A new determination of the c quark mass from non-analytic reconstruction

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CPAN

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Phys. Rev. D **85**, 054008 (2012)
Work in progress

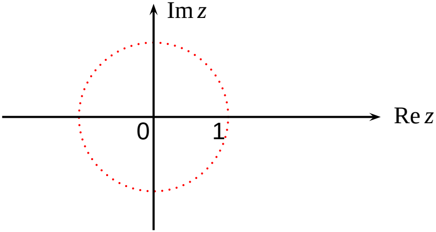
IVth CPAN Jornadas — Granada

Part 1

Non-analytic Reconstruction Method

Hypothesis

Let us consider of a 2 point Green's function, a form factor or more generally a complex function Π:



- Π is analytic on a disk $|z| < 1$: $\Pi(z) \underset{|z|<1}{=} \sum_{n=0}^{\infty} C(n)\, z^n$
- Π admits a branching cut $[1, \infty[$ and has the threshold expansion:

$$\Pi(z) \underset{z\to 1}{\sim} \sum_{p,k} A^{TH}(p,k)\,(1-z)^p \log^k(1-z)$$

- Π has the OPE expansion:

$$\Pi(z) \underset{z\to -\infty}{\sim} \sum_{p,k} B^{OPE}(p,k)\,\frac{1}{z^p}\log^k(-4z)$$

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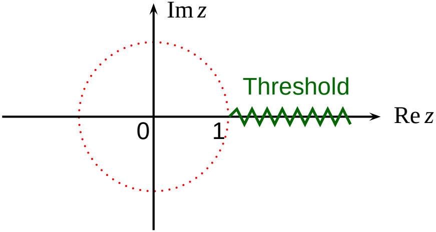

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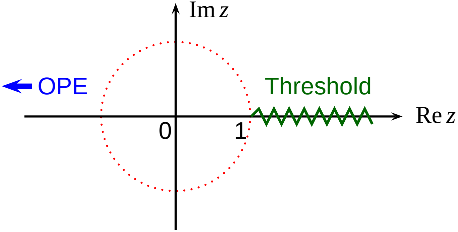

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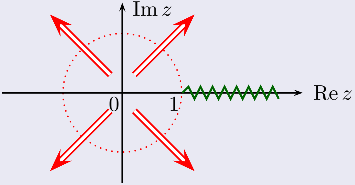
How to do analytic continuation ?

From the infinite number of Taylor coefficients: this is mathematics



$$\Pi(z) \underset{|z|<1}{=} \sum_{n=0}^{\infty} C(n)\, z^n$$

By resumation or by construction order by order, with an infinite number of $C(n)$ the analytic continuation can be easily obtained.

From the finite number of Taylor coefficients

$$\Pi(z) \underset{|z|<1}{=} \sum_{n=0}^{N^*} C(n)\, z^n$$

There is no way from a finite number N^* of $C(n)$, one needs the threshold and/or the OPE.

- Padé approximants under certain conditions.
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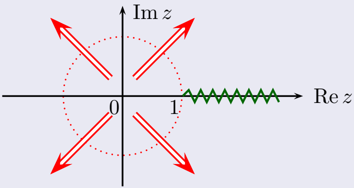

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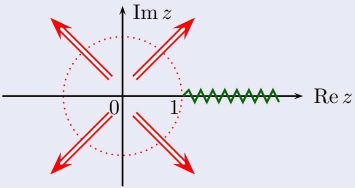

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The Non-Analytic Reconstruction

- It is an analytic reconstruction of the function.
- It is systematic and convergent order by order.
- It is a controlled approximation.
- All is based on a mathematical theorem.

The method assures that

The approximation

$$\Pi(z) = \sum_{n=0}^{N^*} \Omega(n)\omega^n + \sum_{p,\ell} (-)^\ell \left[\alpha_{k,\ell}\, \mathrm{Li}^{(\ell)}(p,\omega) - \beta_{k,\ell}\, \mathrm{Li}^{(\ell)}(p,-\omega) \right] + \mathcal{E}(N_k^*,\omega)$$

- $\omega \doteq \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}$ (conformal variable)
- N^* is the number of known Taylor coefficients.
- The reorganization of the Taylor coefficients is given by

$$\Omega(n) = (-1)^n \sum_{p=1}^{n} \frac{(-1)^p\, 4^p\, \Gamma(n+p)}{\Gamma(2p)\Gamma(n+1-p)}\, C(p)$$

- $\mathrm{Li}^{(\ell)}(s,\omega) = \frac{d^\ell}{ds^\ell}\left[\frac{\omega}{\Gamma(s)} \int_0^1 \frac{dt}{1-\omega t} \log^{s-1}\left(\frac{1}{t}\right) \right] \underset{|\omega|<1}{=} (-1)^\ell \sum_{n=1}^{\infty} \frac{\log^\ell n}{n^s}\, \omega^n$

- the error function $\mathcal{E}(N_k^*,\omega)$ is systematic and known.

Main result provided by the Converse Mapping Theorem

$$\Pi(z) = \sum_{n=0}^{N^*} \Omega(n)\omega^n + \sum_{p,\ell} (-)^\ell \left[\alpha_{k,\ell}\,\mathrm{Li}^{(\ell)}(p,\omega) - \beta_{k,\ell}\,\mathrm{Li}^{(\ell)}(p,-\omega)\right] + \mathcal{E}(N_k^*,\omega)$$

$\alpha_{k,\ell}$ **is only a linear function of** $A^{TH}(p,k)$ threshold coefficients

$\beta_{k,\ell}$ **is only a linear function of** $B^{OPE}(p,k)$ OPE coefficients

The reconstruction converges order by order

A perfect application example: Heavy-Quark Correlators

The correlators ($\mathsf{s} = \overline{\psi}\psi$, $\mathsf{p} = i\,\overline{\psi}\gamma_5\psi$, $\mathsf{v}_\mu = \overline{\psi}\gamma_\mu\psi$ et $\mathsf{a}_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi$)

$$q^2\,\Pi^{\mathsf{s,p}}(q^2) \doteq i\int d^4x\, e^{iqx}\,\langle 0|\mathrm{T}\begin{bmatrix}\mathsf{s}(x)\ \mathsf{s}(0)\\ \mathsf{p}(x)\ \mathsf{p}(0)\end{bmatrix}|0\rangle$$

$$\left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\,\Pi^{\mathsf{v,a}}(q^2) + q_\mu q_\nu\,\Pi_L^{\mathsf{v,a}}(q^2) \doteq i\int d^4x\, e^{iqx}\,\langle 0|\mathrm{T}\begin{bmatrix}\mathsf{v}_\mu(x)\ \mathsf{v}_\nu(0)\\ \mathsf{a}_\mu(x)\ \mathsf{a}_\nu(0)\end{bmatrix}|0\rangle$$

may be decomposed as

$$\Pi(q^2) = \Pi^{(0)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)\Pi^{(1)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^2\Pi^{(2)}(q^2) + \left(\frac{\alpha_s}{\pi}\right)^3\Pi^{(3)}(q^2) + \mathcal{O}(\alpha_s^4)$$

Π obeys an once-subtracted dispersion relation

$$\Pi(q^2) = q^2\int_0^\infty \frac{d\xi}{\xi}\,\frac{1}{\xi - q^2 - i\varepsilon}\,\frac{1}{\pi}\mathrm{Im}\,\Pi(\xi + i\varepsilon)\ .$$

$\mathrm{Im}\,\Pi(\xi \leqslant 4m^2) = 0$, where m is the heavy quark pole mass.

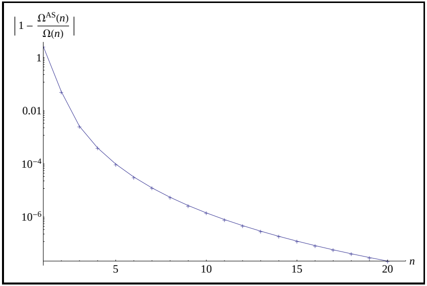
$\Pi^{\mathrm{v}(0)}$: a warming-up example

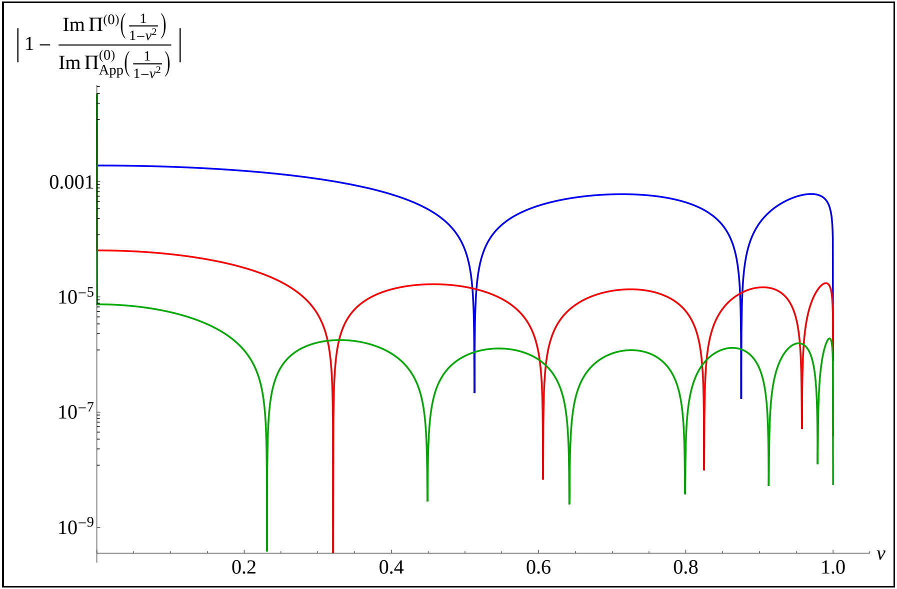
The function $\Pi^{\mathrm{v}(0)}$ is known explicitly, as

$$\Pi^{\mathrm{v}(0)}(z) = \frac{3}{16\pi^2}\left[\frac{20}{9} + \frac{4}{3\,z} - \frac{4(1-z)(1+2\,z)}{3\,z}\,\frac{2\,\frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}\,\log\left(\frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}\right)}{\left(\frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}\right)^2-1}\right]$$

It is the easy to obtain some terms at $z=0$, $z=1$ and $z=-\infty$, and then generate

$$\alpha_{n,\ell} = 0 \text{ and } \beta_{n,\ell} = (-1)^n\left[-\frac{1}{2\pi^2}\,\frac{1}{n} + \frac{9}{32\pi^2}\,\frac{1}{n^5} + \ldots\right]$$




$\left| 1 - \frac{\mathrm{Im}\,\Pi^{(0)}\left(\frac{1}{1-v^2}\right)}{\mathrm{Im}\,\Pi^{(0)}_{\mathrm{App}}\left(\frac{1}{1-v^2}\right)} \right|$
0.001
10^{-5}
10^{-7}
10^{-9}
0.2
0.4
0.6
0.8
1.0
v

$\Pi^{\mathrm{v}(2)}$

Taylor Expansion

$$\Pi^{\mathrm{v}(2)}(z) \underset{|z|<1}{=} \sum_{n=1}^{\mathbf{30}} C(n)\, z^n$$

$N^* = \mathbf{30}$ and $z \doteq \dfrac{q^2}{4m^2}$

Threshold expansion

$$\Pi^{\mathrm{v}(2)}(z) \underset{z\to 1}{\sim} \frac{A(-\frac{1}{2},0)}{\sqrt{1-z}} + \left\{A(0,2)\ \log^2(1-z) + A(0,1)\ \log(1-z) + K^{(2)}\right\}$$
$$+ \left\{A(\tfrac{1}{2},1)\ \log(1-z) + A(\tfrac{1}{2},0)\right\}\sqrt{1-z} + \dots$$

$K^{(2)}$ is unknown.

OPE expansion

$$\Pi^{v(2)}(z) \underset{z\to-\infty}{\sim} \left\{B(0,2)\ \log^2(-4z) + B(0,1)\ \log(-4z) + B(0,0)\right\}$$
$$+\left\{B(-1,2)\ \log^2(-4z) + B(-1,1)\ \log(-4z) + B(-1,0)\right\}\frac{1}{z}$$
$$+\left\{B(-2,3)\ \log^3(-4z) + B(-2,2)\ \log^2(-4z)\right.$$
$$\left. + B(-2,1)\ \log(-4z) + B(-2,0)\right\}\frac{1}{z^2} + \ldots$$

From the Threshold and the OPE expansions, one can obtain easily

$$\Omega^{AS}(n) = \alpha_{0,0} + \left\{\alpha_{1,0} + \alpha_{1,1}\ \log n\right\}\frac{1}{n} + \alpha_{2,0}\frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\log^{\ell_1} n\right)$$
$$+ (-1)^n \left[\left\{\beta_{1,0} + \beta_{1,1}\log n\right\}\frac{1}{n} + \left\{\beta_{3,0} + \beta_{3,1}\log n\right\}\frac{1}{n^3}\right.$$
$$\left. + \left\{\beta_{5,0} + \beta_{5,1}\log n + \beta_{5,2}\log^2 n\right\}\frac{1}{n^5} + \mathcal{O}\left(\frac{1}{n^7}\log^{\ell_2} n\right)\right]$$

where the α's and the β's are known analytically by identification

$$\begin{cases} \alpha_{0,0} = 2\ A(-\frac{1}{2},0) \simeq 3.44514 \\ \alpha_{1,0} \simeq -0.492936 \\ \alpha_{1,1} = 2.25 \\ \alpha_{2,0} \simeq 3.05433 \end{cases}$$

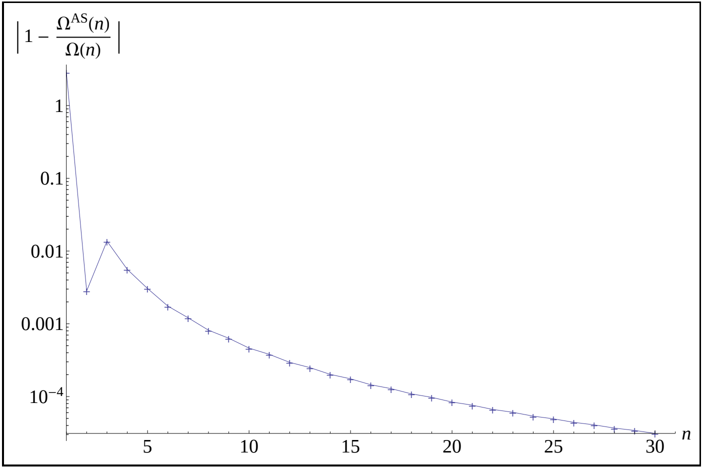
and

$$\begin{cases} \beta_{1,0} \simeq 0.33723 \\ \beta_{1,1} \simeq 0.211083 \\ \beta_{3,0} \simeq 0.183422 \end{cases} \qquad \begin{cases} \beta_{3,1} \simeq -0.620598 \\ \beta_{5,0} \simeq -1.89016 \\ \beta_{5,2} \simeq 1.38684 \end{cases}$$

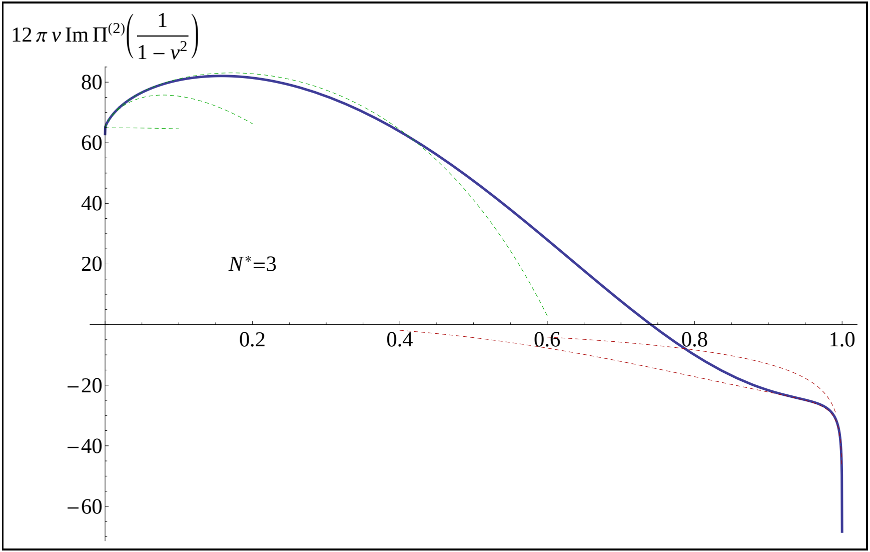
and we estimate the error,

$$[\text{Coef. } \mathcal{E}(N^*,\omega)]_{n>N^*(=30)} \cong \left\{\begin{array}{c} +1 \\ -0 \end{array}\right\} \frac{\log^{1.5} n}{n^3} \pm (-1)^n\, \mathcal{O}\left(\frac{\log^{\ell_2} n}{n^7}\right)$$

In the case of Π^2 we know $N^* = 30$ terms in the Taylor expansion.



Remembering that $\Pi^{v(2)}(z) = \widehat{\Pi}^{v(2)}\left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$, one can reconstruct the imaginary part, (into velocity $\sqrt{1-1/z}$)



From this analytic expression of $\Pi^{v(2)}$, one can extract the value of the constant $K^{(2)}$

$$K^{(2)} = -\frac{\alpha_{0,0}}{2} + \left(\frac{\pi^2}{12} + \frac{\gamma_E^2}{2} + \gamma_1\right)\alpha_{1,1} + \frac{\pi^2}{6}\alpha_{2,0} - \zeta'(2)\,\alpha_{2,1} - \beta_{1,0}\log 2$$
$$+\left(-\frac{\log^2 2}{2} + \gamma_E \log 2\right)\beta_{1,1} - \frac{3\zeta(3)}{4}\,\beta_{3,0} + \left(\frac{\zeta(3)\log 2}{4} + \frac{3\zeta'(3)}{4}\right)\beta_{3,1}$$
$$-\frac{15}{16}\zeta(5)\,\beta_{5,0} + \left(\frac{\zeta(5)\log 2}{16} + \frac{15\zeta'(5)}{16}\right)\beta_{5,1}$$
$$+\left(\frac{\zeta(5)\log^2 2}{16} - \frac{\zeta'(5)\log 2}{8} - \frac{15\zeta''(5)}{16}\right)\beta_{5,2} + \sum_{n=1}^{N^*}\left[\Omega(n) - \Omega^{AS}(n)\right] + \mathcal{E}(N^*,1)$$

$$K^{(2)} = 3.783\ ^{+0.004}_{-0.000}$$

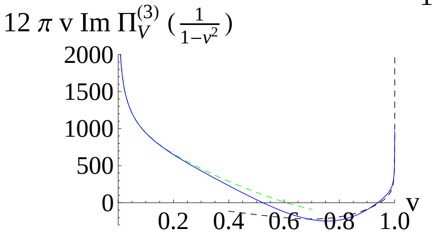
A. H. Hoang et al., Nucl. Phys. B 813, 349 (2009).	3.81 ± 0.02
P. Masjuan and S. Peris, Phys. Lett. B 686, 307 (2010)	3.71 ± 0.03

All channels at order α^3

D.G., P. Masjuan and S. Peris, hep-ph 1104.3425, Phys. Rev. D 85, 054008 (2012)

It is easy to apply the method to all the channels at order α^3, and obtaining the following α's and β's for each:

Ω^{χ}_{AS}	v		a		p		s	
	$n_l=3$	$n_l=4$	$n_l=3$	$n_l=4$	$n_l=3$	$n_l=4$	$n_l=3$	$n_l=4$
$\alpha^{\chi}_{-1,0}$	10.5456	10.5456	0	0	10.5456	10.5456	0	0
$\alpha^{\chi}_{0,0}$	-11.0769	-12.6382	0	0	-6.4835	-8.0448	0	0
$\alpha^{\chi}_{0,1}$	31.0063	28.7095	0	0	31.0063	28.7095	0	0
$\alpha^{\chi}_{1,0}$	36.3318	33.0585	1.4622	1.4622	40.6575	36.7189	2.1932	2.1932
$\alpha^{\chi}_{1,1}$	37.1514	33.8404	0	0	51.8488	48.3155	0	0
$\alpha^{\chi}_{1,2}$	10.1250	8.6805	0	0	10.1250	8.6805	0	0
$\beta^{\chi}_{1,0}$	-0.1819	-0.0555	-0.1819	-0.0555	9.9493	6.9861	9.9493	6.9861
$\beta^{\chi}_{1,1}$	-2.4852	-2.1312	-2.4852	-2.1312	-43.1187	-39.6735	-43.1186	-39.6735
$\beta^{\chi}_{1,2}$	-0.8795	-0.7444	-0.8795	-0.7444	1.6381	1.6688	1.6381	1.6688
$\beta^{\chi}_{1,3}$	0	0	0	0	5.1027	4.6298	5.1027	4.6298
$\beta^{\chi}_{3,0}$	-10.4385	-9.7282	26.2458	22.9826	3.1298	1.1687	93.7790	83.0590
$\beta^{\chi}_{3,1}$	-4.7750	-4.2501	-19.8617	-18.4878	-53.4944	-50.0465	-137.2835	-129.2810
$\beta^{\chi}_{3,2}$	3.8270	3.4724	-6.8349	-6.1103	0.8960	1.1335	-24.9630	-22.4605
$\beta^{\chi}_{3,3}$	0	0	2.5513	2.3149	5.0337	4.6960	15.1011	14.0879
$\beta^{\chi}_{5,0}$	-70.9277	-63.8573	100.2171	89.1103	-115.8498	-108.3750	440.1394	399.7520
$\beta^{\chi}_{5,1}$	56.3093	53.6862	-72.4918	-68.9185	62.1988	60.2675	-512.9781	-487.4843
$\beta^{\chi}_{5,2}$	20.9951	19.0619	-29.3263	-26.2676	38.4395	35.8466	-129.9058	-118.3556
$\beta^{\chi}_{5,3}$	-7.5506	-7.0439	10.1019	9.3589	-9.9903	-9.4668	60.1732	56.5763


$12\,\pi\,v\,\mathrm{Im}\,\Pi_V^{(3)}\left(\frac{1}{1-v^2}\right)$
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500
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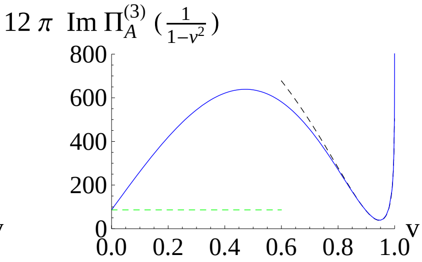
v


$12\,\pi\;\mathrm{Im}\,\Pi_A^{(3)}\left(\frac{1}{1-v^2}\right)$
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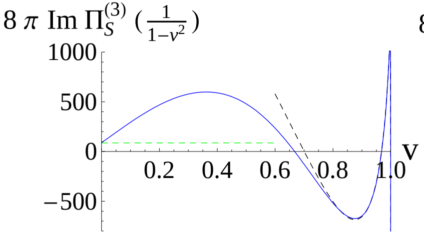
$8\,\pi\;\mathrm{Im}\,\Pi_S^{(3)}\left(\frac{1}{1-v^2}\right)$
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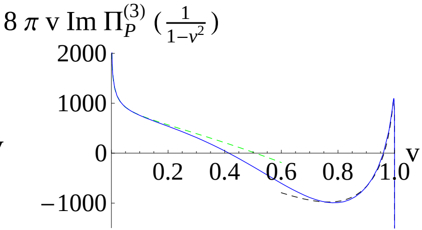
1.0
v


$8\,\pi\,v\,\mathrm{Im}\,\Pi_P^{(3)}\left(\frac{1}{1-v^2}\right)$
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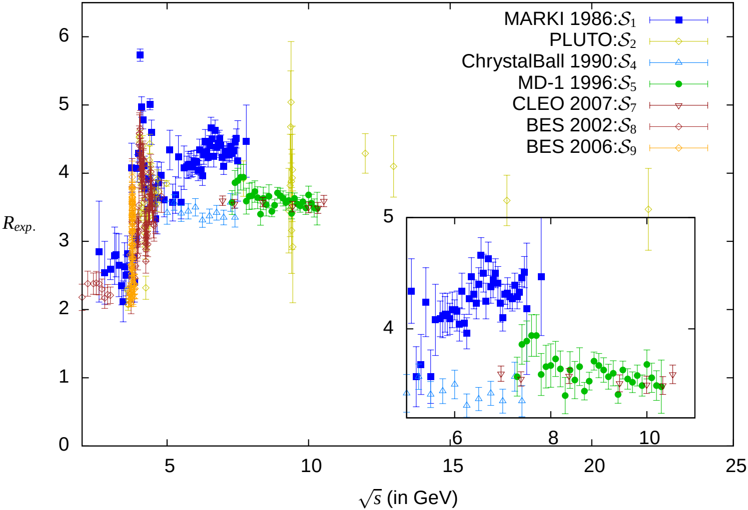
Part 2

Application to the determination of the c quark mass

State of the art

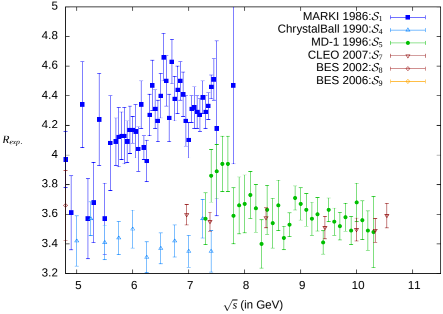


How to extract m_c from the data ?

Fitting procedure at order α_s^2

With the theoretical expression at order α_s^2

$$R_{\text{th.}}(s) = \left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] N_c \left[1 + \frac{\alpha_s(s)}{\pi} + 1.525\left(\frac{\alpha_s(s)}{\pi}\right)^2\right] + 12\pi\left(\frac{2}{3}\right)^2 \text{Im}\left[\Pi^{(0)} + \frac{4}{3}\frac{\alpha_s(s)}{\pi}\Pi^{(1)} + \left(\frac{\alpha_s(s)}{\pi}\right)^2 \Pi^{(2)}\right]$$



We use a χ^2 regression for

$$\chi^2(m_c) \doteq \sum_j \left(\frac{R_{\text{exp.}}(s_j) - R_{\text{th.}}(s_j)}{\sigma_{\text{exp.}}(s_j)}\right)^2 + \left(\frac{R_{\text{exp.}}(s_j) - R_{\text{th.}}(s_j)}{\sigma_{\text{th.}}(s_j)}\right)^2$$

and

$$\sigma^2_{\text{th.}}(s) = \frac{256\pi^2}{9}\left|\text{Im}\left[\left(\frac{\alpha_s(s)}{\pi}\right)^2 \mathcal{E}^{(2)}(N_2^*, \omega)\right]\right|^2$$

VERY Preliminary results

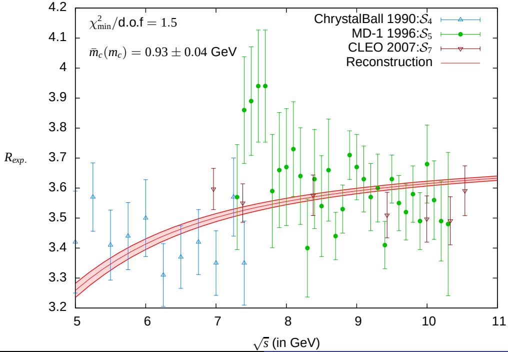
One finds for considering all data (except MARK I) for the range 5 to 11 GeV with $\chi^2_{\min.}$/d.o.f. $= 1.5$ and $\alpha_s(m_Z) = 0.113$

$$\bar{m}_c(\bar{m}_c)\Big|_{\text{pert.}} = 0.93\ (\pm 0.04)_{\text{stat.+ syst.}}$$

To compare to

$$\bar{m}_c(\bar{m}_c)\Big|_{\text{pert.}} = 0.95\ (\pm 0.02)_{\text{pert. + stat. syst.}}$$

in B. Dehnadi, A. H. Hoang, V. Mateu and S. M. Zebarjad, arXiv:1102.2264



Conclusion

- We proved that it is possible to reconstruct in a systematic way a full function with a located cut from its Taylor expansion around 0, its threshold and OPE expansions.
- We show that it is possible to control the systematic error.
- Regarding the application to the heavy-quarks correlators, one can apply it to control in a systematic way the evaluation of the mass of the c quark. But we need to understand better the way to implement the contributions from the non-perturbative contributions and the influence on the selected window
- This method is general enough to be applied to other situations (with similar analytic structure):
  - *Symbols* $\mathcal{S}$ and integrals involved in $\mathcal{N} = 4$ SYM theories.
  - Applicable since one has partial information and need for local constraints.