.e. studying cases with arbitrary w, not only $1/3$ and -1);
- studying even higher orders of the perturbation theory, where one has to define more coefficients parametrizing the equation of state of the solid, and investigate constraints on them;
- considering more general space-time metrics including also space-times beyond a cosmological context, which may require also a non-perturbative approach;
- $\cdots$?