Lectures on gravitational collapse and black holes

Adrián del Río.
Institute for Gravitation and the Cosmos & Physics Department,
The Pennsylvania State University, University Park, PA 16802 U.S.A

Abstract

These notes comprise a basic course on gravitational collapse and black holes, which is based on a series of blackboard lectures given to Ph.D students at the Theoretical Physics Department of the University of Valencia during October 2020. The structure and contents of these notes have been greatly influenced by [1, 2], supplemented with some material from [3, 4, 5]. The purpose of these notes is to provide a more accessible and direct bibliographic source for Master and Ph.D students, trying to give more detailed and basic explanations and to fill gaps when necessary. These notes are accompanied by a set of basic exercises, that can also be found in indico together with these notes, and whose aim is to provide the students an opportunity to get more familiarized with all these topics with practical calculations, while at the same to supplement some points not covered in the course by lack of time. While the intention of these notes is to be self-consistent and elementary, a first course on General Relativity is highly recommended.

The Ph.D course mentioned above was divided in 10 lectures of 2.5 hours each. Lecture 1 covered sections (1.1)-(1.2) below; Lecture 2 covered sections (1.3)-(1.4); Lecture 3 section (1.5); Lecture 4 sections (2.1)-(2.3); Lecture 5 sections (2.4)-(2.5); Lecture 6 section (2.6); Lecture 7 section (2.7); Lecture 8 sections (3.1)-(3.4); Lecture 9 sections (3.5)-(3.6); and finally Lecture 10 sections (3.7)-(3.8).

Contents

1 Stars and gravitational collapse

The central topic of this course is about black holes (BHs). What is a black hole? Different researchers within the gravity community can conceive this notion in different ways [6]. In these lectures we will widely regard a black hole as any exact solution g_{ab} of the Einstein's field equations that possess a "horizon", to be specified below. As we will see in Sections II and III, these mathematical solutions exist and, under certain assumptions, they are unique. Are black holes physically relevant, or just artificial predictions of General Relativity? If they are physically relevant, how are black holes supposed to originate in our Universe? In the first topic of this course, we will learn that astrophysically relevant BHs result as the endpoint of gravitational collapse of sufficiently massive stars.

1.1 White dwarfs and the Chandrasekhar mass limit

An ordinary star can be thought of as an approximately spherically-symmetric, self-gravitating distribution of matter, mainly composed of light elements such as hydrogen atoms, that is supported by internal thermal pressure, and emitting in turn electromagnetic radiation.

During most of its life, the star remains in an equilibrium state: both thermal pressure and energy lost by radiation are supplied by nuclear reactions of light elements. Eventually, however, nuclear fuel runs out (lighter elements in the star merge to produce heavier elements, until an iron nucleus forms, which do not produce further nuclear reactions). What happens then? What are the possible end states of the star?

Let us consider a star of mass M and radius R. To analyze the evolution of this star, we need to study the competition between its gravitational energy E_g and its internal (thermodynamical) energy E_{int}:

$$E_{\text{Total}} = E_g + E_{\text{int}} \,. \tag{1}$$

To carry out this study let us assume that the star can be modeled by an ideal gas, with pressure P, temperature T and particle density n. Thus, we know from standard courses in thermodynamics that the corresponding equation of state is then $P(T) = nk_BT$, where k_B is Boltzmann constant. Similarly, the energy of the gas reads $E_{\text{int}} \sim cNk_BT$ with $N = nR^3$ the total number of particles composing the star, and c a numerical factor (equal to 3/2 or to 5/2 for monoatomic and diatomic gases, respectively). On the other hand, let us assume that we can describe the gravitational field of this star with Newtonian gravity. In this case we can roughly estimate $E_g \sim -G\frac{M^2}{R}$ with G denoting Newton's constant.

When nuclear fusion ends, the star loses energy by radiation (heat transfer) and consequently $E_{\text{int}} \to 0$. From above, this implies, in turn, that $T \to 0$ and $P(T) \to 0$. If the internal pressure vanishes, then there is no longer anything that can balance the gravitational force, and the star starts contracting by its own gravitational attraction. What happens then? Interestingly, when the temperature T becomes sufficiently small, a non-thermal pressure emerges by quantum fluctuations due to fermion degeneracy.

Reminder. A free fermion gas becomes degenerated when all particles in the gas occupy their lowest energy states, which is not $E = 0$ for all the particles because of the Pauli Exclusion Principle. Because $E \neq 0$, we have a non-trivial pressure $P = -\frac{\partial E}{\partial V} \neq 0$, as predicted by Statistical Physics. This is a pressure of quantum origin. The fermion gas becomes degenerated when its temperature T decreases below the critical (Fermi temperature) value $T_F \sim \frac{n^{2/3}}{m}$, where n is the particle density of the gas, m is the mass of the particles. Notice, in particular, that since $m_e << m_p$, then $T_e >> T_p$, where the subscripts e and p refer to electrons and protons, respectively. In other words, a gas of electrons becomes degenerated first.

A natural question: can the electron degeneracy pressure support a star from collapsing?

Let us further assume that the mass of the star M is not particularly high, say, of the order of the solar mass $M_\odot$. In this case, the electrons of the star have low energy and, consequently, we can estimate the average energy of one electron by the non-relativistic formula $\langle E_e \rangle \sim \frac{\langle p_e \rangle^2}{2m_e}$. If the fermion gas is degenerated, then there must be, roughly speaking, one electron per Compton volume: $n_e = \frac{N}{V} \sim \frac{1}{\lambda_e^3}$, so $\lambda_e \sim n_e^{-1/3}$. Using the De Broglie formula we can obtain

$$p_e = \frac{2\pi\hbar}{\lambda_e} \sim \hbar\, n_e^{1/3} \longrightarrow E_{\text{int}} = n_e R^3 \langle E_e \rangle \sim \frac{\hbar^2 n_e^{5/3}}{m_e} R^3 \,, \tag{2}$$

where $\hbar$ is Planck's constant. Given $m_e << m_p$, we can approximate the total mass of the star as $M = R^3(n_e m_e + n_p m_p) \simeq n_p m_p R^3$. If we consider the simplifying assumption that the star is basically composed of hydrogen atoms, we can further claim that $M \simeq n_e m_p R^3$. Therefore, the Fermi energy reads $E_{\text{int}} \sim \frac{\hbar^2 M^{5/3}}{R^2 m_e m_p^{5/3}} \equiv \frac{\beta}{R^2}$ for some constant β. The total energy acquires then the following radial dependence:

$$E_{\text{Total}}(R) \sim -\frac{\alpha}{R} + \frac{\beta}{R^2} \,, \tag{3}$$

for some constant α. The radial profile of this function can be seen in Fig.1. As we can see, there exists an equilibrium point corresponding to minimum energy. More precisely:

$$\frac{dE_{\text{Total}}(R)}{R} = 0 \longrightarrow R_{\min} = \frac{2\beta}{\alpha} = \frac{\hbar^2 M^{-1/3}}{G m_e m_p^{5/3}} \,. \tag{4}$$

In other words, the presence of this electron degeneracy pressure can indeed prevent the gravitational collapse of the star. In particular, the star becomes what is known as a White Dwarf. This is, a small ($R_{\min}(M_\odot) \sim 6000$ km $\sim R_{\text{Earth}}$), cold ($T \sim 0$) and dead (no nuclear reactions) star, supported against its own gravitational attraction by electron degeneracy pressure. The particle density of the resulting star is considerably large:

$$n_e \sim \frac{M}{m_p R_{\min}^3} = \left(\frac{m_e G}{\hbar^2} (M m_p^2)^{2/3} \right)^3 \sim 10^{36} m^{-3} \,. \tag{5}$$

For comparison, notice that the particle density of the Sun is about $n_\odot \sim 10^{30} m^{-3}$.

Can the electron degeneracy pressure always prevent the gravitational collapse of a star?

Let us recall that the non-relativistic limit assumed above requires $\langle p_e \rangle << m_e c$. This implies

$$\frac{\langle p_e \rangle}{m_e} \sim \frac{\hbar n_e^{1/3}}{m_e} << c \longrightarrow n_e << \left(\frac{m_e c}{\hbar} \right)^3 \,. \tag{6}$$

Using (5) we obtain the inequality

$$M << \frac{1}{m_p^2} \left(\frac{\hbar c}{G} \right)^{3/2} \,. \tag{7}$$

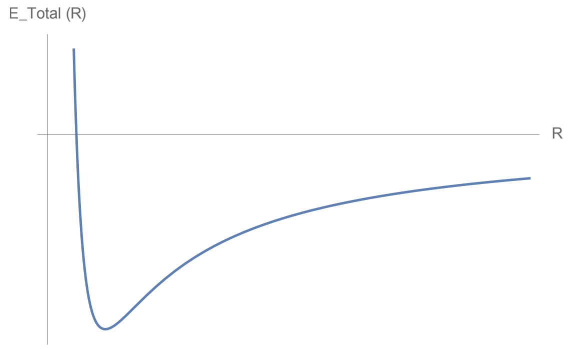

Figure 1: Radial profile of the total energy density of a spherically-symmetric, solar-mass star composed only of hydrogen atoms.

In other words, our results above only hold for sufficiently low values of the star's mass. For higher values of M, the electrons in the star become relativistic, and we should use instead

$$\langle E_e \rangle \sim \langle p_e \rangle c \sim \hbar\, c\, n_e^{1/3} \, . \tag{8}$$

The internal energy can thus be written as

$$E_{\rm int} \sim n_e R^3 \langle E_e \rangle \sim \hbar c R^3 n_e^{4/3} \sim \hbar c \left(\frac{M}{mp} \right)^{4/3} \frac{1}{R} \equiv \frac{\gamma}{R} \, , \tag{9}$$

and the total energy reads $E_{\rm Total} = -\frac{\alpha}{R} + \frac{\gamma}{R}$. The extremization of this function yields

$$\frac{dE_{\rm Total}}{dR} = \frac{\alpha - \gamma}{R^2} = 0 \quad \text{iff} \quad \alpha = \gamma \quad \text{iff} \quad M \sim \frac{1}{m_p^2} \left(\frac{\hbar c}{G} \right)^{3/2} =: M_c \, . \tag{10}$$

This result allows us to rewrite

$$E_{\rm Total} = \frac{M^{4/3} G}{R} \left[M_c^{2/3} - M^{2/3} \right] \, . \tag{11}$$

We have two different situations.

(a) If $M < M_c$, then $E_{\rm Total}(R) > 0$ for all $R > 0$, so $E_{\rm Total}(R_*) = 0$ is a minimum of energy: stability. In particular, it is a stable point and the system will evolve towards $E_{\rm Total} \to 0$. This evolution can only occur at the expense of increasing R, which in turn implies $n_e \to 0$, until the electrones become non-relativistic again, and we can apply the previous study.

(b) If $M > M_c$, then $E_{\rm Total}(R) < 0$ for all $R > 0$, so $E_{\rm Total}(R_*) = 0$ is a maximum of energy: instability. In this case $E(R) \to -\infty$, or equivalently $R \to 0$, is energetically favorable: gravitational collapse occurs.

Therefore, if the mass of the star is sufficiently high, electron degeneracy pressure is not enough to prevent the collapse and a white dwarf does not form. The critical mass for the formation of a white dwarf is

$$M_c \sim \frac{M_{\rm Pl}^3}{m_p^2} \sim 1.85 M_\odot \, , \tag{12}$$

where $M_{\rm Pl} = \sqrt{\frac{\hbar c}{G}}$ is the Planck mass. A more rigorous calculation gives $M_c \simeq 1.46 M_\odot$ (see exercise 2 of Exercise List). This is known as the Chandrasekhar mass limit.

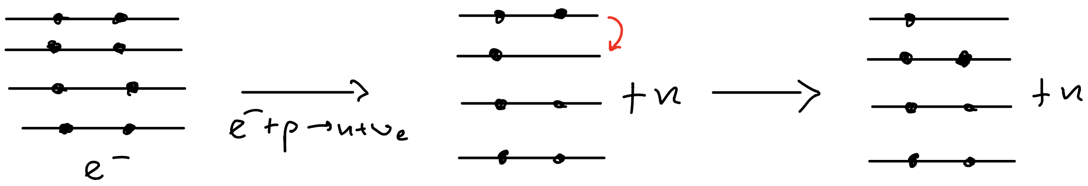

Figure 2: Process of neutron star formation: electrons and protons combine to produce neutrons via inverse beta decay.

1.2 Neutron stars: need of General Relativity

In the previous section we learnt that electron degeneracy pressure can manage to support the gravitational pressure of a star if the mass does not overcome the Chandrasekhar mass limit $M_c \simeq 1.46 M_\odot$. What happens when the mass is higher than this critical value? Is gravitational collapse ineludible?

Roughly speaking, a star is mostly made of electrons and protons (hydrogen atoms). When nuclear fuels runts out and the temperature decreases significantly, $T \sim 0$, the star is forced to collapse further if $M > M_c$. At low temperature, the gas of electrons is degenerated, and therefore the electron energy is of the order of the Fermi energy: $E_e \simeq E_F$. Since

$$E_F \sim \frac{n_e^{2/3}}{k_B m_e} = \frac{N_e^{2/3}}{k_B m_e R^2}\,, \tag{13}$$

the Fermi energy increases as the star collapses, $R \to 0$. Initially $E_e \simeq m_e c^2$ (non-relativistic energies). As the star collapses, $E_e \simeq E_F$ will progressively increase. When the electron energy reaches the value $E_e \simeq 3 m_e c^2 \simeq (m_n - m_p) c^2$, β inverse decay takes place:

$$e^- + p \to n + \nu_e\,, \tag{14}$$

where e^-, p, n and ν_e denote, respectively: electron, proton, neutron, and electronic neutrino. However, the reverse reaction ($n + \bar{\nu}_e \to e^- + p$, with $\bar{\nu}_e$ an antineutrino) cannot occur because the neutrinos ν_e escape from the star. Similarly, β decay ($n \to e^- + p + \bar{\nu}_e$) cannot occur either because, after one electron dissappears, the Fermi levels refill much quicker than the typical β decay rate ($\sim$ 10 minutes). See Fig2 for an illustration of this process.

In conclusion, if a White Dwarf does not form because $M > M_c$, then the star loses most of its electrons and protons during the process of gravitational collapse. As a result, the star becomes primarily made of neutrons. Now, since neutrons are also fermions:

Can neutron degeneracy pressure support a star from collapsing, and form a new star?

If we consider the same assumptions as in the previous section: (i) Newtonian gravity and (ii) ideal gas, then the maximum mass that neutron degeneracy pressure could support would be given by $0.7 M_\odot$. This is lower than the Chandrasekhar mass limit for a white dwarf, and consequently a neutron star could never be formed. However, these two assumptions are naive:

(i) The collapse has proceeded further than in the white dwarf, as a result of which the comopactness of the star is considerably bigger. As a consequence, the gravitational field of the resulting neutron star is too strong: we need to employ General Relativity rather than Newtonian gravity.

(ii) Densities are so high that neutrons are subject to short-range strong interactions (typical nuclear matter densities of neutron stars are $\sim 10^{17}$ kg/m^3). In other words, they are not free anymore and the ideal gas approximation fails to accurately describe the physics of the matter content.

Therefore, to figure out if a star can form by the degeneracy pressure of neutrons, we need to follow a more rigorous approach. Let us model the spacetime of a star in equilibrium by a spherically-symmetric, static metric. We can always choose a suitable coordinate system such that this metric reads (see exercise 3 of Exercise List)

$$ds^2 \equiv g_{ab}\, dx^a \otimes dx^b = -e^{\nu(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,, \tag{15}$$

where $d\Omega^2$ is the usual line element on the 2-sphere, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Let us further describe the matter content of the star by a perfect fluid. Perfect fluids are idealized fluids with no shear stresses, viscosity (resistance under deformations), or heat conduction. They are fully described by an energy density ρ, pressure p, and field's velocity u^a, the latter being tangent to the congruence of curves describing the motion of the fluid in spacetime. For a static perfect fluid, the corresponding stress-energy tensor takes the simple form

$$T_{ab} = (\rho(r) + p(r))u_a u_b + p(r) g_{ab} \,. \tag{16}$$

Using these two expressions, Einstein's equations $G_{ab}[g] = 8\pi G T_{ab}$ (or, equivalently, the contracted Bianchi identities $\nabla_a T^{ab} = 0$) leads to the following ordinary differential equations (ODEs):

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \,, \tag{17}$$

$$\frac{dP}{dr} = -G(\rho(r) + p(r))\frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))} \,. \tag{18}$$

See exercise 4 of Exercise List for more details. These hydrostatic equations in General Relativity are known as TOV (Tolman-Oppenheimer-Volkoff) equations. They determine the equilibrium configurations of non-rotating stars. Mathematically, they constitute a boundary value problem: to get a particular solution for an equilibrium star, we need to solve the ODEs above with boundary data $\{p(0) = p_c, m(0) = 0\}$ for some positive value p_c, the latter representing the central pressure of the star. Different values of the central pressure lead to different stars. Once a particular solution is obtained, the radius of the star is defined as that real number R that accomplishes $p(R) = 0$, while the mass of the star is defined by $M := m(R)$. Astrophysical observations indicate that the typical radius of a Neutron star is of the order of 10km (tiny!).

The system above contains only two equations for three variables $(m(r), \rho(r), p(r))$. Therefore, we need one extra equation to really solve it: an equation of state of the form $P = P(\rho, s, X)$, where s is the entropy per particle, and X is the fraction of particles in the star. For self-gravitating neutron stars we have $T \sim 0$, and Nerst heat Theorem ensures that (i) $s = 0$ everywhere, as well as (ii) $X =$ neutrons remains constant. Therefore, the problem reduces to finding a suitable $P = P(\rho)$ that accurately describes the matter content in the interior of the star[1].

[1]This is not the case for ordinary stars like the Sun, since s is not uniform and has to be determined from the equations of radiative equilibrium.

Unfortunately, this is a difficult question. As commented above, the ideal gas equation does not work anymore. Strong interactions at such high densities are poorly understood and $P = P(\rho)$ is largely unknown. As a result, the maximum allowed mass for neutron stars is somewhat uncertain, of the order of $2 - 3M_{\odot}$. Gravitational-wave observations are expected to unravel the specific equation of state $P = P(\rho)$, which will teach us about how matter behaves for very high and low densities and temperatures, respectively.

Even though we ignore the form of the actual equation of state for neutron stars, an illustrative analytical example can give us interesting insights on how self-gravitating matter behaves. A simple model is given by the incompressible fluid, for which $\rho = \rho_c =$ constant. The solution of TOV equations can be worked out in closed analytical form, and gives (See exercise 5 of Exercise List):

$$p(r) = \rho_0 c^2 \frac{\sqrt{1 - Rr^2/R_0^3} - \sqrt{1 - R/R_0}}{3\sqrt{1 - R/R_0} - \sqrt{1 - Rr^2/R_0^3}}\,, \tag{19}$$

where $R = 2MG/c^2$, and R_0 is the resulting radius of the star, connected with the central pressure by $p_c = \frac{1-\sqrt{1-R/R_0}}{3\sqrt{1-R/R_0}-1}$. The total mass M of the star cannot exceed a certain critical value M_c if the body is to remain in hydrostatic equilibrium. The critical value M_c is determined when $p_c = \infty$, which yields

$$M_c \sim 5M_{\odot}\,, \tag{20}$$

for nuclear matter density. Notice that this simple model, though highly unrealistic, does provide a remarkably good estimate of the mass upper bound for actual neutron stars. More physically realistic models of stars can also be considered by using Polytrope equations of state (see section 5.6.5. of [5]).

The existence of upper mass limits is common in General Relativity and is due to the non-linear nature of this theory. For spherically symmetric spacetimes, in particular, Buchdahl's Theorem states that $M_c = \frac{4R}{9G}$ is an upper bound for any star (see exercise 6 of Exercise List). The specific maximum mass allowed for neutron stars provides a decisive method for distinguishing neutron stars from actual black holes (to be specified in later sections) using astrophysical observations. It is therefore of high physical relevance.

To conclude, if $M_{\odot}$ is sufficiently small, the original star can reach a final equilibrium state as a white dwarf or neutron star, supported by degeneracy pressure. However, if $M_{\odot}$ is above cold matter upper mass limits, equilibrium can never be reached again and complete gravitational collapse of the matter of the star will occur.

1.3 Brief review on test particles and geodesics

In General Relativity, a spacetime is described by a 4-dimensional differentiable manifold M endowed with a Lorentzian metric g_{ab} (with $\det g_{ab} < 0$; signature convention $-,+,+,+$) that satisfies the Einstein's field equations

$$R_{ab}[g] - \frac{1}{2} g_{ab} R[g] = 8\pi G\, T_{ab}\,. \tag{21}$$

Given some g_{ab}, it is always possible to construct the Levi-Civita connection ∇_a, which is the unique torsion-free ($\nabla_a \nabla_b g = \nabla_b \nabla_a g$ for any function g) and metric-compatible ($\nabla_a g_{bc} = 0$) connection. In a given coordinate system, this connection is characterized by the Christoffel symbols Γ^a_{bc}, which satisfy $\Gamma^a_{bc} = \Gamma^a_{cb}$ (torsion-free property). Then, using this connection one defines the Riemann curvature tensor of this spacetime by the non-conmutative property $[\nabla_a, \nabla_b]v_c = R_{abc}{}^d v_d$, where v_d is any co-vector on M. Taking traces $R_{ab} := R_{adb}{}^d$, $R := g^{ab}R_{ab}$ we obtain the Ricci tensor and scalar curvature, respectively.

In this framework, gravity is the manifestation of a non-trivial spacetime curvature, produced by some energy distribution, as determined by the Einstein's field equations. To probe the spacetime curvature, it is useful to invoke the concept of test particles. A test particle is an idealized body that does not influence the spacetime geometry through which it moves (even though it may have non-vanishing mass).

The trajectory of these test particles are described by smooth curves on the spacetime's manifold M. A curve on a manifold is a smooth map $\gamma : I \subset \mathbb{R} \to M$. Given a coordinate system $\{x^a\}$, we can label each point of the image of the curve, $\gamma(t) \in M$, as

$$x^a(t) \equiv x^a(\gamma(t)) = (x^0(t), x^1(t), x^2(t), x^3(t)) \in \mathbb{R}^4 \,. \tag{22}$$

The 4-velocity of the test particle is determined by the tangent vector to the curve, $u^a(x(t)) \equiv u^a = \frac{dx^a(\gamma(t))}{dt}$. The existence of a Lorentzian metric allows us to classify curves into 3 classes:

$$\text{timelike curve if} \qquad g_{ab}(x(t))u^a(x(t))u^b(x(t)) < 0, \;\; \forall t \,, \tag{23}$$
$$\text{spacelike curve if} \qquad g_{ab}(x(t))u^a(x(t))u^b(x(t)) > 0, \;\; \forall t \,, \tag{24}$$
$$\text{null curve if} \qquad g_{ab}(x(t))u^a(x(t))u^b(x(t)) = 0, \;\; \forall t. \tag{25}$$

It is easy to show that the causal character of a curve on a single point fully determines the causal character at any other point of the curve (see exercise 7 of Exercise List).

Freely-falling (unaccelerated) massive test particles follow timelike geodesics.

A geodesic is a curve in a spacetime that extremizes the length between two points with respect to the metric g_{ab} (roughly speaking, it is the "straightest path that connect two points in a curved geometry"). Let us define the length of timelike curves between two points $A, B \in M$ by

$$\int_{\tau_A}^{\tau_B} d\tau = \int_{\lambda_A}^{\lambda_B} d\lambda \sqrt{-g_{ab}(x(\lambda))u^a(x(\lambda))u^b(x(\lambda))} \,, \tag{26}$$

where λ is an arbitrary parametrizaton of the curve, and $\gamma(\tau_A) = A$, $\gamma(\tau_B) = B$. It is not difficult to show that this notion is invariant under reparametrizations of the curve. We call $\int_{\tau_A}^{\tau_B} d\tau$ the proper time between A and B.

The equation of motion for test particles is obtained via the action principle (extremization). The action for a test particle of mass m moving on a timelike curve γ is simply taken to be proportional to its proper time

$$I[x] = -mc^2 \int_{\tau_A}^{\tau_B} d\tau = -mc^2 \int_{\lambda_A}^{\lambda_B} d\lambda \sqrt{-g_{ab}(x(\lambda))\dot{x}^a(x(\lambda))\dot{x}^b(x(\lambda))} \,, \tag{27}$$

where $\dot{x}^a = \frac{dx^a}{d\lambda}$. The factor mc^2 is included to have dimensions of action. To obtain the equation of motion for the test particle's wordline $x^a(\lambda)$ we extremize this action:

$$\frac{\delta I}{\delta x^a} = 0 \to \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^a} = \frac{\partial \mathcal{L}}{\partial x^a} \,. \tag{28}$$

Working out the second equation above yields

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0\,, \tag{29}$$

where $\Gamma^a_{bc} = \frac{1}{2}g^{ae}(\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc})$ are the Christoffel symbols. This is called the geodesic equation. The result obtained depends on a given coordinate frame x^a. The geodesic equation can actually be rewritten in a more geometric fashion by recalling that the Levi-Civita connection acts on any vector V^a as: $\nabla_a V^b = \partial_a V^b + \Gamma^a_{bc}V^c$. Keeping in mind that $\dot{x}^a\partial_a = \frac{d}{d\lambda}$, then the equation above reads

$$\nabla_u u^a = u^b\nabla_b u^a = 0\,, \tag{30}$$

where $u^a \equiv \dot{x}^a$.

Freely-falling (unaccelerated) massless test particles follow null geodesics.

To unify both $m = 0$ and $m \neq 0$ cases we need an alternative form of the action. Let us consider now

$$I[x,e] = \frac{1}{2}\int_{\lambda_A}^{\lambda_B} d\lambda[e^{-1}(\lambda)\dot{x}^a\dot{x}^b g_{ab} - m^2 e(\lambda)]\,, \tag{31}$$

where $e(\lambda)$ is an auxiliary function. It is easy to show that this action yields the same equations of motion as the previous one, i.e. both are equivalent:

$$\frac{\delta I[x,e]}{\delta e} = 0 \longrightarrow e = \frac{1}{m}\sqrt{-g_{ab}\dot{x}^a\dot{x}^b} = \frac{1}{m}\frac{d\tau}{d\lambda} \longrightarrow ds^2 \equiv g_{ab}\dot{x}^a\dot{x}^b = -m^2e^2 = -d\tau^2 \, ⊲ \tag{32}$$

$$\frac{\delta I[x,e]}{\delta x^a} = 0 \longrightarrow \left(e^{-2}\dot{e}\dot{x}^a g_{ab} - e^{-1}\ddot{x}^a g_{ab} - e^{-1}\dot{x}^a\frac{d}{d\lambda}g_{ab}\right) + \frac{1}{2}e^{-1}\dot{x}^a\dot{x}^c\partial_c g_{ac} = 0 \tag{33}$$

$$\longrightarrow \ddot{x}^c + \Gamma^c_{ab}\dot{x}^a\dot{x}^b = e^{-1}\dot{e}\,\dot{x}^c\,. \tag{34}$$

Combining both results we get

$$\frac{dx^c}{d\tau} = \frac{\partial\lambda}{\partial\tau}\dot{x}^c = \frac{1}{me}\dot{x}^c\,, \tag{35}$$

$$\frac{d^2x^c}{d\tau^2} = \frac{1}{m}\frac{\partial\lambda}{\partial\tau}\left(-\frac{\dot{e}}{e^2}\dot{x} + \frac{\ddot{x}}{e}\right) = \frac{1}{m^2e^2}\left[-\frac{\dot{e}}{e}\dot{x} + \ddot{x}\right]\,, \tag{36}$$

and

$$\frac{d^2x^c}{d\tau^2} + \Gamma^c_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = 0\,, \tag{37}$$

which is the geodesic equation again.

The freedom in the choice of the function $e(\lambda) = \frac{1}{m}\frac{d\tau}{d\lambda}$ is equivalent to the freedom in choosing a parametrization λ for the curve. A curve $x^a(\lambda)$ is called geodesic if $u^a = \dot{x}^a$ satisfies $u^b\nabla_b u^a = f(x)u^a$ for any function $f(x)$. A natural choice of parametrization is, precisely, the one that achieves

$$u^b\nabla_b u^a = 0\,. \tag{38}$$

A geodesic that satisfies this equation is said to be affine parametrized, and the corresponding λ is called an affine parametrization.

For timelike geodesics ($m \neq 0$), $\dot{e} = 0$ leads to $\frac{1}{m}\frac{d\tau}{d\lambda}$ =const, so $\lambda = \frac{\tau}{me} + \lambda_0$. In other words, the affine parameter is essentially the proper time of the curve (modulo an affine transformation).

The advantage of working with the action (31) instad of (27) is that the massless limit $m \to 0$ is now available. The equation of motion for massless test particles is obtained by extremization:

$$\frac{\delta I[x,e]}{\delta e} = 0 \quad \longrightarrow \quad g_{ab}\dot{x}^a\dot{x}^b = 0 \longrightarrow ds^2 = 0\,, \quad \text{(null curve)} \tag{39}$$

$$\frac{\delta I[x,e]}{\delta x^a} = 0 \quad \longrightarrow \quad u^b\nabla_b u^a = \frac{\dot{e}}{e}u^a\,. \quad \text{(geodesic)} \tag{40}$$

Again, we have a freedom to choose $e(\lambda)$. The choice $e(\lambda) = e_0$ constant is called affine parametrization.

Summary. The equation of motion for freely-falling (or unaccelerated) test particles is given by the geodesic equation

$$u^b\nabla_b u^a = 0, \quad ds^2 = \sigma d\tau^2\,, \tag{41}$$

where $\sigma = 0$ and $\sigma = -1$ for $m = 0$ (null curves) and $m \neq 0$ (timelike curves), respectively.

1.4 Brief review on isometries and Killing vectors

Let us evaluate the effect of a general coordinate transformation (diffeomorphism) on the action (31). Under an infinitesimal coordinate transformation $x^a \to x^a - \alpha k^a$, $e \to e$, for some $\alpha \in \mathbb{R}$, the action transforms as

$$\begin{aligned} I[x,e] \quad \to \quad & \frac{1}{2}\int_{\lambda_A}^{\lambda_B} d\lambda[e^{-1}(\lambda)(\dot{x}^a - \alpha\dot{k}^a)(\dot{x}^b - \alpha\dot{k}^b)g_{ab}(x-\alpha k) - m^2 e(\lambda)] \\ & = \frac{1}{2}\int_{\lambda_A}^{\lambda_B} d\lambda[e^{-1}(\lambda)\dot{x}^a\dot{x}^b g_{ab} - m^2 e(\lambda)] \\ & +\alpha\frac{1}{2}\int_{\lambda_A}^{\lambda_B} d\lambda e^{-1}(\lambda)\left[-2\dot{x}^a\dot{x}^c\partial_c k^b g_{ab} - \dot{x}^a\dot{x}^b k^c\partial_c g_{ab}(x)\right] + O(\alpha^2) \\ & = I[x,e] - \frac{\alpha}{2}\int_{\lambda_A}^{\lambda_B} d\lambda e^{-1}(\lambda)\dot{x}^a\dot{x}^b\mathcal{L}_k g_{ab}(x) + O(\alpha^2)\,, \end{aligned} \tag{42}$$

where in the second equality we expanded $g_{ab}(x - \alpha k) = g_{ab}(x) - \alpha k^c\partial_c g_{ab}(x) + O(\alpha^2)$, and in the third equality we introduced the Lie derivative,

$$\mathcal{L}_k g_{ab} = k^c\partial_c g_{ab} + g_{bc}\frac{\partial k^c}{x^a} + g_{ac}\frac{\partial k^c}{x^b} = 2\nabla_{(a}k_{c)}\,. \tag{43}$$

As we can see, the action $I[x,e]$ for geodesics remains invariant if and only if the coordinate transformation is an *isometry*, i.e. $\mathcal{L}_k g_{ab} = 0$. This is called the Killing equation, and any vector k^a that satisfies $\mathcal{L}_k g_{ab} = 0$ is called a Killing Vector Field (KVF). From a geometric viewpoint, a KVF is the generator of an isometry: its local flow preserves the spacetime metric, or equivalently g_{ab} remains invariant along the integral curves of k^a.

By Noether's theorem, if the action is invariant under a 1-parameter continuous group of transformations, then there exists a conserved charge associated to geodesics. This charge is calculated as

$$Q_k = \frac{\partial L[x]}{\partial \dot{x}^a}\delta\dot{x}^a - F\,, \tag{44}$$

where $\frac{dF}{dt} = \delta L$. Since $\delta L = 0$, F is just a constant, which can be fixed to zero without loss of generality. Thus,

$$Q_k = e^{-1}\dot{x}^b g_{ab}(-\alpha k^b) = m\frac{dx^b}{d\tau}g_{ab}(-\alpha k^a)\,, \tag{45}$$

so $Q_k = m\frac{dx^a}{d\tau}k_a \equiv p^a k_a$ is a constant of motion. It is easy to check that, indeed, Q_k is conserved along geodesics:

$$\begin{aligned}\frac{d}{d\lambda}Q_k &= \frac{d\tau}{d\lambda}\frac{d}{d\tau}Q_k = \frac{d\tau}{d\lambda}x'^a\nabla_a Q_k = m\frac{d\tau}{d\lambda}x'^a\nabla_a(x'^b p_b)\\ &= m\frac{d\tau}{d\lambda}\left[(x'^a\nabla_a x'^b)k_b + x'^a x'^b\nabla_{(a}k_{b)}\right] = 0\,, \end{aligned}\tag{46}$$

where in the last equality we made use of the Killing equation as well as the geodesic equation.

Reminder. Given a coordiante system $\{x^a\}$ and the associated canonical basis of vectors $\{\frac{\partial}{\partial x^a}\}$, any vector field can be written as $k = k^a\frac{\partial}{\partial x^a}$, where k^a are the components of the vector field k in this basis.

For any vector field k, one can always choose a set of coordinates $\{y^a\}$ such that $k = \frac{\partial}{\partial y^0}$. Then, the Lie derivative in this coordinate system reads $\mathcal{L}_k g_{ab} = k^c\partial_c g_{ab} = \frac{\partial g_{ab}}{\partial y^0}$. Thus, k is a KVF if and only if the metric, written in this coordinates, $g_{ab}(y^a)$, is independent of the coordinate y^0. The conservation law is simply the statement that the component 0 of the linear momentum of the geodesic remains constant, $Q_k = p^a k_a = p^{y^0} = \text{const.}$

Example. Killing vector fields of 3+1 Minkowski space $(\mathbb{R}^4, \eta_{ab})$.

In cartesian coordinates $\{t, x, y, z\}$ the flat Minkowski metric is written as $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. From the statement above, we conclude that the basis vectors $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ are KVF. In fact, they are the generators of translations, which are known to be isometries of $(\mathbb{R}^4, \eta_{ab})$. The corresponding conserved charges are, respectively, the energy $E \equiv p^0$, and the components of linear momentum: p^x, p^y, p^z.

In spherical coordinates $\{t, r, \theta, \phi\}$, the Minkowski metric reads $ds^2 = -dt^2 + dr^2 + r^2(d\theta_z^2 + \sin^2\theta_z d\phi_z^2)$, in a frame where the angle θ_z is measured from the z axis. Thus, the vector field $\frac{\partial}{\partial\phi_z} = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$ is a KVF. Similarly , we can change the frame and conclude that $\frac{\partial}{\partial\phi_x} = -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}$, $\frac{\partial}{\partial\phi_y} = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$ are also KVF. These three vectors are the generators of spatial rotations. The corresponding conserved charges are the components of the angular momentum vector, $L_z = -xp_y + yp_x$, etc.

Doing a similar analysis one can conclude that $t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}$, $t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}$, $t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}$ are also KVF, which are the generators of boosts.

The set of all Killing vector fields on a spacetime form a vector space on the real numbers. For 4-dimensional spacetimes, the maximum dimension of this vector space is 10 (see exercise 8 of Exercise List). The argument above shows that Minkowski space is a maximally symmetric spacetime.

As a final remark, if instead of test particles we consider test classical fields with stress-energy tensor T_{ab}, then the existence of a KVF leads to a conserved Noether current, see exercise 9 of Exercise List.

1.5 Oppenheimer-Snyder gravitational collapse

The Oppenheimer-Snyder metric is an exact solution of Einstein's field equations that describes the process of gravitational collapse of a simple stellar model and eventual formation of a black hole. It models the collapsing star as a spherically symmetric, preasure-free distribution of matter (a "dust" fluid) with uniform density. Since no internal force is able to stop the gravitational collapse, internal pressures and rotation effects play no essential role, and therefore this toy model is expected to capture the relevant physics of any actual process of gravitational collapse.

The model is obtained by matching two solutions of Einstein's field equations, corresponding to the interior and the exterior metric of the star. We therefore have to distinguish two situations.

(a) Interior of the star.

Let u^a represent the vector field tangent to a congruence of observers comoving with the particles composing the star during the gravitational collapse, and τ the proper time measured by this fluid. Physically, $u = \frac{\partial}{\partial \tau}$ describes the fluid's 4-velocity. We will assume a perfect fluid of dust ($p = 0$) and with uniform density ρ. Roughly speaking, uniformity means that there exists a particular coordinate frame $\{\tau, \vec{x}\} \in \mathbb{R}^4$ for which $\rho = \rho(\tau)$. The stress-energy tensor of this perfect fluid takes then the simple form

$$T_{ab} = \rho \, u_a u_b \,. \tag{47}$$

Because of all these simplifications, the metric solution of the Einstein's field equations with this source term will be spatially homogeneous and isotropic. These two properties constrain the form of the metric severely and can actually be exploited to extract the general form of the metric.

First of all, the hypersurfaces of homogeneity, $\Sigma_{\tau_0} = \{(\tau, \vec{x}) \in \mathbb{R}^4 / \tau = \tau_0\}$ must be orthogonal to the fluid's 4-velocity u^a, otherwise the isotropy condition would be violated. The metric solution g_{ab} induces a Riemannian metric h_{ab} (i.e. positive-definite) on each leaf Σ_{τ_0} via $h_{ab} = g_{ab} + u_a u_b$.

Secondly, due to homogeneity and isotropy, we have 6 isometries on each Riemannian space $(\Sigma_\tau, h_{ab}(\tau))$. Each of them is therefore a maximally symmetric spatial hypersurface. This allows us to write the 3-dimensional Riemann tensor as (see exercise 8 of Exercise List).

$${}^3R_{abcd} = \frac{K}{6}[h_{ac}h_{bd} - h_{ad}h_{bc}] \,, \tag{48}$$

where K is a constant of dimensions L^{-2} for some length scale L. As we can see, the spatial geometry of each $(\Sigma_\tau, h_{ab}(\tau))$ is severely restricted. As a result, the metric solution of the Einstein's field equations can be written in the form

$$ds^2_{\text{interior}} = -d\tau^2 + a^2(\tau) d\Omega^2_K \,, \tag{49}$$

for some function $a = a(\tau)$ known as scale factor. We have three possibilities for $d\Omega^2_K$: (1) flat space $K = 0$, (2) three-dimensional spheres $K = 1/L^2$, and (3) 3-dimensional hyperboloids $K = -1/L^2$. The freedom to reescale the spatial metric $h_{ab} \to \frac{1}{\sqrt{|K|}} h_{ab}$, and to absorb a constant in $a = a(\tau)$, allows us to restrict to the cases $K \in \{-1, 0, 1\}$.

Notice that the most general solution to the interior of the star is of the form (49), where only two unknowns, $a(\tau)$ and K, remain. These two unknowns are then determined by using Einstein's equations. There are only two independent ones:

$$G_{\tau\tau} = 8\pi G T_{\tau\tau} \quad \longrightarrow \quad \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{K}{a^2}\,, \tag{50}$$

$$\nabla_a T^{a\tau} = 0 \quad \longrightarrow \quad \rho(\tau)a^3(\tau) = \rho(0)a^3(0)\,,\dagger \tag{51}$$

where one dot means derivative with respect to τ, and $\tau = 0$ is the time when gravitational collapse begins. If the star is initially at rest, and has a finite radius, then $\dot{a}(0) = 0$ implies $K = 1$ from the first equation above. Furthermore, the same equation tells us that the constant $\rho(0)$ is $\frac{3}{8\pi a^2(0)}$, which, together with the second equation above, tells us that $\rho(\tau) \sim 1/a^3(\tau)$.

Since $K = 1$ in (49), we are dealing with a closed Fridman-Lemaitre-Robertson-Walker (FLRW) spacetime metric. In suitable coordinates, it can be expressed as

$$ds^2_{\text{interior}} = -d\tau^2 + a^2(\tau)(d\chi^2 + \sin^2\chi d\Omega^2)\,, \tag{52}$$

where $d\Omega^2$ is the line element on the homogeneous sphere $\mathbb{S}^2$. The specific form of the scale factor $a(\tau)$ can be solved in full closed form from the Einstein's equations above, and is given in parametric form:

$$a(\eta) = \frac{a(0)}{2}(1 + \cos\eta), \quad \text{where} \quad \tau(\eta) = \frac{a(0)}{2}(1 + \sin\eta)\,. \tag{53}$$

This solution tells us that the collapse begins at $\eta = 0$, with $a(0) = \sqrt{\frac{3}{8\pi G\rho(0)}}$, and ends at $\eta = \pi$, with $\tau = \frac{\pi a(0)}{2}$, when the scale factor vanishes, $a = 0$.

(b) Exterior of the star.

We are interested in a spherically symmetric, vacuum ($T_{ab} = 0$) solution of the Einstein's field equations that can differentiably match the solution found for the matter of the collapsing star.

Birkhoff's theorem (see exercise 10 of Exercise List): if (M, g_{ab}) is a spherically symmetric solution of the vacuum Einstein's field equations, then it is locally isometric to a portion of the Schwarzschild solution.

In coordinates $\{t, r, \theta, \phi\}$ adapted to a static observer $u = \frac{\partial}{\partial t}$ at spatial infinity, the Schwarzschild metric takes the form

$$ds^2_{\text{exterior}} = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2\,, \tag{54}$$

for some positive constant M, and where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of the homogeneous sphere in standard spherical coordinates. The domain of these coordinates is $t \in \mathbb{R}$, $(\theta, \phi) \in \mathbb{S}^2$, but $r > R$, where R represents the radius of the star.

(c) Matching.

Can (52) and (54) be joined smoothly (or at least C^2) at the surface of the collapsing star? The matching conditions impose non-trivial constraints on the fluid of the star and not necessarily all models may be successful. To give an example, a perfect fluid with uniform density but non-vanishing pressure in the interior of the star is not allowed, see exercise 32.4 of [7] for more details.

A sufficient condition for the matching to be differentiable is

$$h_{ab}^{\rm interior} = h_{ab}^{\rm exterior}\,, \quad K_{ab}^{\rm interior} = K_{ab}^{\rm exterior}\,, \tag{55}$$

where $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$ is the extrinsic curvature of the timelike hypersurface Σ representing the surface of the star ($\mathcal{L}_n$ denotes the Lie derivative with respect to the normal vector n). These are called junction (or Darmois-Israel) conditions.

Let us fix coordinates $\{\tau, \theta, \phi\}$ on Σ, where τ is the proper time of observers comoving with the star's surface, $u = \frac{\partial}{\partial\tau}$. In the coordinate frame adapted to observers comoving with the fluid of the star, $\{\tau, \chi, \theta, \phi\}$, the surface of the star is characterized by the hypersurface $\Sigma = \{\chi = \chi_0\}$, while for the coordinate frame adapted to inertial observers at spatial infinity, $\{t, r, \theta, \phi\}$, the surface of the star is the hypersurface $\Sigma = \{r = R(t), \tau = T(t)\}$, for a pair of unknown functions R, T. To determine these two functions, we apply the joint conditions above.

(c.1) First joint condition.

The induced metric on Σ from (52) is

$$ds^2_\Sigma = -d\tau^2 + a^2(\tau)\sin^2\chi_0 d\Omega^2\,. \tag{56}$$

Let us consider now a generic coordinate transformation $t = t(\tau, \chi)$, $r = r(\tau, \chi)$ to write the exterior metric (54) as

$$ds^2 = -(f\dot{t}^2 - f^{-1}\dot{r}^2)d\tau^2 - 2(f\dot{t}\,t' - f^{-1}\dot{r}\,r')d\tau d\chi - (ft'^2 - f^{-1}r'^2)d\chi^2 + r^2(\tau,\chi)d\Omega^2\,, \tag{57}$$

where $f(\tau, \chi) := 1 - \frac{2M}{r(\tau,\chi)}$; and dot and prime denote a derivative with respect to τ and χ, respectively. Then, the induced metric on Σ takes the form

$$ds^2 = -(F\dot{T}^2 - F^{-1}\dot{R}^2)d\tau^2 + R^2(\tau)d\Omega^2\,, \tag{58}$$

where we defined the functions $T(\tau) := t(\tau, \chi_0)$, $R(\tau) := r(\tau, \chi_0)$, $F(\tau) := f(\tau, \chi_0)$ and $R(\tau) := r(\tau, \chi_0)$. To obtain the matching we need to find explicit expressions for the coordinate transformations $R(\tau)$ and $T(\tau)$. The result of imposing the equivalence between the two expressions above yields

$$\begin{aligned} R(\tau) &= a(\tau)\sin\chi_0\,, && (59)\\ F\dot{T}^2 - F^{-1}\dot{R}^2 &= 1\,. && (60) \end{aligned}$$

We see that $R(\tau)$ is fully determined, while $\dot{T}$ is fixed by $F\dot{T} = \sqrt{\dot{R}^2 + F} \equiv \beta(R, \dot{R})$.

(c.2) Second joint condition.

Let $n_a = a\nabla_a\chi$ be the unit vector normal to Σ. Indeed, using (52) it is easy to see that $n_a n^a = 1$ and that it is orthogonal to the tangent vector $u = \frac{\partial}{\partial\tau}$, $g_{ab}n^a u^b = 0$. The extrinsic curvature can be calculated from

$$K_{ab} = h_a^c h_b^d \nabla_c n_d\,, \tag{61}$$

where $h_{ab} = n_a n_b + g_{ab}$ is the induced metric on Σ. In the interior region I, using (52), we can obtain:

$$\begin{aligned} K^I_{\tau\tau} &= u^a u^b K^I_{ab} = u^a u^b \nabla_a n_b = u^a\nabla_a(n^b n_b) - u^a n_b \nabla_a u^b = -a^b n_b = 0\,, && (62)\\ K^I_{\theta\theta} &= h^a_\theta h^b_\theta \nabla_a n_b = \nabla_\theta n_\theta = \partial_\theta n_\theta - \Gamma^\chi_{\theta\theta} a = a\sin\chi\cos\chi\,, && (63) \end{aligned}$$

$$K^I_{\phi\phi} \;=\; h^a_\phi h^b_\phi \nabla_a n_b = \nabla_\phi n_\phi = \partial_\phi n_\phi - \Gamma^\chi_{\phi\phi} a = a \sin^2\theta \sin\chi \cos\chi \,, \tag{64}$$

which produces $K^{I,\theta}_\theta = g^{\theta\theta} K^I_{\theta\theta} = \frac{1}{a}\cot\chi_0$ and $K^{I,\phi}_\phi = g^{\phi\phi} K^I_{\phi\phi} = \frac{1}{a}\cot\chi_0$. The last equality in (62) follows from

$$a^b n_b = n_b u^a \nabla_a u^b = n_b u^a (\partial_a u^b + \Gamma^b_{ac} u^c) = \Gamma^b_{\tau\tau} n_b = a \Gamma^\chi_{\tau\tau} = 0 \,. \tag{65}$$

On the other hand, in region II, we have

$$K^{II}_{\tau\tau} \;=\; -a^b n_b \,, \tag{66}$$

$$K^{II}_{\theta\theta} \;=\; -\Gamma^a_{\theta\theta} n_a = -\Gamma^r_{\theta\theta} = (-2M + R)\dot{T} = RF\dot{T} \,, \tag{67}$$

$$K^{II}_{\phi\phi} \;=\; -\Gamma^a_{\phi\phi} n_a = -\Gamma^r_{\phi\phi} = (-2M + R)\sin^2\theta \dot{T} = RF\sin^2\theta\dot{T} \,, \tag{68}$$

which produces $K^{II,\theta}_\theta = g^{\theta\theta} K^{II}_{\theta\theta} = \frac{F\dot{T}}{R} = \frac{\beta}{R}$ and $K^{II,\phi}_\phi = g^{\phi\phi} K^{II}_{\phi\phi} = \frac{F\dot{T}}{R} = \frac{\beta}{R}$. Imposing now the equality of the results found above for both regions I and II yields

$$a^b|_\Sigma \;=\; 0 \,, \tag{69}$$

$$\frac{\beta}{R} \;=\; \frac{\cot\chi_0}{a} \,. \tag{70}$$

The constraint (69) tells us that the surface of the collapsing star follows a geodesic also from the viewpoint of the exterior metric. If u^a is a geodesic, since $k = \partial/\partial t$ is a Killing vector field of the Schwarzschild metric, $\mathcal{L}_k g_{ab} = \partial_t g_{ab} = 0$, then $E := -g_{ab} u^a k^b = g_{tt}\dot{T} = F\dot{T} = \beta$ is a constant along the geodesic. This is actually evident from (70), which requires $\beta = \cos\chi_0$ after using (59). Recalling the definition $F\dot{T} = \sqrt{\dot{R}^2 + F} \equiv \beta \equiv E$ we solve

$$\dot{T} = \frac{\cos\chi_0}{1 - \frac{2M}{a(\tau)\sin\chi_0}} \,. \tag{71}$$

This formula can be integrated numerically to give the coordinate transformation $T = T(\tau)$ up to an irrelevant constant of integration.

(d) Physical interpretation of the OS solution.

(d.1) What is the physical meaning of the free parameter M in the exterior Schwarzschild metric? Using (60) and the definition $\beta = F\dot{T}$, we can get $\beta^2 - \dot{R}^2 = F$, or $\cos^2\chi_0 - \dot{R}^2 = F$, and $1 - F = \sin^2\chi_0 + \dot{R}^2$. Using now (59),

$$\dot{a}^2 + 1 = \frac{\dot{R}^2 + \sin^2\chi_0}{\sin^2\chi_0} = \frac{1-F}{\sin^2\chi_0} = \frac{2M}{R\sin^2\chi_0} = \frac{2Ma^2}{R^3} \,. \tag{72}$$

Finally, using (50), we get

$$M = \frac{4\pi}{3}\rho(\tau) R^3(\tau) \,. \tag{73}$$

Thus, the parameter M is just the mass of the collapsing star, which happens to remain constant during the entire evolution (this is because, in spherical symmetry, there is no energy loss by gravitational wave emission).

(d.2) The matching condition (69) requires that particles initially at rest on the surface of the star must move along timelike geodesics in the Schwarzschild geometry. How does the gravitational collapse of the star look like to exterior, static observers?

Recall that the proper time of inertial observers at spatial infinity is t. Using (60) and doing a coordinate transformation we get

$$E^2 - F = \dot{R}^2 = \dot{T}^2(\partial_t R)^2 = \frac{E^2}{F^2}(\partial_t R)^2 \,. \tag{74}$$

Solving this equation for $\partial_t R$ yields

$$\partial_t R = -\frac{1}{E}\left(1 - \frac{2M}{R}\right)\sqrt{\frac{2M}{R} - 1 + E^2}\,, \tag{75}$$

where the negative sign is chosen so as to physically describe the contraction of the star, $\partial_t R < 0$. Equation (75) is a nonlinear ODE. We can use standard techniques to infer the qualitative behaviour of the solutions. More precisely, the equation has two fixed points, $\partial_t R = 0$: (i) $R \equiv R_{\rm min} = 2M$ and (ii) $R \equiv R_{\rm max} = \frac{2M}{1-E^2}$. The latter is unstable, while the former is stable. Thus, the solution of (75) will tend to $R_{\rm min}$, regardless of the initial condition. The time of reaching this point, from the viewpoint of inertial observers at infinity, is

$$\Delta t = \int_{R_{\rm max}}^{2M} (-\partial_t R)^{-1} dR = \infty! \tag{76}$$

In other words, the collapse begins at $R \equiv R_{\rm max} = \frac{2M}{1-E^2}$ with zero velocity, then R decreases monotonically and approaches $R \equiv R_{\rm min} = 2M$ asymptotically as $t \to \infty$.

The above result can be somewhat surprising. A static observer at spatial infinity sees the star contract at most to $R = 2M$ but no further. Does the gravitational collapse of the star really end here? To really understand what is happening, we need to change to a frame adapted to an observer on the surface of the star. In particular, let us measure time now with the affine parameter τ of ingoing radial and timelike geodesics, $u = \frac{\partial}{\partial \tau}$. It is straightforward to see that (75) provides

$$\dot{R} = -\sqrt{\frac{2M}{R} - 1 + E^2} = -\sqrt{1 - E^2}\sqrt{\frac{R_{\rm max}}{R} - 1} < 0\,. \tag{77}$$

Notice that now there is only one single fixed point at $R = R_{\rm max}$, which is unstable. Since $\dot{R} < 0$, the star will contract until $R = 0$, as perceived by observers on the surface. In particular, an observer on the surface of the star will see the star falling from $R = R_{\rm max}$ through $R = 2M$ in finite proper time. Moreover, it reaches $R = 0$ in about

$$\Delta\tau = -\int_{R_{\rm max}}^{0} dr \frac{1}{\sqrt{1 - E^2}\sqrt{\frac{R_{\rm max}}{R} - 1}} = \frac{\pi M}{(1 - E^2)^{3/2}} < \infty\,. \tag{78}$$

When $R = 0$ the OS star's density becomes infinite and geodesics followed by dust particles inside the star cannot be extended beyond $\Delta\tau$ in (78). They are said to be "incomplete" geodesics, and that the spacetime reaches a singularity. Moreover, we will see in later sections that at this point the spacetime curvature blows up.

1.6 Emergence of singularities after collapse

As we have just seen, the fate of gravitational collapse in the Oppenheimer-Snyder model happens to be the development of a singularity in the form of geodesic incompleteness.

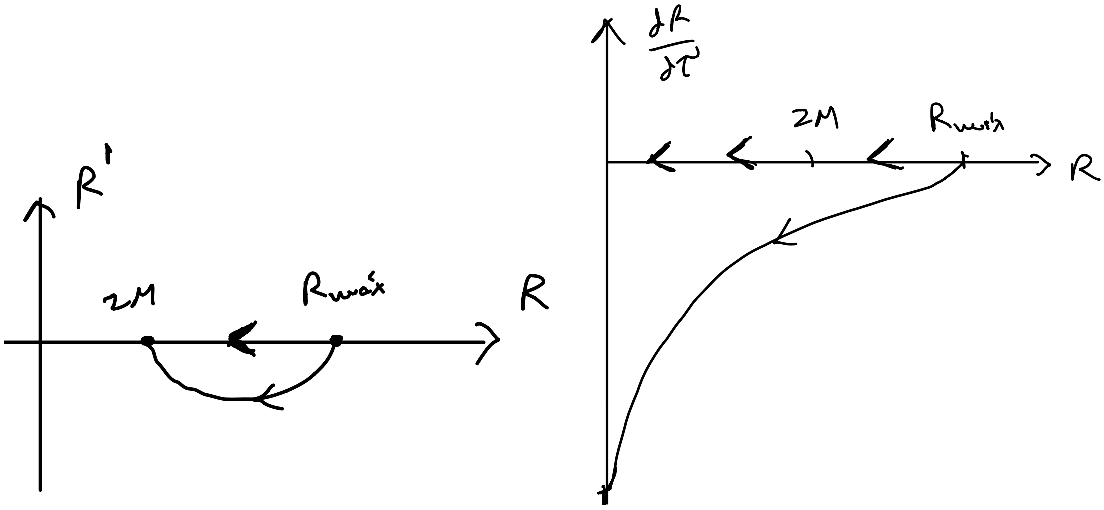

Figure 3: Phase diagrams of the nonlinear ODEs (75) (left) and (77) (right).

One may wonder whether this singularity is simply a consequence of the high symmetry assumed in the model. Does the effect of including rotation in the star, or a more physically realistic equation of state, avoid the emergence of this singularity? After the publication in 1939 by Oppenheimer-Snyder, this was a major concern in general relativity. Since the emergence of a singularity is physically absurd, it was generally believed that more realistic physical models of gravitational collapse would result in a more reasonable picture. This was the dominant view until 1965, when Penrose proved the celebrated singularity theorem, which asserts that the development of singularities is in fact the general result for any gravitational collapse, and actually quite common in general relativity.

Singularity Theorem (Penrose, 1965)....

This theorem establishes that, for "normal" matter satisfying the (classical) "energy conditions", if a trapped surface forms (we will see in more detail what this is in later sections), then the formation of singularities is unavoidable in General Relativity. In a sense, this theorem establishes the limits of validity of Einstein's theory of gravity. Since singularities are not physically real, it is widely expected that this problem will be smoothed out only when quantum effects of the spacetime are taken into account.[2] This is still today an open problem.

Note: as mentioned in the previous section, the OS model is just a toy model that captures the main physical ingredients of gravitational collapse by using a simplified stellar model. More physically realistic models are given by scalar fields [8].

[2]As a matter of fact, the classical energy conditions are known to be violated in the quantum regime, so this bypasses the assumptions of the theorem above.

2 The Schwarzschild black hole solution

2.1 Birkhoff's uniqueness theorem

Birkhoff's theorem ensures that the unique spherically symmetric solution of the vacuum Einstein's equations is the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2 \,, \quad R_{ab}[g] = 0 \,. \tag{79}$$

For a proof, see exercise 10 of Exercise List. As illustrated in the previous subsection with the Oppenheimer-Snyder model, this metric describes the endopoint of gravitational collapse of non-rotating (spherically-symmetric), sufficiently massive stars, therefore it is physically relevant. Despite the apparently simple form, it displays a non-trivial causal structure, which is manifested in the existence of spacetime horizons. In this second section we will address this topic in detail.

2.2 Eddington-Finkelstein coordinates: future and past event horizons

The Schwarzschild line element is singular at $r = 2M$. Therefore, it can only describe the exterior of the collapsing star until the surface's radius reaches the value $R = 2M$. However, an observer on the star's surface will claim that "nothing special happens when the radius of the star reaches $R = 2M$, the collapse continues until $r = 0$ in finite time". In other words, from the viewpoint of the interior of the star, the spacetime is perfectly well-behaved at $r = 2M$. What is going on here?

The exterior spacetime of the star after crossing $r = 2M$ is not defined by (79). Let us attempt to extend the spacetime across $r = 2M$ by following ingoing geodesics. Take radial ($d\Omega^2 = 0$) and null ($ds^2 = 0$) geodesics for infalling observers. These null geodesics determine the light-cone structure, i.e. the causal structure of the spacetime. Then (79) implies

$$dt^2 = \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} =: dr_*^2 \to d(t \pm r_*) = 0 \,, \tag{80}$$

where $r_* = r + 2M \log\left|\frac{r}{2M} - 1\right|$ is the Regge-Wheeler ("tortoise") coordinate[3]. This result shows that the combinations $t \pm r_*$ remain constant along radial null geodesics. Motivated by this result, let us define the advanced time coordinate $v := t + r_* \in \mathbb{R}$. With this new coordinate the metric (79) takes the form

$$ds^2 = (1 - \frac{2M}{r})(-dt^2 + dr_*^2) + r^2 d\Omega^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2drdv + r^2 d\Omega^2 \,. \tag{81}$$

Interestingly, the new metric (81) is non-singular at $r = 2M$. Then, the range of the radial coordinate r can now be analytically extended to all $r > 0$, in contrast to the original Schwarzschild metric which was initially defined only for $r > 2M$. The set of coordinate functions $\{v, r, \theta, \phi\}$ is called ingoing Eddington-Finkelstein coordinates.

In Schwarzschild coordinates, the spheres $\{r = 2M\}$ were merely a coordinate singularity. However, we have just found that the spacetime can actually be extended beyond the

[3]If $r \in (2M, \infty)$, then $r_* \in (-\infty, \infty)$.

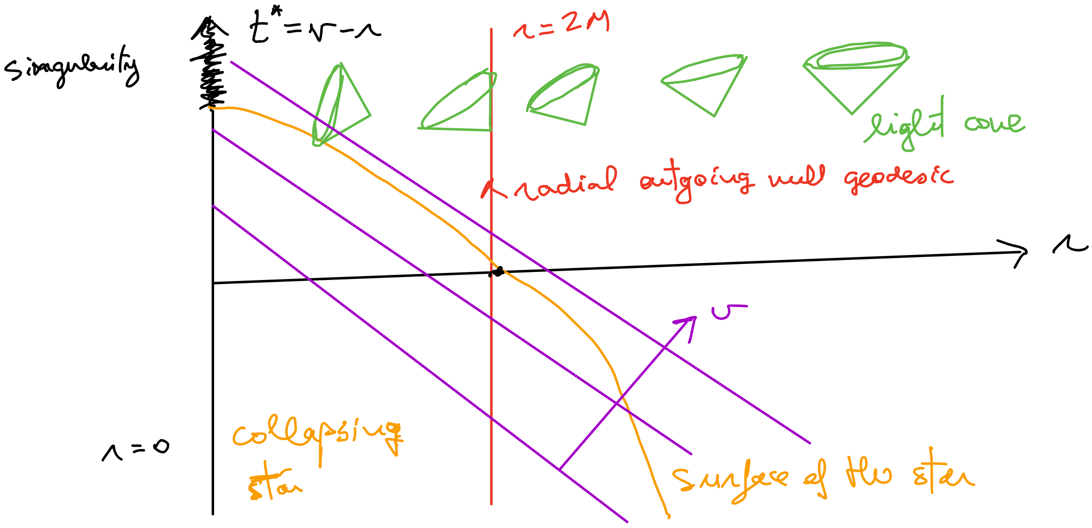

Figure 4: Finkelstein diagram illustrating the causal structure of the Schwarzschild spacetime around $r = 2M$ in ingoing Eddington-Finkelstein coordinates.

original Schwarschild manifold [4]. Let us then investigate in detail the causal structure of the spacetime around $r = 2M$.

Proposition. No future-directed ($dv > 0$), timelike or null worldlines ($ds^2 < 0$) can ever reach the exterior region $r > 2M$ from the inner region $r \leq 2M$. Proof:

$$2drdv = -\left[(-ds^2) + \left(\frac{2M}{r} - 1\right) dv^2 + r^2 d\Omega^2\right] \leq 0\,. \tag{82}$$

Then, since $dv > 0$ we must have $dr \leq 0$, with $dr = 0$ iff it is an ingoing radial null geodesic ($ds^2 = 0$, $d\Omega = 0$) at $r = 2M$.

This result shows that nothing can prevent the star from collapsing to $r = 0$ after it has crossed the surface $r = 2M$. This is illustrated in the Finkelstein diagram of Figure 4. More precisely, a particle entering $r < 2M$ will never be able to escape to the exterior region $r > 2M$ and will approach the singularity $r = 0$ in a finite proper time (see exercise 13 of Exercise List). Because of this, the region $r < 2M$ in ingoing Eddington-Finkelstein coordinates is called a black hole, and the corresponding boundary $r = 2M$ is called the future event horizon. The hypersurface $r = 2M$ is a one-way membrane.

Is this result compatible with the property of time reversibility of Einstein's field equations? Let us define now the retarded time $u = t - r_* \in \mathbb{R}$. The set of coordinate functions $\{u, r, \theta, \phi\}$ is called outgoing Eddington-Finkelstein coordinates. The Schwarschild metric (79) takes now the form:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2drdu + r^2 d\Omega^2\,. \tag{83}$$

Again, the metric can be analytically extended to all $r > 0$, as with (81). However, the region $r < 2M$ in outgoing Eddington-Finkelstein coordinates is not the same as the region $r < 2M$ in the ingoing Eddington-Finkelstein coordinate system. This is clearly seen from the fact that:

[4] Alternatively, we could have considered timelike geodesics to extend the spacetime beyond the original Schwarzschild manifold. This leads to so-called Painlevé-Gullstrand coordinates.

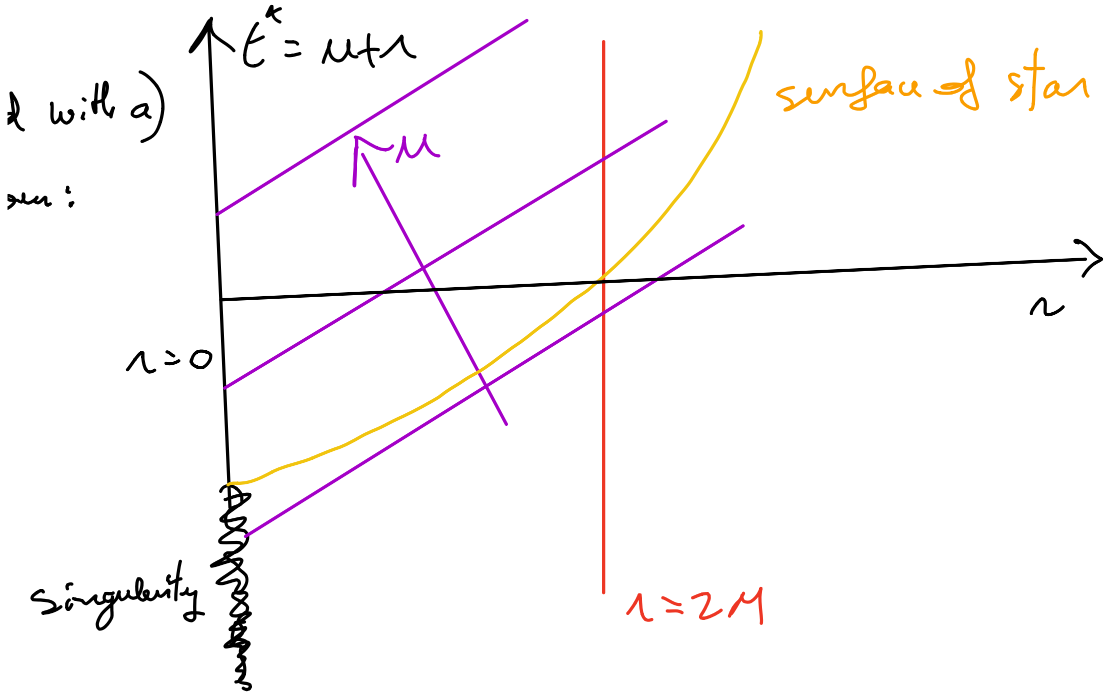

Figure 5: Finkelstein diagram illustrating the causal structure of the Schwarzschild spacetime around $r = 2M$ in outgoing Eddington-Finkelstein coordinates.

Proposition. All future-directed ($du > 0$), timelike or null worldlines ($ds^2 < 0$) will always reach the exterior region $r > 2M$ from the inner region $r \leq 2M$. Proof:

$$2drdu = (-ds^2) + \left(\frac{2M}{r} - 1\right) du^2 + r^2 d\Omega^2 \geq 0\,. \tag{84}$$

Then, since $du > 0$ we must have $dr \geq 0$, with $dr = 0$ iff it is an outgoing radial null geodesic ($ds^2 = 0$, $d\Omega = 0$) at $r = 2M$.

In other words, a particle in $r < 2M$ will always escape to the exterior region $r > 2M$, crossing the hypersurface $r = 2M$. This is illustrated in the Finkelstein diagram of Figure 5, which can be thought of as the time-reversed diagram appearing in Figure 4. Because of this, the region $r < 2M$ in outgoing Eddington-Finkelstein coordinates is called a white hole, and the corresponding boundary $r = 2M$ is called the past event horizon. The hypersurface $r = 2M$ is again a one-way membrane, but with opposite direction. A white hole is just the time-reverse of a black hole. A star with surface in $r < 2M$ would expand through $r = 2M$.

Both black hole and white hole are exact solutions of general relativity (consequence of time-reversibility of the field equations). However, white holes do not have a physical counterpart, unlike black holes which are the result of gravitational collapse.

2.3 Kruskal-Szekeres coordinates: eternal black holes

In the previous subsection we argued that the Schwarzschild coordinates $\{t, r\}$ only cover the exterior region $r > 2M$ of the spacetime manifold. Because of this, we introduced two new time coordinates to extend the manifold beyond $r = 2M$:

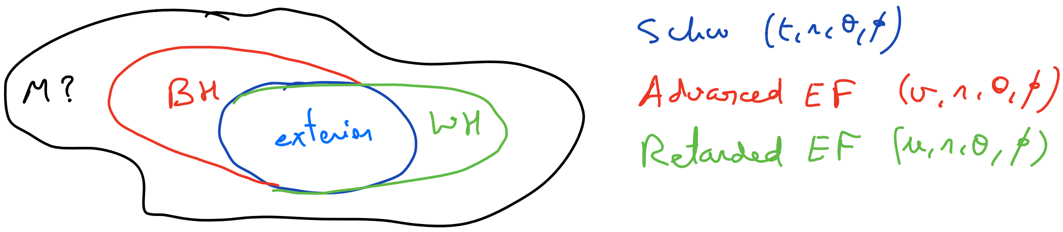

Figure 6: Pictorial diagram illustrating manifold extensions by analytically extending the range of certain coordinate functions.

a) Ingoing EF coordinates (v, r), with advanced time $v = t + r_*$, uncovered the black hole interior $\{v \in \mathbb{R}, r < 2M\}$, but did not cover the white hole region.

b) Outgoing EF coordinates (u, r), with retarded time $u = t - r_*$, uncovered the white hole interior $\{u \in \mathbb{R}, r < 2M\}$, but did not cover the black hole region.

Is it possible to uncover more hidden regions of the spacetime? What is the maximum spacetime that we can reveal by changing coordinates?

Let us first work with both advanced and retarded coordinates, $\{u, v, \theta, \phi\}$. In these coordinates the Schwarzschild line element reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du dv + r^2 d\Omega^2 \,. \tag{85}$$

Now, let us introduce a set of two new null coordinates by

$$U := -e^{-u/(4M)} < 0 \,, \tag{86}$$
$$V := e^{v/(4M)} > 0 \,. \tag{87}$$

Then (85) can be written as

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV + r^2 d\Omega^2 \,. \tag{88}$$

The set $\{U, V, \theta, \phi\}$ is called Kruskal-Szekeres coordinates. The old radial coordinate is now a function $r = r(U, V)$ defined implicitly by

$$UV = -e^{\frac{r_*(r)}{2M}} = -\frac{r - 2M}{2M} e^{\frac{r}{2M}} \,. \tag{89}$$

Notice that, by definition, $U < 0$ and $V > 0$. However, the metric (88) is actually regular for any $U \in \mathbb{R}$ and $V \in \mathbb{R}$. Therefore, the spacetime can be analytically extended to both $U \geq 0$ and $V \leq 0$, and reveal new spacetime regions not covered by Schwarzschild. Again, the extended metric will still be a solution of the vacuum Einstein's equations, $R_{ab} = 0$, since a mere change of coordinates does not spoil a tensorial equation.

In this new set of coordinates, the horizon, originally located at the points $\{r = 2M\}$, is equivalent to the condition $UV = 0$, or $U = 0$ and $V = 0$. This leads to 4 causally distinguished spacetime regions, as illustrated in the Kruskal diagram of Figure 7. These are:

Region I. This is the spacetime originally covered by the Schwarzschild coordinates $\{t, r, \theta, \phi\}$ (because $U < 0$ or $V > 0$ can be converted back to t, r via (86) and (87)).

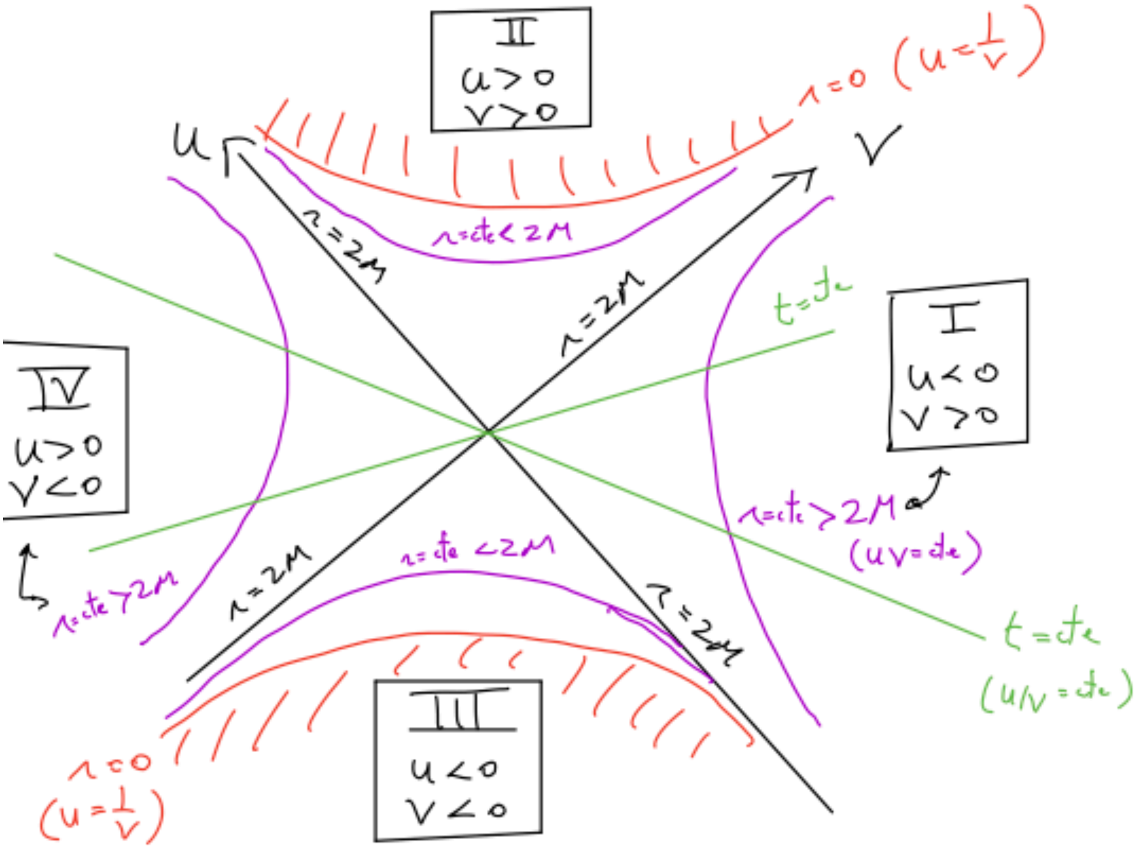

Figure 7: Kruskal diagram of the Schwarzschild black hole. The $\{t = \text{const}\}$ hypersurfaces of the original Schwarschild metric correspond to $U/V = \text{const}$, thus straight lines in the diagram. The $\{r = \text{const}\}$ hypersurfaces correspond to $UV = \text{const}$, thus hyperbolas. In particular, the curvature singularity $r = 0$ corresponds to $UV = 1$. From (89) we infer that regions I and IV contain the points $\{r > 2M\}$ while regions II, III contain the points $\{r < 2M\}$.

The $\{U = \text{const}\}$ const and $\{V = \text{const}\}$ hypersurfaces represent outgoing and ingoing radial null geodesics: $0 = ds^2 = -\frac{32M^3}{r}e^{-\frac{r}{2M}}dUdV + 0 \to U = \text{const}, V = \text{const}$.

Region I $\cup$ II. This is the region covered by the ingoing EF coordinate system $\{v, r, \theta, \phi\}$, and which is relevant for describing gravitational collapse. In particular, region II is the black hole, and timelike geodesics terminate at the curvature singularity.

Region I $\cup$ III. This is the region covered by the outgoing EF coordinate system $\{u, r, \theta, \phi\}$. In particular, region III is the white hole.

Region IV. This is a new spacetime region not covered by any coordinate system used before.

The region III $\cup$ IV does not exist physically: in realistic scenarios this is replaced by a regular metric describing the interior of the collapsing star, with $R_{ab} \neq 0$. The mathematical existence of regions III and IV is a consequence of the time reversibility in Einstein's equations, however realistic black holes formed by gravitational collapse are not time-symmetric. In contrast, in the picture of Figure 88 the black hole has not been produced

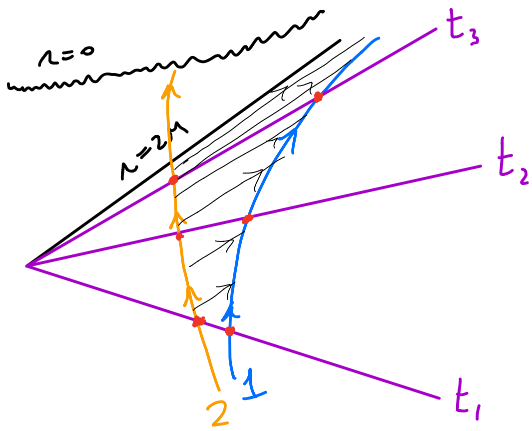

Figure 8: Gravitational collapse as seen by different observers. The blue line denotes the trajectory of a static observer, thus uniformly accelerated (non-freely falling). The orange curve denotes the trajectory of a freely falling observer, comoving with the star. The freely falling observer communicates with the static observer at infinity by sending light signals. The time measured by the freely falling observer in reaching $r = 2M$ is finite, but this process takes an infinite amount of time according to the static observer. More precisely, a light-ray emitted by the freely falling observer just before $r = 2M$ reaches the static observer asymptotically in the limit $t \to \infty$.

as the result of a collapse. It has always existed ($R_{ab} = 0$ everywhere in the spacetime): it is an eternal black hole.

The Kruskal spacetime is the maximal analytical extension of the Schwarzschild spacetime (cannot be extended further). Furthermore, it is unique (maximal Cauchy development), up to diffeomorphisms.

Despite that most of the Kruskal diagram is not physically realizable, it is still useful to depict the causal structure of actual black holes. For instance, Figure 8 illustrates how a static observer at infinity sees the process of gravitational collapse. According to this observer, the star takes an infinite amount of time to cross the surface $r = 2M$ (this is a geometric interpretation of the formula (76)).

Light-rays emitted by a freely falling observer (see figure 8) are redshifted and attenuated exponentially in time: $\nu(t) \sim \nu_0 e^{-\kappa t}$, $I(t) \sim I_0 e^{-t/M}$. In practice, the surface of the star turns effectively black in a finite amount of time (exercise: Zeldovich-Novikov).

2.4 Null hypersurfaces

As shown above, the event horizon $\{r = 2M\}$ plays a distinguished role in the Schwarzschild geometry. Mathematically, it is a null hypersurface. What does this mean?

Definition. A hypersurface Σ is a $n - 1$ dimensional submanifold of a n-dimensional manifold M.

The implicit function theorem on manifolds tells us that a hypersurface can be specified by setting a differentiable function equal to a constant. Some examples:

i) Let $t : \mathbb{R}^4 \to \mathbb{R}$ be the usual time coordinate function in Minkowski space. Then $\Sigma_{t_0} = \{p \in \mathbb{R}^4 / t(p) = t_0 = \text{const}\} \equiv \{t = t_0\} = \mathbb{R}^3 \subset \mathbb{R}^4$ is a hypersurface, and $\mathbb{R}^4 = \cup_{t\in\mathbb{R}}\Sigma_t$. We say that $\mathbb{R}^4$ is foliated by the hypersurfaces Σ_t.

ii) Let $r : \mathbb{R}^3 \to \mathbb{R}^+$ be the usual radial coordinate function in an Euclidean space. Then $\Sigma_{r_0} = \{\vec{x} \in \mathbb{R}^3 / r(\vec{x}) = r_0 = \text{const}\} \equiv \{r = r_0\} = \mathbb{S}^2 \subset \mathbb{R}^3$ is a hypersurface, and $\mathbb{R}^3 - \{(0,0,0)\} = \cup_{r>0}\Sigma_r$. We say that $\mathbb{R}^3$ (minus a point) is foliated by the 2-spheres Σ_r.

Let $S = S(x)$ be a smooth function on our spacetime manifold M, and consider a family of hypersurfaces $S(x) = \text{const}$. Let us also consider the vector field $\ell = \ell^a \frac{\partial}{\partial x^a} := h(x) g^{ab}\nabla_b S \frac{\partial}{\partial x^a}$, with $h(x) \neq 0$ an arbitrary function. Then

$$g_{ab}\ell^a k^b = k^a \nabla_a S = 0\,, \tag{90}$$

for any tangent vector field k to the $S(x) = \text{const}$ hypersurfaces. Therefore, ℓ is a normal vector to the hypersurfaces $S(x) = \text{const}$.

Example. In Minkowski space, $\ell = \frac{\partial}{\partial t}$ is normal to the $\{t = t_0\}$ hypersurfaces, $n = \frac{\partial}{\partial r}$ is normal to the $\{r = r_0\}$ hypersurfaces, etc. In components, we can write $\ell^a = -\eta^{ab}\nabla_b t$, $n^a = \eta^{ab}\nabla_b r$.

Causal classification:

If ℓ is timelike ($\ell^a(x)\ell_a(x) < 0$ for all x), the hypersurface is spacelike.

If ℓ is spacelike ($\ell^a(x)\ell_a(x) > 0$ for all x), the hypersurface is timelike.

If ℓ is null ($\ell^a(x)\ell_a(x) = 0$ for all x), the hypersurface is null.

Since $\ell^a(x)\ell_a(x)$ is a scalar, the causal character is a coordinate invariant, it is a geometric property.

Example. Let us consider the Schwarzschild spacetime (M, g_{ab}) in EF ingoing coordinates, $\{v, r, \theta, \phi\}$. The metric g_{ab} is given by (81). Let us consider the hypersurface

$$\mathcal{N}_{r_0} = \{x^a \in M / S(x^a) = r - 2M = r_0 - 2M\}\,. \tag{91}$$

The corresponding normal vector has the form[5]

$$\ell = h(r) g^{ab}\nabla_a S \frac{\partial}{\partial x^b} = h(r) g^{rb}\partial_r S \partial_b = h(r)\left[\left(1 - \frac{2M}{r}\right)\partial_r + \partial_v\right]\,, \tag{92}$$

and satisfies

$$\ell^a\ell_a = g^{ab}\nabla_a S \dagger \nabla_b S h^2 = g^{rr} h^2 = \left(1 - \frac{2M}{r}\right) h^2\,. \tag{93}$$

This is, $\mathcal{N}_{2M}$ is a null hypersurface ($\ell^a\ell_a|_{r=2M} = 0$) and $\ell = h(2M)\frac{\partial}{\partial v}$ is its normal.

Let $\mathcal{N}$ be a null hypersurface with null normal ℓ. Since:

(i) A vector field k^a is tangent to $\mathcal{N}$ iff $k^a\ell_a = 0$,

(ii) $\ell^a\ell_a = 0$,

then, ℓ^a is actually tangent to $\mathcal{N}$ (because it's orthogonal to the normal). By Frobenius theorem there exists an integral curve in $\mathcal{N}$, say with a parametrization $x^a(\lambda)$, such that $\ell^a = \frac{dx^a}{d\lambda}$.

[5]In matrix notation, $[g_{ab}] =$ and the inverse is $[g^{ab}] =$.

Proposition. A curve parametrized by $x^a(\lambda)$, with null tangent vector $\ell^a = \frac{dx^a}{d\lambda}$, is a geodesic (although λ might not be an affine parameter).

Proof. Let $\mathcal{N}$ be the member $S = 0$ of the family of (not necessarily null) hypersurfaces $S = \text{const}$. If $\ell^a = hg^{ab}\nabla_b S$, then

$$\begin{aligned}
\ell^a\nabla_a\ell^b &= \ell^a\nabla_a(h)g^{cb}\nabla_c S + hg^{ab}\ell^c\nabla_c\nabla_a S & (94)\\
&= \ell^b\ell^a\nabla_a\log|h| + hg^{ab}\ell^c\nabla_a\nabla_c S \quad (\text{torsion} - \text{free connection}) & (95)\\
&= \ell^b\frac{d\log|h|}{d\lambda} + h\ell^c\nabla^b(h^{-1}\ell_c) \quad (\text{chain rule}) & (96)\\
&= \ell^b\frac{d\log|h|}{d\lambda} + \ell^c\nabla^b\ell_c - \ell^c\ell_c\nabla^b\log|h| & (97)\\
&= \ell^b\frac{d\log|h|}{d\lambda} + \frac{1}{2}\nabla^b(\ell^c\ell_c) - \ell^c\ell_c\nabla^b\log|h|\,. & (98)
\end{aligned}$$

If $\mathcal{N}$ is null then $\ell^a\ell_a|_{\mathcal{N}} = 0$ and the last term vanishes. Notice however that $\nabla^b(\ell^c\ell_c) \neq 0$ in general unless $\ell_a\ell^a = 0$ for all hypersurfaces $S = \text{const}$, i.e. if they are all null. We can write

$$\nabla_a(\ell^c\ell_c) = c_1\ell_a + c_2t_a^1 + c_3t_a^2 + c_4t_a^3\,, \tag{99}$$

where $\{t_a^i\}_{i=1,2,3}$ is a basis of tangent vector fields on $\mathcal{N}$. Since $\ell^a\ell_a|_{\mathcal{N}} = 0$, $\ell^a\ell_a$ is constant on $\mathcal{N}$, so $t_i^a\nabla_a(\ell^b\ell_b) = 0$ for $i = 1, 2, 3$. This leads to $c_2 = c_3 = c_4 = 0$. Therefore, we conclude that $\ell^a\nabla_a\ell^b|_{\mathcal{N}} \propto \ell^b$, and consequently $x^a(\lambda)$ is a geodesic with tangent vector ℓ^a. □

Remark. The arbitrary function $h(x)$ can always be chosen such that $\ell^a\nabla_a\ell^b|_{\mathcal{N}} = 0$, and so that λ is an affine parameter.

Definition. The congruence of null geodesics $x^a(\lambda)$ with affine parameter λ, for which the tangent vectors $\ell^a = \frac{dx^a}{d\lambda}$ are normal to a null hypersurface $\mathcal{N}$, are called the generators of $\mathcal{N}$. The union of all these geodesic spans the hypersurface $\mathcal{N}$ (i.e. for each $p \in \mathcal{N}$, there exists one –and only one– geodesic passing through p).

Example. Let us consider the hypersurface $\mathcal{N} = \{U = a\}$, $a \in \mathbb{R}$, in Kruskal spacetime covered with coordinates $\{U, V, \theta, \phi\}$. The normal vector reads

$$\ell = h(p)g^{ab}\nabla_a S\partial_b = h(p)g^{Ub}\partial_b = h(p)g^{UV}\partial_V = -\frac{h(p)r}{16M^3}e^{\frac{r}{2M}}\frac{\partial}{\partial V}\,. \tag{100}$$

Since $U = 0$ implies $r = 2M$, on the future event horizon the normal vector has the form

$$\ell|_{\mathcal{N}_0} = -\frac{h(p)}{8M^2}e\frac{\partial}{\partial V}\,. \tag{101}$$

On the other hand, the norm gives

$$\ell^a\ell_a = g^{VV}h^2 = 0 \quad (g^{VV} = 0)\,, \tag{102}$$

so $\mathcal{N}_a$ is a null hypersurface for any $a \in \mathbb{R}$. As a consequence, $\nabla_a(\ell^b\ell_b) = 0$ and from the calculation given in the proof above we infer

$$\ell^a\nabla_a\ell^b = \ell^b\frac{d\log|h|}{dV}\,. \tag{103}$$

Therefore, if we choose $h = -8M^2e^{-1}$ then $\ell|_{\mathcal{N}_0} = \frac{\partial}{\partial V}$ is a null vector field normal to $\{U = 0\}$, and V is an affine parameter for the generators of $\mathcal{N}_0 = \{U = 0\}$.

Similarly, one can see that (i) $\{V = \text{const}\}$ are all null hypersurfaces, (ii) $n = \frac{\partial}{\partial U}$ is a null vector field normal to $\{V = 0\}$, which is the null hypersurface of the past event horizon, and (iii) $n^a\nabla_a n^b = 0$, i.e. U is an affine parameter for the generator of $\{V = 0\}$.

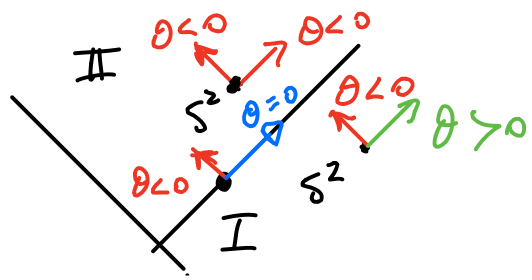

Figure 9: Illustration of trapped surfaces in the Kruskal diagram. Each point of the Kruskal diagram is a 2-sphere. Those in region II are trapped surfaces (both ingoing and outgoing null rays have negative expansion parameter) while those in region I are not (outgoing light-rays have positive expansion parameter). The 2-spheres on the future event horizon are only marginally trapped surfaces because $\theta = 0$ for outgoing light-rays.

2.5 Event horizon vs Apparent horizon

In the previous subsection we saw that the event horizon is a null hypersurface. Another important property of the $r = 2M$ surface has to do with the behaviour of outgoing light-rays.

Proposition. The expansion θ of a congruence of outgoing light rays (i.e. null curves with $U =$ const in the Kruskal spacetime) changes sign at $r = 2M$.

Proof. As we have seen in the example above, the affinely parametrized tangent vector field to outgoing light-rays is given by $\ell_a = -\nabla_a U$ (the minus sign is required to obtain $\ell = \frac{\partial}{\partial V}$). The expansion parameter is computed as

$$\theta = \nabla_a \ell^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} \ell^a) \, . \tag{104}$$

The metric determinant gives $\sqrt{-g} = |g_{UV}| r^2 \sin\theta$, and $\ell^a = g^{ab}\ell_b = -g^{aU}$, and $\ell^V = |g_{UV}|^{-1}$. Thus

$$\theta = \frac{2\partial_V r}{r|g_{UV}|} = \cdots = -\frac{U}{2Mr} \, , \tag{105}$$

which is positive (negative) in the Kruskal region I (II). □

Proposition. The expansion θ of a congruence of ingoing light rays (i.e. null curves with $V =$ const in the Kruskal spacetime) is negative, $\theta < 0$.

Proof. Just change $U \leftrightarrow V$ in the proof above and you will find $\theta = -\frac{V}{2Mr} < 0$ in both Kruskal regions I and II. □

This result motivates the notion of apparent horizon.

Definition. A trapped surface S on a spacelike hypersurface Σ is a closed, 2-dimensional surface, such that for both ingoing and outgoing congruences of future-directed null geodesics orthogonal to S, the expansion θ is negative everywhere on S.

Example. Each 2-sphere $\{U = U_0, V = V_0\}$ in region II of the Kruskal diagram, for a pair of fixed positive numbers $U_0 > 0$, $V_0 > 0$, is a trapped surface. See Figure9 for an illustration.

Definition. The collection of all trapped surfaces S in Σ is called a trapped region $\mathcal{R}$, and its boundary $\partial\mathcal{R}$ is called the apparent horizon.

We also say that the apparent horizon is a marginally trapped surface, in the sense that the expansion paramter $\theta = 0$ for the congruence of outgoing null geodesics.

Example. Any 2-sphere at $r = 2M$ in the Kruskal spacetime is an apparent horizon.

Definition. The union of all apparent horizons from all $\Sigma \subset M$ in the foliation froms a 3-dimensional hypersurface, which is also called apparent horizon (or trapping surface).

In the Schwarszchild eternal black hole, both event and apparent horizon coincide. This is because of the underlying stationarity of the spacetime. For more general black hole spacetimes, the two hypersurfaces can actually differ.

Example. Vaidya spacetime (non-stationary black hole spacetime). In EF ingoing coordinates, the metric can be written as

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \,, \tag{106}$$

where

$$m(v) = \begin{cases} m_1, & v \leq v_1 \\ m_{12}(v), & v_1 \leq v \leq v_2 \\ m_2, & v_2 \leq v \end{cases} \tag{107}$$

and m_{12} is a monotonically increasing function. This metric is a solution of the Einstein's field equations with a given source term T_{ab}, see exercise 14 of Exercise List.

Proposition. The apparent horizon in the Vaidya spacetime is the hypersurface $AH = \{(v, r, \theta, \phi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 / \, r = 2m(v)\}$.

Proof. Outgoing null geodesics are parametrized by $u =$ const, where $u = v - 2r$, so the tangent vector is

$$k_a = -f \nabla_a u = -f \nabla_a v + 2\nabla_a r \,, \tag{108}$$

for some function f. It is easy to see that $k^a k_a|_{AH} = 0$, $k^b \nabla_b k^a = ... = \frac{2m(v)}{r^2} k^a \equiv \kappa k^a$. The latter equation tells us that $k^a = \frac{dx^a}{d\lambda}$ is not affinely parametrized. However, $k^a_* = e^{-\Gamma} k^a$ with $\frac{d\Gamma}{d\lambda} = \kappa(\lambda)$ is affinely parametrized:

$$k^b_* \nabla_b k^a_* = (\kappa k^a_* - \nabla_b \Gamma \, k^b k^a_*) e^{-\Gamma} = 0 \,. \tag{109}$$

We can now evaluate the expansion parameter using the traditional formula:

$$\theta = \nabla_a k^a_* = e^{-\Gamma}(\nabla_a k^a - \Gamma_{;a} k^a) = e^{-\Gamma}\left(\nabla_a k^a - \frac{d\Gamma}{d\lambda}\right) = e^{-\Gamma}(\nabla_a k^a - \kappa) \,. \tag{110}$$

It is easy to see that $\nabla_a k^a = ... = \frac{2(r - m(v))}{r^2}$, so

$$\theta = e^{-\Gamma} \frac{2}{r^2}(r - 2m(v)) \,, \tag{111}$$

so the marginally trapped surface $\theta = 0$ corresponds to $r = 2m(v)$. □

Proposition.

The apparent horizon is null for $v < v_1$, $v > v_2$, and spacelike for $v_1 < v < v_2$.

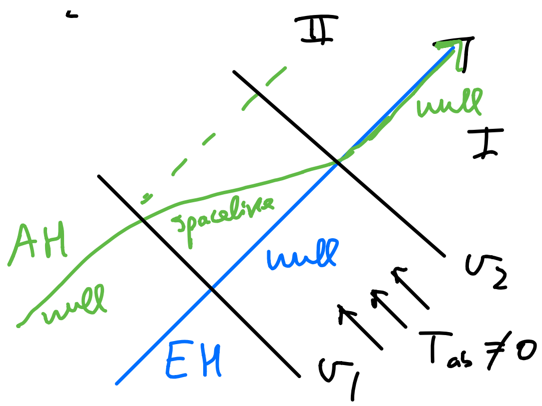

Figure 10: Apparent horizon vs Event horizon in a Vaidya spacetime of metric (106). The event horizon is always null, while the apparent horizon migrates, during a transitory period, from a null hypersurface to another one.

Proof. The function $\phi(v, r) := r - 2m(v)$ defines the location of the apparent horizon. The causal character is determined from

$$g^{ab}\nabla_a\phi\nabla_b\phi = -4\frac{dm}{dv} = \begin{cases} = 0, & v < v_1, \quad v > v_2 \\ < 0, & v_1 < v < v_2 \end{cases} \quad \square \tag{112}$$

Remark. Since the formal definition of the event horizon (to be given in later sections) requires it to be null, one concludes that the notion of event horizon disagrees with the notion of apparent horizon (although the apparent horizon lies within the event horizon). See figure 10 for an illustration.

The apparent horizon is a quasi-local notion of a horizon, while the event horizon is a global notion ("teleological", it requires knowledge of the full spacetime, including the future, to be defined).

2.6 Killing horizons and surface gravity

Before moving on, let us briefly review some basic concepts on differential geometry:

Definition. A 1-parameter group of diffeomorphisms on a manifold M is a smooth map $\phi : \mathbb{R} \times M \to M$, such that

(a) $\forall t \in \mathbb{R}$, the map $\begin{array}{rcl} \phi_t : M & \to & M \\ x & \to & \phi(t, x) \end{array}$ is a diffeomorphism.

(b) $\forall x \in \mathbb{M}$, the map $\begin{array}{rcl} \phi_x : \mathbb{R} & \to & M \\ t & \to & \phi(t, x) \end{array}$ satisfies $\phi_x(0) = x$ (thus $\phi(0) = \mathbb{I}$).

(c) $\forall s, t \in \mathbb{R}$, the composition map is given by $\phi_s \circ \phi_t = \phi_{s+t}$.

Proposition. Given any smooth vector field X on the manifold M, there exists a (local) 1-parameter group of diffeomorphisms for any $x \in M$, which is constructed from the integral curves of X. This is called the flux of a vector field.

Definition. Given any point $x \in M$, the set $\phi_x(\mathbb{R}) = \{\phi_x(t) \,/\, t \in \mathbb{R}\} \subset M$, is called the orbit of the vector field X that passes through $x \in M$.

In general relativity, a 1-parameter group of diffeomorphisms is physically relevant when it leaves the spacetime metric invariant, i.e. when it corresponds to an isometry.

The vector field $k = \frac{\partial}{\partial t}$ is a Kiling Vector field (KVF) of the Schwarzschild metric (79), $\mathcal{L}_k g_{ab} = \partial_t g_{ab} = 0$, and it is timelike for $r > 2M$: $g_{ab}k^a k^b = g_{tt} = \frac{2M}{r} - 1 < 0$. Physically, the associated 1-parameter group of diffeomorphisms represents the time translation symmetry of the spacetime metric (i.e. the invariance under $t \to t + c$, $c \in \mathbb{R}$). More precisely, if $x^a = (t, r, \theta, \phi)$ is a generic point of the spacetime in Schwarzschild coordinates, an infinitesimal "displacement" in the spacetime of the form $x^a \to x^a + ck^a + O(c^2)$, driven or "generated" by the KVF $k^a = (1, 0, 0, 0)$, leaves the spacetime metric invariant. Because of this property, k^a is called the infinitesimal generator of this isometry.

This isometry extends to the entire Kruskal spacetime, and it is generated by the vector field

$$\tilde{k} = \frac{1}{4M}\left(V\frac{\partial}{\partial V} - U\frac{\partial}{\partial U}\right), \tag{113}$$

which satisfies $\mathcal{L}_{\tilde{k}} g_{ab} = 0$. To see that $\tilde{k}$ is indeed an extension of k to the entire Kruskal spacetime, let $x^a = (U, V, \theta, \phi)$ be a generic point in the Kruskal spacetime. An infinitesimal transformation along the integral curves of $\tilde{k}$ produces $x^a \to x^a + c\tilde{k}^a + O(c^2)$. If $\delta t = c$, then

$$\delta U = -\frac{c}{4M}U, \qquad \delta V = \frac{c}{4M}V, \tag{114}$$

$$\delta\theta = 0, \qquad \delta\phi = 0, \tag{115}$$

and we can read off $\tilde{k}^a = (-\frac{1}{4M}U, \frac{1}{4M}V, 0, 0)$. Thus, in region I of the Kruskal spacetime we can write

$$\tilde{k}|_I = \frac{1}{4M}\left[\frac{\partial}{\partial \log V} - \frac{\partial}{\partial \log(-U)}\right] = \frac{\partial}{\partial v} + \frac{\partial}{\partial u} = \frac{\partial t}{\partial v}\frac{\partial}{\partial t} + \frac{\partial t}{\partial u}\frac{\partial}{\partial t} = \frac{\partial}{\partial t} = k\,. \tag{116}$$

Geometrically, time translation is described by the flow of $\tilde{k}^a$. To determine the integral curves $\gamma(\lambda)$ of $\tilde{k}^a$, we have to solve the equation $\tilde{k}(\gamma(\lambda)) = \dot{\gamma}(\lambda)$. If $\gamma(\lambda) = (U(\lambda), V(\lambda), 0, 0)$, this equation produces $-\frac{U}{4M} = \dot{U}$, and $\frac{V}{4M} = \dot{V}$. which can be solved to give

$$U(\lambda) = U_0 e^{-\lambda/4M}, \tag{117}$$

$$V(\lambda) = V_0 e^{\lambda/4M}. \tag{118}$$

In particular, the solution satisfies $U = \frac{c}{V}$ for some constant $c \in \mathbb{R}$. These integral curves can be consulted in Figure 11. The orientability of the flow is determined by $\delta\lambda > 0$ (in particular, all curves tend to $U(\lambda) \to 0$, $V(\lambda) \to \pm\infty$).

Notice that the causal character of the KVF (113) does not remain constant:

$$\tilde{k}^a\tilde{k}_a = g_{ab}\tilde{k}^a\tilde{k}^b = 2g_{UV}\tilde{k}^u\tilde{k}^V = 2\left(-\frac{16M^3}{r}e^{-r/2M}\right)\left(\frac{-U}{4M}\right)\left(\frac{V}{4M}\right) \tag{119}$$

$$= \frac{2M}{r}e^{-r/2M}UV = -\frac{2M}{r}\frac{r-2M}{2M} = -\left(1 - \frac{2M}{r}\right). \tag{120}$$

Therefore:

$\tilde{k}$ is timelike in regions I and IV ($r > 2M$).

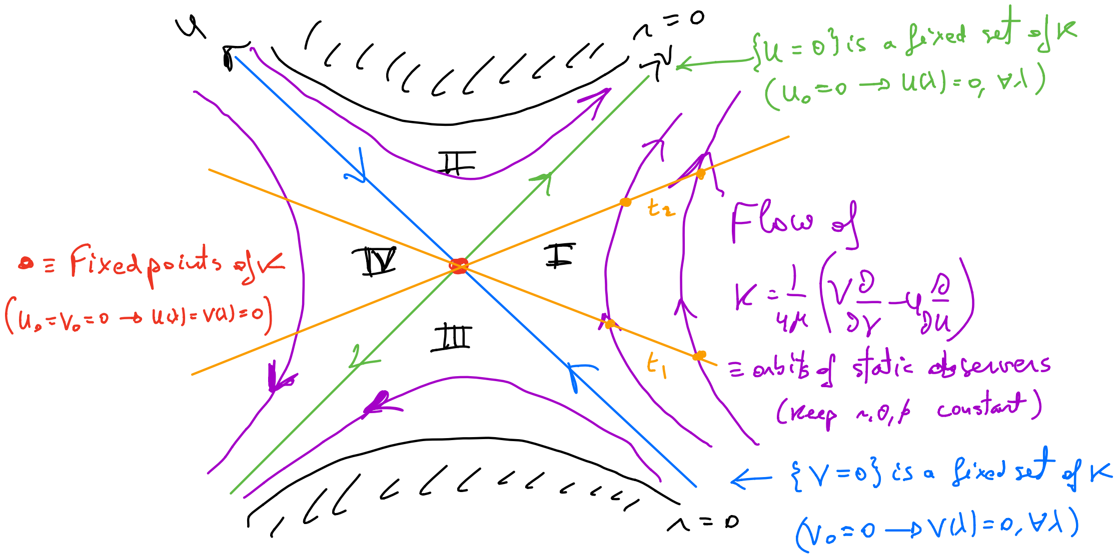

Figure 11: Integral curves of the Killing Vector field (113).

$\tilde{k}$ is spacelike in regions II and III ($r < 2M$).

$\tilde{k}$ is null on $\{U = 0\}$ and $\{V = 0\}$ ($r = 2M$).

In other words, the stationary KVF (113) changes its causal character whenever we cross the event horizon (a null hypersurface). This motivates the following definition.

Definition. Let $(k, \mathcal{N})$ be a vector field and a hypersurface, respectively. If k is a killing vector field, it is normal to $\mathcal{N}$, and it becomes null at precisely $\mathcal{N}$, then we say that $\mathcal{N}$ is a Killing horizon of k (in particular, the hypersurface $\mathcal{N}$ is null).

The notion of killing horizons play a fundamental role in the context of stationary black holes, as it provides a quasi-local notion of a black hole in equilibrium.

To every killing horizon we can associate a quantity called surface gravity. Let ℓ be the normal vector to a null hypersurface $\mathcal{N}$. By a previous proposition, ℓ is tangent to a geodesic in $\mathcal{N}$: $\ell^a \nabla_a \ell^b = 0$ (assume affine parametrization for simplicity). If $\mathcal{N}$ is a killing horizon of a KVF k, then $k^a = f(x)\ell^a$ on $\mathcal{N}$ for some function $f(x)$ (since it must be normal to $\mathcal{N}$). Then,

$$k^a \nabla_a k^b = \ell^b k^a \nabla_a f + f k^a \nabla_a \ell^b = k^b k^a \nabla_a \log|f| \equiv \kappa k^b \,, \tag{121}$$

where in the second equality we used the geodesic equation to get rid of the second piece. The quantity $\kappa := k^a \nabla_a \log|f|$ is called surface gravity.

Proposition. The surface gravity of a killing horizon $(k, \mathcal{N})$, defined by the geodesic equation $k^a \nabla_a k^b = \kappa k^b$, satisfies $\kappa^2 = -\frac{1}{2} \nabla^a k^b \nabla_a k_b|_{\mathcal{N}}$.

Proof. First of all, k^a is normal to $\mathcal{N}$, so by Frobenius theorem:

$$k_{[a} \nabla_b k_{c]}|_{\mathcal{N}} = 0 \,. \qquad \text{(hypersurface orthogonality)} \tag{122}$$

Now, for any killing vector field we have

$$\nabla_a k_b = \nabla_{(a} k_{b)} + \nabla_{[a} k_{b]} = \nabla_{[a} k_{b]} \,, \tag{123}$$

so

$$k_a \nabla_b k_c|_{\mathcal{N}} + (k_b \nabla_c k_a - k_c \nabla_b k_a)|_{\mathcal{N}} = 0\,. \tag{124}$$

Multiply now by $\nabla^b k^c$:

$$k_a \nabla^b k^c \nabla_b k_c|_{\mathcal{N}} = -2(\nabla^b k^c) k_b \nabla_c k_a|_{\mathcal{N}} = -2\kappa k^c \nabla_c k_a|_{\mathcal{N}} = -2\kappa^2 k_a|_{\mathcal{N}}\,. \tag{125}$$

Therefore, for any $p \in \mathcal{N}$ with non-vanishing $k^a(p) \neq 0$, we must have $\kappa^2 = -\frac{1}{2}\nabla^a k^b \nabla_a k_b|_{\mathcal{N}}$. □

Note: it will turn out that all points at which $k^a|_{\mathcal{N}} = 0$ are limit points of orbits of k^a for which $k^a \neq 0$, so by continuity this formula is valid even when $k^a|_{\mathcal{N}} = 0$.

Lemma. For any KVF k^a we have $\nabla_a \nabla_b k^c = R^c{}_{bad} k^d$, where $R^c{}_{bad}$ is the Riemann tensor of the spacetime.

Proof. See exercise 8 of Exercise List). □

Proposition. Given a killing horizon $\mathcal{N}$ with killing vector field k^a, the surface gravity κ is constant along the integral curves of k^a.

Proof. For any tangent vector field t^a to $\mathcal{N}$, we have

$$\begin{aligned} t^a \nabla_a \kappa^2 &= -\nabla^a k^b t^c \nabla_c \nabla_a k_b|_{\mathcal{N}} \qquad \text{(tangency)} && (126)\\ &= -\nabla^a k^b t^c R_{bac}{}^d k_d|_{\mathcal{N}}\,. \qquad \text{(Lemma)} && (127)\end{aligned}$$

The killing vector field k^a is tangent to $\mathcal{N}$ because it is both normal and null on $\mathcal{N}$. If we take $t^a = k^a \equiv \frac{dx^a}{d\alpha}$ as a particular case in the formula above, then

$$\frac{d\kappa^2}{d\alpha} = k^a \nabla_a \kappa^2 = -\nabla^a k^b k^c R_{bacd} k^d = 0\,, \tag{128}$$

where the last equality follows from antisymmetry. □

In the following we will always restrict to non-degenerate killing horizons, for which $\kappa \neq 0$. These are the physically realistic scenarios. Thus, the integral curves of k^a on $\mathcal{N}$ are geodesics not affinely parametrized, and so they are not the generators of $\mathcal{N}$. In other words, they do not necessarily span $\mathcal{N}$ completely. Is κ constant on all $\mathcal{N}$?

Proposition. Let κ be the surface gravity of a killing horizon $(k^a, \mathcal{N})$. Suppose $\kappa \neq 0$ on one orbit of k^a in $\mathcal{N}$, then this orbit coincides with only half of an orbit of the null generator $x^a(\lambda)$ of $\mathcal{N}$.

Proof. Let $k^a = \frac{dx^a}{d\alpha}$ and consider a generator $x^a(\lambda)$ of $\mathcal{N}$ with tangent vector $\ell^a = \frac{dx^a}{d\lambda}$. By definition it satisfies the affinely parametrized geodesic equation, $\ell^a \nabla_a \ell^b = 0$. If $\alpha = \alpha(\lambda)$ on an orbit of k^a:

$$k|_{\text{orbit}} = k^a \partial_a|_{\text{orbit}} = \frac{d\lambda}{d\alpha}\frac{d}{d\lambda}|_{\text{orbit}} = f\ell\,, \tag{129}$$

where $f := \frac{d\lambda}{d\alpha}$ and $\ell^a \partial_a = \frac{d}{d\lambda}$. Using $\kappa = \frac{d}{d\alpha}\log|f|$ and $\frac{\partial\kappa}{\partial\alpha} = 0$ (see proposition above), then $\frac{d\lambda}{d\alpha} = f(\alpha) = f_0 e^{\kappa\alpha}$. The constant f_0 can be fixed by using the freedom to shift α by a constant, $\alpha \to \tilde{\alpha}$:

$$f(\tilde{\alpha}) = f(\alpha + \kappa^{-1}\log|k/f_0|) = \pm\kappa e^{\kappa\alpha} \to \frac{d\lambda}{d\tilde{\alpha}} = \frac{d\lambda}{d\alpha} = \pm\kappa e^{\kappa\alpha}\,. \tag{130}$$

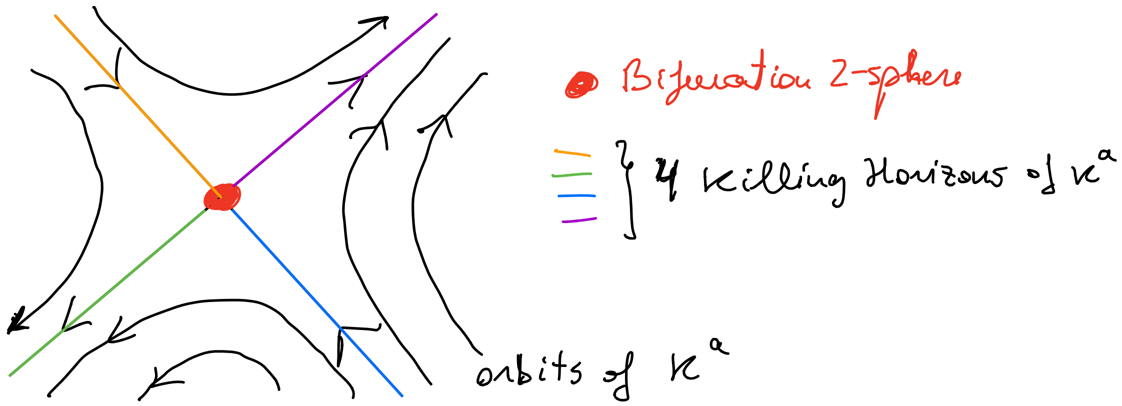

Figure 12: Bifurcate killing horizon of the Kruskal spacetime.

Solving this equation yields $\lambda = \pm e^{\kappa\alpha} + C$, for some constant C. We can set $C = 0$ without loss of generality. Then $\lambda(\alpha) = \pm e^{\kappa\alpha}$. As α ranges from $-\infty$ to ∞ we cover the $\lambda > 0$ or $\lambda < 0$ portions of the orbit of $x^a(\lambda)$ in $\mathcal{N}$, but not both at the same time. □

The proof of the previous proposition shows that $\lambda = 0$ is a fixed point of the orbits of the KVF k^a. We call it a bifurcation 2-sphere (Boyer-Kruskal axis in Kruskal spacetime).

Definition. The set comprised by the bifurcation 2-sphere and $\{U > 0, V = 0\} \cup \{U < 0, V = 0\} \cup \{U = 0, V > 0\} \cup \{U = 0, V < 0\}$ is called a bifurcate killing horizon.

We proved in one of the propositions above that the surface gravity κ is constant in each of these 4 branches of $\mathcal{N}$. Is the same constant value for the 4 branches?

Proposition. If $\mathcal{N}$ is a bifurcate killing horizon of k^a, with bifurcation 2-sphere $\mathcal{B}$, then κ^2 is constant on the full $\mathcal{N}$.

Proof. As we saw in one of the propositions above, κ^2 is constant on each orbit of the killing vector field k^a, and the value of this constant is the value of κ^2 at the limit point of the orbit on $\mathcal{B}$, so κ^2 will be constant on $\mathcal{N}$ iff it is constant on the 2-sphere $\mathcal{B}$. Since $k^a|_{\mathcal{B}} = 0$ we infer from equation (127) that $t^a \nabla_a \kappa^2 = 0$ on the full bifurcation 2-sphere $\mathcal{B}$. And because t^a can be any tangent vector field on $\mathcal{B}$, κ^2 is indeed constant on this 2-sphere. Hence it is constant on the full bifurcate killing horizon $\mathcal{N}$, as just argued. □

Example. The bifurcate killing horizon in the Kruskal spacetime is given by $\mathcal{N} = \{U = 0\} \cup \{V = 0\}$. This is a null hypersurface, as we have previously checked. The associated KVF is given by (113), which is the generator of time translations. On $\mathcal{N}$ we have

$$k = \begin{cases} \frac{1}{4M} V \frac{\partial}{\partial V} & \text{on} \quad \{U = 0\} \\ -\frac{1}{4M} U \frac{\partial}{\partial U} & \text{on} \quad \{V = 0\} \end{cases} = f\ell\,, \tag{131}$$

where

$$f = \begin{cases} \frac{V}{4M} & \text{on} \quad \{U = 0\} \\ -\frac{U}{4M} & \text{on} \quad \{V = 0\} \end{cases} \tag{132}$$

and

$$\ell = \begin{cases} \frac{\partial}{\partial V} & \text{on} \quad \{U = 0\} \\ \frac{\partial}{\partial U} & \text{on} \quad \{V = 0\} \end{cases} \tag{133}$$

Therefore, k is normal to $\mathcal{N}$. In conclusion, $\mathcal{N}$ is a killing horizon of k^a.

If $x^a(\lambda)$, with $\lambda \in \{U, V\}$, are the generators of $\mathcal{N}$, then $\ell^a = \frac{dx^a}{d\lambda}$ and $\ell^a \nabla_a \ell^b = 0$. Then the surface gravity can be calculated as

$$\kappa = k^a \partial_a \log |f| = \begin{cases} \frac{1}{4M} V \frac{\partial}{\partial V} \log \frac{V}{4M} = \frac{1}{4M} & \text{on} \quad \{U = 0\} \\ -\frac{1}{4M} U \frac{\partial}{\partial U} \log \frac{U}{4M} = -\frac{1}{4M} & \text{on} \quad \{V = 0\} \end{cases} \tag{134}$$

so $\kappa^2 = \frac{1}{(4M)^2}$ is indeed a constant on $\mathcal{N}$. The orbits of k^a lie either entirely in $\{U = 0\}$ or in $\{V = 0\}$, and are fixed points on $\mathcal{B}$. This is the reason behind the two different signs obtained for κ on the two branches of $\mathcal{N}$.

Remark 1: on the normalization of k^a.

If $\mathcal{N}$ is a killing horizon of the killing vector field k^a with surface gravity κ, then $\mathcal{N}$ is also a killing horizon of the killing vector field ck^a with surface gravity $c\kappa$, where $c \in \mathbb{R}$ is any constant. This shows that the surface gravity κ is not only a property of the killing horizon $\mathcal{N}$ alone, but it also depends on the normalization of the killing vector field k^a. In general spacetimes, there is no natural normalization of k^a, since $k^a k_a = 0$ on $\mathcal{N}$ by definition. However, in asymptotically flat spacetimes (to be specified in detail in the next subsection), there is a canonical normalization at (spatial) infinity, $r \to \infty$ (keeping t, θ, ϕ constant). To give an example, in Kruskal spacetime the time translation KVF k^a is normalized according to $k_a k^a (r \to \infty) = -1$. This is, $k = \frac{\partial}{\partial t}$. It is easy to check that this gives $\kappa = \frac{1}{4M}$. See also exercise 20 of Exercise List.

Remark 2: Event horizon vs Killing horizon.

The notion of killing horizon differs from the notion of event horizon (to be given in more precise terms in the next subsection). *Hawking Rigidity Theorem:* "in a stationary, asymptotically flat black hole spacetime, the event horizon is a killing horizon for some killing vector field". However, we can find killing horizons not related to any event horizon. For example, in Minkowski space ($\mathbb{R}^4$, $ds^2 = -dt^2 + d\vec{x}^2$) there are no event horizons, but the null hypersurfaces $\mathcal{N}_\pm = \{x \pm t = 0\}$ are killing horizons of the KVF that generates boosts in, for instance, the x-direction, $\chi = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}$. Indeed, $\chi^a \chi_a = -x^2 + t^2$, so:

χ^a is timelike for $|x| > t$,

χ^a is null for $|x| = t$,

χ^a is spacelike for $|x| < t$.

This is not even the unique killing horizon that we can find: $k = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$ is null everywhere and is a killing vector field, so $\{x \pm t + c = 0\}$ are all killing horizons for any $c \in \mathbb{R}$.

2.7 Asymptotic flatness and Carter-Penrose diagrams.

In several instances during this section 2 we have mentioned that the Schwarzschild metric is "asymptotically flat", i.e. that for sufficiently large distances away from the horizon, the metric approaches, in some vague sense, the metric of Minkowski spacetime. This property was easy to identify here because we worked in a particular coordinate system that is easy to interpret physically. This intuitive notion can actually be formalized in a more geometric manner, independently of the choice of coordinates, and is in fact useful to study any isolated source, such as stars, black holes or binary systems.

In astrophysics one is often interested in describing the individual gravitational field of isolated sources in order to, for instance, analyze the gravitational wave flux emitted by the source at, ideally, infinity. For this purpose, a rigorous mathematical and geometrical scheme is needed in general relativity that ignores the influence of distant matter on this spacetime, and that manages to approach smoothly the Minkowski spacetime without knowledge of any preferred coordinate system. This is precisely the framework of asymptotically flat spacetimes, for which the metric becomes flat at large distances from the source in a precise and controlled manner. These class of spacetimes represent, ideally, isolated systems in general relativity.

What does "large" distance mean? A notion of asymptotic flatness (AF) necessarily requires knowledge of the global picture of the spacetime. Therefore, to study these class of spacetimes, we need to provide first of all a precise definition of the notion of infinity.

Let (M, g_{ab}) be a generic spacetime. Intuitively, infinity can be thought of as those points that can be asymptotically reached by following geodesics $\gamma(\lambda)$ of infinite affine parametrization, so that we can asymptotically reach these points by making $\lambda \to \infty$. In practical calculations, to work around infinity it is convenient to bring these points to a finite location. Since the causal structure of the spacetime, determined by the causal character of curves, remains invariant under conformal transformations (see exercise 16 of Exercise List)

$$g_{ab}(x) \to \tilde{g}_{ab}(x) = \Lambda^2(x) g_{ab}(x), \quad \Lambda(x) \neq 0, \quad x \in M\,, \tag{135}$$

the key idea is to build a new (unphysical) metric by choosing Λ such that all points that are at infinite affine length following null geodesics in the original metric, are located at finite affine parameter in the new metric.

This can be done as follows. Consider a null geodesic $\tilde{\gamma}$ with respect to the unphysical metric $\tilde{g}_{ab}$, with affine parametrization $\tilde{\lambda}$. In coordinates the geodesic equation reads

$$\frac{d^2x^a}{d\tilde{\lambda}^2} + \tilde{\Gamma}^a_{bc}\frac{dx^b}{d\tilde{\lambda}}\frac{dx^c}{d\tilde{\lambda}} = 0\,. \tag{136}$$

It is not difficult to see that

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \Lambda^{-1}(2\nabla_{(b}\Lambda\delta^a_{c)} - g_{bc}\nabla^a\Lambda)\,, \tag{137}$$

$$\frac{dx^a}{d\tilde{\lambda}} = \frac{d\lambda}{d\tilde{\lambda}}\frac{dx^a}{d\lambda}\,, \tag{138}$$

$$\frac{d^2x^a}{d\tilde{\lambda}^2} = \frac{d^2\lambda}{d\tilde{\lambda}^2}\frac{dx^a}{d\lambda} + \left[\frac{d\lambda}{d\tilde{\lambda}}\right]^2\frac{d^2x^a}{d\lambda^2}\,, \tag{139}$$

therefore

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = -\frac{1}{\lambda'}\left[\frac{\lambda''}{\lambda'} + 2\frac{\Lambda'}{\Lambda}\right]\frac{dx^a}{d\lambda} = -\frac{1}{\lambda'}\frac{d\log(\lambda'\Lambda^2)}{d\tilde{\lambda}}\frac{dx^a}{d\lambda}\,. \tag{140}$$

Notice how null geodesics for $\tilde{g}_{ab}$ transform to null geodesic for g_{ab}, but this is not true in general for timelike or spacelike geodesics. If we now impose λ to be an affine parameter, then

$$\frac{d\lambda}{d\tilde{\lambda}} = \frac{c}{\Lambda^2}\,, \tag{141}$$

for some constant $c \in \mathbb{R}$. The solution is

$$\lambda(\tilde{\lambda}) = c\int^{\tilde{\lambda}} \frac{dx}{\Lambda(x)}\,. \tag{142}$$

To identify the points $x^a(\tilde{\lambda})$ of "infinity" we need to take $\lambda \to \infty$ while keeping $\tilde{\lambda} < \infty$. This requires $\Lambda(x^a(\lambda)) \to 0$ as $\lambda \to \infty$. Since $\Lambda(x) \neq 0$ for all $x \in M$ by construction, these points are not part of our original spacetime manifold. Therefore, we will define a new set of points, $p \in \mathscr{J}$, called "the points of infinity", that will be characterized by the property $\Lambda(p) = 0$. These points are not part of the original spacetime manifold, $p \notin M$, but can be attached smoothly to form a larger (unphysical) spacetime manifold $\tilde{M}$. We call this a conformal compactification of the spacetime:

$$\begin{array}{ccc} \text{Physical spacetime} & & \text{Unphysical spacetime} \\ (M, g_{ab}) & \longrightarrow & (\tilde{M} = M \cup \mathscr{J}, \tilde{g}_{ab} = \Lambda^2 g_{ab}) \end{array}$$

This mathematical operation brings the points of infinity to a finite distance, and attaches them to the physical manifold as a boundary. The points of infinite actually constitute a hypersurface of $\tilde{M}$:

$$\mathscr{J} = \{p \in \tilde{M} \,/\, \Lambda(p) = 0\} = \partial\tilde{M}\,. \tag{143}$$

The new manifold $\tilde{M}$ is called the conformal extension of the physical spacetime manifold M, and the new metric $\tilde{g}_{ab}$ is smooth in all $\tilde{M}$.

Notice that curvature tensors are not preserved by conformal transformations in general ($\tilde{R}^a{}_{bcd} \neq R^a{}_{bcd}$), with the only exception of the Weyl tensor. Consequently, $(\tilde{M}, \tilde{g}_{ab})$ is an unphysical spacetime, it is only a good representation of the causal structure of the physical spacetime (M, g_{ab}).

Since the process of conformal compactification may look abstract at first glance, we will illustrate how it works with basic examples.

Example 1: conformal compactfication of Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$.

Let us write the spacetime metric in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{144}$$

where $t \in \mathbb{R}$, $r > 0$, $(\theta, \phi) \in \mathbb{S}^2$. Since radial null geodesics follow $u, v =$ const curves, with $u = t - r \in \mathbb{R}$ and $v = t + v \in \mathbb{R}$ the retarded and advanced times, respectively, let us write the metric in these coordinates:

$$ds^2 = -dudv + \frac{(u-v)^2}{4}d\Omega^2\,, \tag{145}$$

where $d\Omega^2$ is shorthand for $d\theta^2 + \sin^2\theta d\phi^2$. To bring infinity to a finite location, let us compactify these null coordinates via the coordinate transformation

$$u = \tan\tilde{U} \quad \to \quad -\frac{\pi}{2} < \tilde{U} < \frac{\pi}{2}\,, \tag{146}$$

$$v = \tan\tilde{V} \quad \to \quad -\frac{\pi}{2} < \tilde{V} < \frac{\pi}{2}\,. \tag{147}$$

Because the domain of the coordinate functions $\tilde{U}$, $\tilde{V}$ is bounded, we call them "compactified null coordinates". Since $r \geq 0$, we have $v \geq u$, $\tan\tilde{V} \geq \tan\tilde{U}$ and therefore $\tilde{V} \geq \tilde{U}$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Doing some algebra we can find:

$$-dudv \;=\; -\left(1 + \tan^2\tilde{U}\right)\left(1 + \tan^2\tilde{V}\right) d\tilde{U}d\tilde{V} = -\frac{d\tilde{U}d\tilde{V}}{\cos^2\tilde{V}^2\cos^2\tilde{V}}\,, \tag{148}$$

$$u - v = \tan\tilde{U} - \tan\tilde{V} = \frac{\sin\tilde{U}\cos\tilde{V} - \sin\tilde{V}\cos\tilde{U}}{\cos\tilde{V}\cos\tilde{V}} = \frac{\sin(\tilde{U}-\tilde{V})}{\cos\tilde{U}\cos\tilde{V}}\,, \qquad (149)$$

so the metric takes the form:

$$ds^2 = \frac{1}{4\cos^2\tilde{U}\cos^2\tilde{V}}\left[-4d\tilde{U}d\tilde{V} + \sin^2(\tilde{U}-\tilde{V})d\Omega^2\right]. \qquad (150)$$

Approaching infinity in the original spacetime, $u \to \pm\infty$ or $v \to \pm\infty$, implies here $|\tilde{U}| \to \frac{\pi}{2}$ or $|\tilde{V}| \to \frac{\pi}{2}$, which is at finite distance in the compactified coordinates. However, in this limit the metric blows up because of the prefactor. If we build a new metric $\tilde{g}_{ab} = \Lambda^2 g_{ab}$ and choose a conformal factor $\Lambda(\tilde{U},\tilde{V}) = 2\cos\tilde{U}\cos\tilde{V}$, we will be able to bring points of infinity to a finite affine parameter distance in the new metric $\tilde{g}_{ab}$.

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -4d\tilde{U}d\tilde{V} + \sin^2(\tilde{U}-\tilde{V})d\Omega^2\,. \qquad (151)$$

Notice that this new unphysical metric is smooth when $|\tilde{U}|,|\tilde{V}| \to \frac{\pi}{2}$. Therefore, it can be extended smoothly to a larger manifold that includes the points of infinity.

Let us add now the "points of infinity" to the original manifold $\mathbb{R}^4$. Due to the inherent causal structure of the spacetime, we have different types infinities: (keep in mind that $\tilde{V} \geq \tilde{U}$, $r = \frac{v-u}{2}$, and $t = \frac{v+u}{2}$)

$$\left(\tilde{U} = -\frac{\pi}{2}, \tilde{V} = \frac{\pi}{2}\right) \longleftrightarrow (u \to -\infty, v \to +\infty) \longleftrightarrow (r \to \infty,\ t\ \text{finite}) \qquad (152)$$

$$\left(\tilde{U} = \pm\frac{\pi}{2}, \tilde{V} = \pm\frac{\pi}{2}\right) \longleftrightarrow (u \to \pm\infty, v \to \pm\infty) \longleftrightarrow (r\ \text{finite}, t \to \pm\infty) \qquad (153)$$

$$\left(\tilde{U} = -\frac{\pi}{2}, |\tilde{V}| \neq \pm\frac{\pi}{2}\right) \longleftrightarrow (u \to -\infty,\ v\ \text{finite}) \longleftrightarrow (r \to \infty, t \to -\infty)/\, r + t\ \text{finite} \qquad (154)$$

$$\left(|\tilde{U}| \neq \frac{\pi}{2}, \tilde{V} = \frac{\pi}{2}\right) \longleftrightarrow (u\ \text{finite}, v \to \infty) \longleftrightarrow (r \to \infty, t \to \infty)/\, r - t\ \text{finite} \qquad (155)$$

The point indicated in equation (152) is called spatial infinity (and denoted by i_0), the two points of (153) are called future(+)/past(-) timelike infinity (denoted by $i^\pm$), the set of points in (154) are called past null infinity (denoted by $\mathscr{J}^-$), and finally the set of points in (155) are called future null infinity (denoted by $\mathscr{J}^+$).

The process of conformal compactification consists then in embedding Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$ into the larger (but unphysical) spacetime $(\tilde{\mathbb{R}}^4, \tilde{\eta}_{ab}) \equiv (\mathbb{R}^4 \cup \mathscr{J}^+ \cup \mathscr{J}^- \cup i^+ \cup i^- \cup i_0, \Lambda^2\eta_{ab})$. Infinity in $(\mathbb{R}^4, \eta_{ab})$ is now the boundary $\Lambda = 0$ of $(\tilde{\mathbb{R}}^4, \tilde{\eta}_{ab})$, and can be reached by geodesics in finite distance.

Now the spacetime is compact. This, together with the underlying spherical symmetry, implies that the full causal structure of the original spacetime can be faithfully represented on a 2-dimensional, bounded picture. To see this, let

$$\tau := \tilde{V} + \tilde{U}\,, \qquad (156)$$

$$\chi := \tilde{V} - \tilde{U}\,, \qquad (157)$$

be new coordinates, with domain in $-\pi \leq \tau \leq \pi$ and $0 \leq \chi \leq \pi$, respectively (recall that $\tilde{V} \geq \tilde{U}$). The unphysical metric is

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -d\tau^2 + d\chi^2 + \sin^2\chi d\Omega^2\,, \qquad (158)$$

with $\Lambda = \cos\tau + \cos\chi$. Since the metric is periodic in χ, $\chi \sim \chi + 2\pi$, χ can be thought of as an angular variable instead of a radial one. Equation (158) is the natural metric on

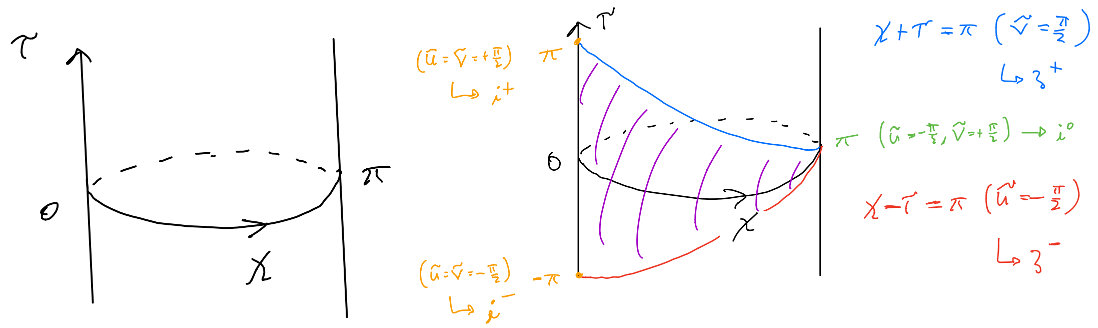

Figure 13: Isometric embedding of the compactified Minkowski space in the Einstein static universe.

the Einstein static universe (ESU), given by the manifold $\mathbb{R}\times\mathbb{S}^3$, where $\mathbb{R}$ stands for time τ and $\mathbb{S}^3$ stands for spatial sections coordinated by (χ, θ, ϕ). In other words, there exists a conformal isometry of Minkowski space $(\mathbb{R}^4, \eta_{ab})$ into the open region O of the ESU $(\mathbb{R}\times\mathbb{S}^3, \tilde{g}_{ab})$ given by (158). The conformal infinity of Minkowski space can be therefore identified with the boundary of this region O in the ESU.

It is possible to construct useful diagrammatic representations of this idea. First, notice that 2-spheres of constant τ and χ in (158) have radius $|\sin\chi|$ (except for $\chi=0$ and $\chi=\pi$, which are simply two points). If we represent each of these 2-spheres as points on a two-dimensional surface, we obtain the so-called ESU cylinder, see left figure of Figure 13. On this cylinder, the compactified Minkowski spacetime $(\tilde{\mathbb{R}}^4, \tilde{\eta}_{ab})$ is isometric to the region $-\pi\leq\tau\leq\pi$, $0\leq\chi\leq\pi$, see right figure of Figure†13.

If we now flatten this portion of the cylinder, we arrive at the Carter-Penrose diagram of the compactified Minkowski spacetime, see Figure 14. In this picture, each point except the vertical line (corresponding to the origin $r=0$) and the points $i^{\pm}$, i_0 (recall that $\chi=0,\pi$, i.e. zero radius for these cases), represents a 2-sphere. Some characteristic properties are:

(a) Ingoing null geodesics begin at past null infinity $\mathscr{I}^-$, while outgoing null geodesics end at future null infinity $\mathscr{I}^+$. They always travel at 45 degrees of inclination in the diagram. If they pass through the (arbitrary) origin of our frame $r=0$, they intersect the vertical line in the diagram.

(b) Both past and future null infinities are null hypersurfaces, as can be checked from their normal vector field: $\tilde{\eta}^{ab}\nabla_a\Omega\nabla_b\Omega|_{\Lambda=0}=0$. They are diffeomorphic to $\mathbb{R}\times\mathbb{S}^2$.

(c) All (inextendible) timelike geodesics begin at past timelike infinity i^- and end at future timelike infinity i^+. There can be non-geodesic timelike curves that end at null infinity, if they become asymptotically null.

(d) All (inextendible) spacelike geodesics both begin and end at spatial infinity i_0, though physically they are irrelevant.

(e) i_0, $i^{\pm}$ are actually points, not 2-spheres (since they correspond to $\chi=0,\pi$ in the Einstein static universe, which are the north and south poles of $\mathcal{S}^3$). Spatial sections of the compactified spacetime are diffemorphic to $\mathbb{R}^3\cup i_0\simeq\mathbb{S}^3$, and are therefore compact without boundary.

The boundary of this conformally compactified spacetime, consisting of the points i_0, $i^{\pm}$ and the null hypersurfaces $\mathscr{I}^{\pm}$, defines a precise notion of infinity for Minkowski space-

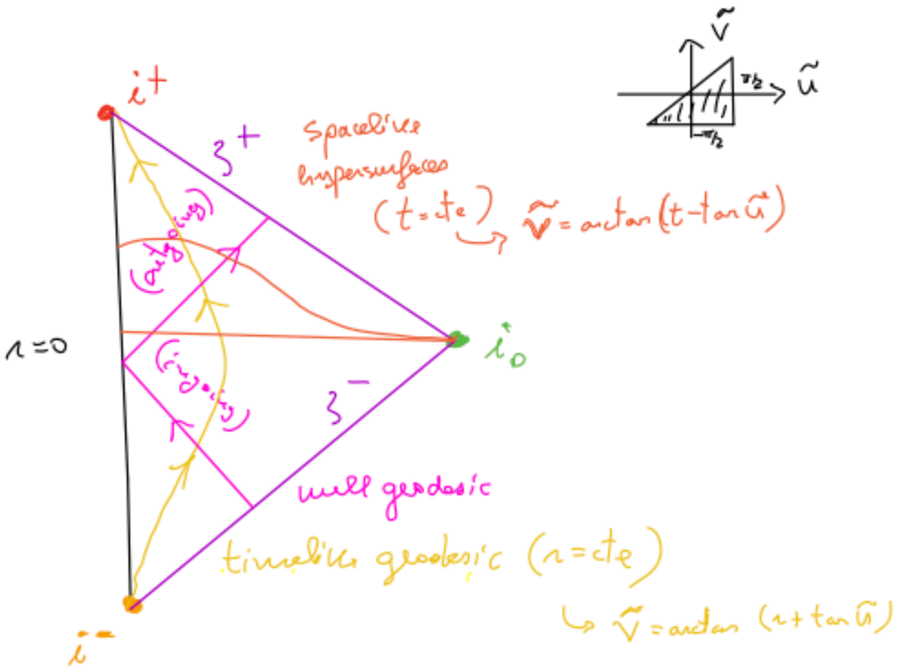

Figure 14: Carter-Penrose diagram for Minkowski spacetime.

time: it is called conformal infinity.

Example 2: conformal compactfication of Kruskal spacetime.

Following the same line of thought as with Minkowski, we start out by writing the Schwarzschild metric in Eddington-Finkelstein coordinantes (u, v, θ, ϕ), where now $u = t - r_* \in \mathbb{R}$, $v = t + r_* \in \mathbb{R}$, with r_* the tortoise radial coordinate. The line element was given in (85), and we recall that it only covers the exterior region outside the event horizon $r > 2M$, region I in the Kruskal diagram. This is enough, as we are only interested in the asymptotically far region away from the horizon.

Again, null geodesics follow $u, v =$ const curves. We introduce compactified null coordinates as before:

$$u = \tan \tilde{u} \quad \to \quad -\frac{\pi}{2} < \tilde{u} < \frac{\pi}{2}\,, \tag{159}$$

$$v = \tan \tilde{v} \quad \to \quad -\frac{\pi}{2} < \tilde{v} < \frac{\pi}{2}\,. \tag{160}$$

Since $r > 2M$, we have $r_* > 0$ and therefore $\tilde{v} \geq \tilde{u}$. Doing a similar calculation as in the previous example, the Schwarzschild line element (85) reads

$$ds^2 = \frac{1}{4\cos^2\tilde{u}\cos^2\tilde{v}}\left[-4\left(1 - \frac{2M}{r}\right)d\tilde{u}d\tilde{v} + 4r^2\cos^2\tilde{u}\cos^2\tilde{v}d\Omega^2\right], \tag{161}$$

with $r_* = \frac{v-u}{2} = \frac{\sin(\tilde{v}-\tilde{u})}{2\cos\tilde{u}\cos\tilde{v}}$. We define the conformally related metric by

$$d\tilde{s}^2 = \Lambda^2 ds^2 = -4\left(1 - \frac{2M}{r}\right)d\tilde{u}d\tilde{v} + \left(\frac{r}{r_*}\right)^2 \sin^2(\tilde{v} - \tilde{u})\, d\Omega^2\,. \tag{162}$$

This expression approaches the conformally compactified Minkowski metric (151) as $r \to \infty$. As a consequence, we have an identical structure for conformal infinity. Namely, we

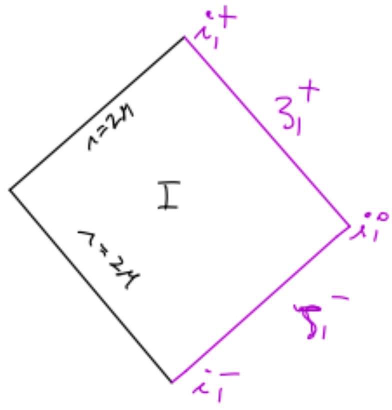

Figure 15: Carter-Penrose diagram for the exterior Schwarzschild black hole (region I in the Kruskal diagram). The asymptotic structure (in purple) is identical to Minkowski spacetime.

have a point for spatial infinity i_1^0, two points for past and future timelike infinities $i_1^{\pm}$, as well as two null hypersurfaces corresponding to past and future null infinities, $\mathscr{J}_1^{\pm}$. All these new points can be attached to the original Schwarzschild manifold in exactly the same way as we did in the example above for Minkowski. The result can be seen in Figure 15. Notice that, in contrast to Minkowski space, not all timelike geodesics terminate at i_1^+, as some of them penetrate through the horizon.

Figure 15 only covers the exterior of the Schwarzschild horizon. To get a full picture, we need to move to Kruskal coordinates (U, V, θ, ϕ), where $U, V \in \mathbb{R}$. The Kruskal metric takes the form given in (88) with (89), and covers all regions in Kruskal diagram 7, i.e. $r > 0$. Again, let us introduce compactified coordinates:

$$U = \tan \tilde{U} \quad \rightarrow \quad -\frac{\pi}{2} < \tilde{U} < \frac{\pi}{2} \, , \tag{163}$$

$$V = \tan \tilde{V} \quad \rightarrow \quad -\frac{\pi}{2} < \tilde{V} < \frac{\pi}{2} \, . \tag{164}$$

Note that, in contrast to the previous case, now $\tilde{V} < \tilde{U}$ is allowed! The Kruskal line element in these coordinates is now:

$$ds^2 = \frac{1}{\cos^2 \tilde{U} \cos^2 \tilde{V}} \left[-\frac{32M^3}{r} e^{-\frac{r}{2M}} d\tilde{U} d\tilde{V} + r^2 \cos^2 \tilde{U} \cos^2 \tilde{V} \, d\Omega^2 \right] , \tag{165}$$

and the conformally related metric $d\tilde{s}^2 = \Lambda^2 ds^2$ is obtained with the conformal factor $\Lambda = \cos \tilde{U} \cos \tilde{V}$.

The Killing horizon is located at $r = 2M$, i.e. at points where $UV = 0$. This corresponds to points where $U = 0$ or $V = 0$. In the compactified coordinates, these points satisfy $\tilde{U} = 0$ or $\tilde{V} = 0$. On the other hand, the curvature singularity is located at $r = 0$, i.e. at points for which $UV = 1$. This leads to $\tan \tilde{U} \tan \tilde{V} = 1$, which is satisfied iff $\sin \tilde{U} \sin \tilde{V} = \cos \tilde{U} \cos \tilde{V}$, or, equivalently, $\cos(\tilde{U} + \tilde{V}) = 0$. Taking into account the domain of $\tilde{U}$ and $\tilde{V}$, the solution is $\tilde{U} + \tilde{V} = \pm \frac{\pi}{2}$.

As in Minkowski, the points of infinity correspond to $|\tilde{U}| = \frac{\pi}{2}$, $|\tilde{V}| = \frac{\pi}{2}$, but unlike Minkowski space, $\tilde{V} < \tilde{U}$ is allowed. Therefore, conformal infinity is “larger” in the

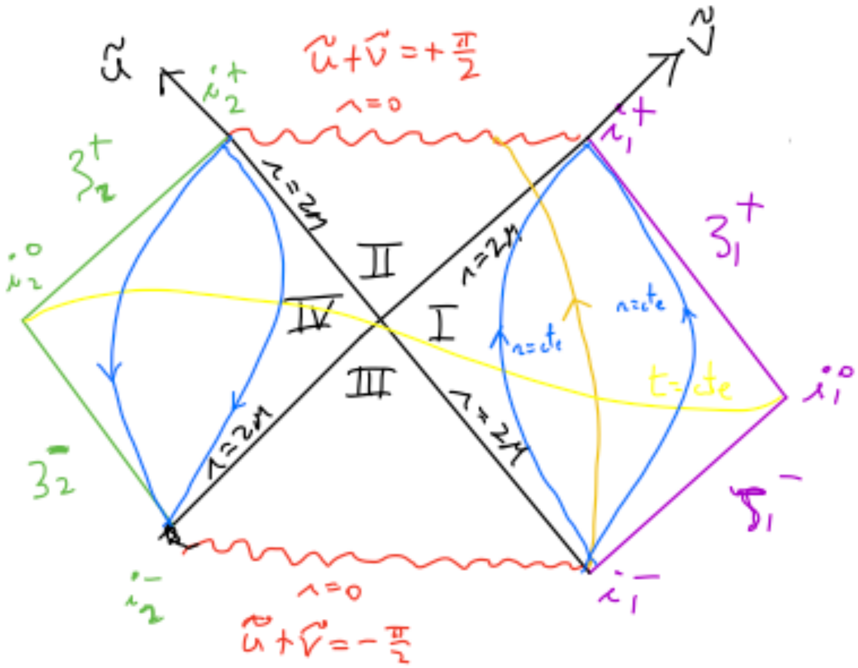

Figure 16: Carter-Penrose diagram for the Kruskal spacetime.

Kruskal spacetime. A similar analysis as in Minkowski, where $\tilde{V} \geq \tilde{U}$, yields the points at infinity

$$\mathscr{I}_1^+ = \left\{\tilde{V} = \frac{\pi}{2}\right\}, \quad \mathscr{I}_1^- = \left\{\tilde{U} = -\frac{\pi}{2}\right\}, \quad i_1^0 = \left\{\tilde{U} = -\frac{\pi}{2}, \tilde{V} = \frac{\pi}{2}\right\}, \tag{166}$$

$$i_1^+ = \left\{\tilde{U} = 0, \tilde{V} = \frac{\pi}{2}\right\} \quad i_1^- = \left\{\tilde{U} = -\frac{\pi}{2}, \tilde{V} = 0\right\}. \tag{167}$$

On the other hand, the new points of infinity allowed by the condition $\tilde{V} < \tilde{U}$, and not present in Minkowski, are given by (exchange $\tilde{V} \leftrightarrow \tilde{U}$)

$$\mathscr{I}_2^+ = \left\{\tilde{U} = \frac{\pi}{2}\right\}, \quad \mathscr{I}_2^- = \left\{\tilde{V} = -\frac{\pi}{2}\right\}, \quad i_2^0 = \left\{\tilde{V} = -\frac{\pi}{2}, \tilde{U} = \frac{\pi}{2}\right\}, \tag{168}$$

$$i_2^+ = \left\{\tilde{V} = 0, \tilde{U} = \frac{\pi}{2}\right\}, \quad i_2^- = \left\{\tilde{V} = -\frac{\pi}{2}, \tilde{U} = 0\right\}. \tag{169}$$

What do these points correspond to? Recall that region IV in the Kruskal diagram 7 is isometric to region I. The isometry is provided by a time reversal transformation: $(U, V) \to (V, U)$. Thus, the second set of points at infinity obtained above constitute, precisely, the conformal infinity structure of region IV in the Kruskal diagram. The full Carter-Penrose diagram for Kruskal spacetime is displayed in Figure 16. Note that the singular points $r = 0$ are not part of the boundary of the unphysical manifold.

Example 3: conformal compactification of a spherically-symmetric gravitational collapse.

Due to Birkhoff's theorem, the exterior spacetime is determined by the Schwarzschild line element. This is the only relevant part of the manifold needed for analyzing the asymptotic distant structure. We already studied this case in the previous example. Therefore, the structure of conformal infinity can be borrowed from the Carter-Penrose diagram of Kruskal, in Figure 16. On the other hand, the interior region depends on the particular

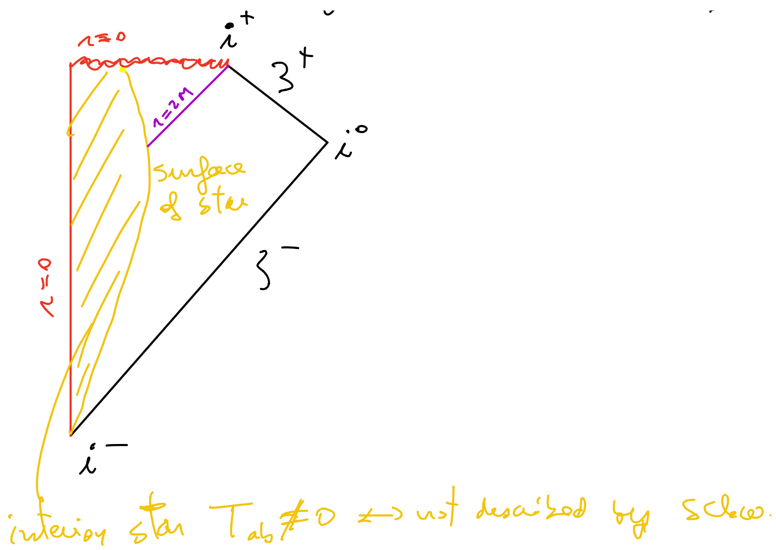

Figure 17: Carter-Penrose diagram for a spherically symmetric gravitational collapse. The exterior diagram is isometric to region I of the Kruskal diagram in Figure 16.

physics underlying the star, but it is nevertheless causally equivalent to Minkowski space. The Carter-Penrose diagram for the gravitational collapse is displayed in Figure 17.

3 The Kerr black hole solution

The Penrose-Hawking singularity theorems show that gravitational collapse is a generic feature of general relativity, not simply a peculiarity of spherical symmetry. Consider a body which undergoes complete gravitational collapse and forms a black hole. We can have two different cases:

(a) Non-rotating body: spherically symmetric case.

By Birkhoff's theorem the spacetime outside the collapsing body is always the Schwarzschild spacetime. Therefore, the final state of the collapse is always a Schwarzschild black hole, i.e. a stationary, non-rotating vacuum solution $R_{ab} = 0$ with an event horizon. This is the case studied in detail in section 2.

(b) Rotating body: non-spherical case (as rotation breaks spherical symmetry).

No analogous theorem exists: the spacetime outside the collapsing body will in general depend on the details of the collapse and can in fact vary with time. Despite that, the spacetime should settle down at some point to an equilibrium state. What is this final stationary state after the collapse?

Since the spacetime has some degree of rotation, the answer cannot be another Schwarzschild solution. This indicates that there should be additional black hole solutions within general relativity. Our goal in this section is to find stationary, rotating black hole solutions of the Einstein's vacuum field equations.

3.1 The Kerr-Newmann solution. Black holes uniqueness theorems.

We assume no spherical symmetry anymore. Since solving the vacuum Einstein's field equations without any hypothesis is actually a highly non-trivial problem, we will restrict to stationary solutions with axial symmetry around the axis of rotation.

Definition. An asymptotically flat spacetime is said to be stationary iff it admits a timelike killing vector field k^a (timelike, at least, near spatial infinity).

A stationary spacetime admits a coordinate system where $k = \frac{\partial}{\partial t}$, where t is a time coordinate. If it is asymptotically flat, there is furthermore a canonical normalization $k^a k_a|_{r\to\infty} \to -1$. By imposing this, t becomes the natural proper time measured by static observers at spatial infinity. Since k^a is a KVF, $\mathcal{L}_k g_{ab} = \frac{\partial}{\partial t} g_{ab}(t, \vec{x}) = 0$. Therefore, a stationary metric has the general form

$$g_{ab}(t, \vec{x}) dx^a dx^b = g_{tt}(\vec{x}) dt^2 + 2 g_{ti}(\vec{x}) dt dx^i + g_{ij}(\vec{x}) dx^i dx^j \,. \tag{170}$$

Definition. A spacetime is said to be static if it is stationary and furthermore there exists a spacelike hypersurface Σ which is orthogonal to the orbits of k^a.

If the spacetime is static, it must be invariant under time reversal $t \to -t$. Imposing $ds^2(t, \vec{x}) = ds^2(-t, \vec{x})$ leads to $g_{ti} dt dx^i = -g_{ti} dt dx^i$ and therefore $g_{ti} = 0$. Thus, the general form for a static metric is

$$g_{ab}(t, \vec{x}) dx^a dx^b = g_{tt}(\vec{x}) dt^2 + g_{ij}(\vec{x}) dx^i dx^j \,. \tag{171}$$

Definition. An asymptotically flat spacetime is axisymmetric if there exists a spacelike KVF m^a for which all integral curves are closed. Such vector field m^a is called an axial KVF.

An axisymmetric spacetime admits a coordinate system such that $m = \frac{\partial}{\partial\phi}$, with the equivalence relation $\phi \sim \phi + 2\pi$ (reflecting the closedness of the orbits of m^a). Again, if the spacetime is also asymptotically flat then we have a canonical normalization, $m^a m_a|_{r\to\infty} \to r^2$.

Israel's theorem. If (M, g_{ab}) is an asymptotically flat, static, vacuum spacetime that is non-singular on and outside an event horizon, then (M, g_{ab}) is the Schwarzschild (Kruskal) black hole solution, and therefore it is spherically symmetric.

This theorem shows that the static property is actually a quite strong requirement to study horizons with rotation. We must therefore give up with static spacetimes. We can relax this condition and still find interesting results:

Carter-Robinson theorem. If (M, g_{ab}) is an asymptotically flat, stationary and axisymmetric vacuum spacetime that is non-singular on and outside an event horizon, then (M, g_{ab}) is a member of the 2-parameter Kerr family:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2 - \frac{4Mra\sin^2\theta}{\Sigma}dtd\phi\,, \dagger \quad (172)$$

where the metric is written in so-called Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$, with $t \in \mathbb{R}$, $r \in (0, \infty)$, $(\theta, \phi) \in \mathbb{S}^2$, and

$$\Sigma = r^2 + a^2\cos^2\theta\,, \quad (173)$$
$$\Delta = r^2 - 2Mr + a^2\,. \quad (174)$$

This is a 2-parameter $\{M, a\}$ solution of the vacuum Einstein's field equations: $R_{ab}[g] = 0$. The parameter $a \equiv J/M$, with dimensions of length (when $G = c = 1$) is called the Kerr parameter, while M is the ADM mass. Without loss of generality, we can restrict to $a \geq 0$ (the transformation $\phi \to -\phi$ changes sign of a), with $a = 0$ recovering the Schwarzschild solution.

The Carter-Robinson theorem (valid for vacuum solutions $R_{ab} = 0$) can be generalized to the Einstein-Maxwell field equations ($G_{ab} = 8\pi T^{\mathrm{EM}}_{ab}$): stationary electrovacuum black holes must belong to the 3-parameter family $\{M, a, e\}$ of Kerr-Newman solutions:

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 + \frac{\Sigma}{\Delta}dr^2 + \frac{\left(r^2 + a^2\right)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \Sigma d\theta^2 - 2a\sin^2\theta\frac{r^2 + a^2 - \Delta}{\Sigma}dtd\phi\,,$$

$$A = \frac{er(dt - a\sin^2\theta d\phi)}{\Sigma}\,,$$

where A is the electromagnetic potential 1-form, and

$$\Sigma = r^2 + a^2\cos^2\theta\,, \quad (175)$$
$$\Delta = r^2 - 2Mr + a^2 + e^2\,. \quad (176)$$

The 2-paramter Kerr family is obtained as a particular case when the electric charge vanishes $e = 0$, i.e. $\{M, a, 0\}$. Similarly, the 2-parameter Reissner-Nordstrom family is obtained when there is no rotation, $a = 0$, i.e. $\{M, 0, e\}$.

It is difficult for any astrophysical body to get or maintain net electric charge. To see this, let us evaluate the ratio between the gravitational and electric forces:

$$\frac{F_e}{F_g} \sim \frac{\frac{1}{4\pi}\frac{qe}{r^2}}{\frac{mM}{r^2}} = \frac{qe}{mM} \sim \pm 10^{18}\frac{|e|}{M}\,. \tag{177}$$

If $|e| \sim M$, F_e dominates over F_g and the astrophysical body selectively attracts particles of opposite charge until it becomes neutral. If $|e|/M \sim 10^{-18}$, then $F_e \sim F_g$ but this case is essentially $e \approx 0$ (1 single electron) for solar-mass objects $M = M_\odot$, which is a negligible electric charge for such a body.

The Carter-Robinson Uniqueness theorem ensures that if a black hole forms after gravitational collapse of a rotating star, then the spacetime is uniquely determined by the total mass M and angular momentum J of the original star. This is known as the no-hair theorem. In other words, all multipole moments of the gravitational field of the original star are radiated away in the form of gravitational waves during the collapse, except for the mass M (which is the electric-type monopole $\ell = 0$) and the angular momentum J (which is the magnetic-type dipole $\ell = 1$).

The Kerr metric has not been matched smoothly to any known form of matter T_{ab} for the interior of a physically realistic star (unlike the spherically symmetric case, i..e the Oppenheimer-Snyderl collapse of section 1). Still, the Ker metric is a good approximation to the metric outside a rotating star for large distances away from the star, where all multipole moments of the star, except the $\ell = 0$ and $\ell = 1$, are negligible. Therefore, it is astrophysically important!

Properties of the Kerr solution:

(1) The Kerr metric is a vacuum solution of the Einstein's field equations, i.e. it is possible to check that (172) satisfies $R_{ab}[g] = 0$. This is however a rather tedious calculation. We recommend the student to check it using a software algebra like xAct.

(2) The Kerr metric (172) is asymptotically flat. More precisely, $g_{ab} \to \eta_{ab}$ as we follow either outgoing/ingoing null geodesics to future/past null infinity; future/past-directed timelike geodesics to future/past timelike infinity; or spacelike geodesics to spatial infinity. To give an example, in Boyer-Lindquist coordinates it is easy to check that the limit $r \to \infty$ while keeping $\{t, \theta, \phi\}$ constant recovers the Minkowski metric in spherical coordinates.

(3) Near spatial infinity, for $r >> M, a$, the metric takes the approximate form

$$\begin{aligned} ds^2 = -\left[1 - \frac{2M}{r} + O(r^{-2})\right] dt^2 + \left[1 + \frac{2M}{r} + O(r^{-2})\right] dr^2 + \left[r^2 + O(r^0)\right] d\Omega^2 \\ - \left[\frac{4J}{r}\sin^2\theta + O(r^{-2})\right] dt d\phi\,. \end{aligned}$$

This is precisely the gravitational field exterior to a spinning sphere of constant density in the weak-field limit (so-called Lense-Thirring metric). The parameters M and J are identified with the total mass and angular momentum of the source.

(4) In Boyer-Lindquist coordinates, the Kerr metric is independent of $\{t, \phi\}$. This implies that $\{k = \frac{\partial}{\partial t}, m = \frac{\partial}{\partial \phi}\}$ are two killing vector fields. Therefore, the Ker metric is stationary and axisymmetric.

The Ker metric is further invariant under a reflection with respect to the equatorial plane, $\theta \to \pi - \theta$. It is also invariant under the transformation $(t, \phi) \to (-t, -\phi)$, which indicates that it is a spinning source. These are all symmetries expected for the geometry of a rotating body, not a spherically symmetric spacetime.

(5) The Kerr metric becomes static (i.e. invariant under $t \to -t$) iff $J = 0$ (no rotation). Therefore, the Kerr metric becomes the Schwarzschild black hole solution with mass M when $a = 0$ (no angular momentum and thus not rotating).

(6) In Boyer-Lindquist coordinates, the Kerr metric is singular when $\Sigma = 0$ or $\Delta = 0$.

(6.a) $\Sigma = r^2 + a^2 \cos^2\theta = 0$ iff $r = 0$ and $\theta = \frac{\pi}{2}$. It is possible to see that curvature invariants blow up on these points:

$$R^{abcd}R_{abcd} = \frac{48M^2(r^6 - 15a^2\cos^2\theta r^4 + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)}{\Sigma^6} \to \infty! \tag{178}$$

Therefore, these points constitute a real singularity, with a ring shape, analogous to the Schwarzschild $r = 0$ curvature singularity (for $a \to 0$ this ring reduces to the single point $r = 0$).

(6.b) $\Delta = (r - r_+)(r - r_-) = 0$ iff $r = r_\pm = M \pm \sqrt{M^2 - a^2}$. We have to distinguish two situations: (i) if $M^2 < a^2$ then $r_\pm \in \mathbb{C}$, so there are no singularities in the spacetime. This spacetime is unphysical for several reasons (existence of naked curvature singularities; existence of closed timelike curves, which is a global violation of causality; etc). (ii) if $M^2 \geq a^2$, the metric is singular at $r = r_\pm \in \mathbb{R}$, but curvature invariants do not blow up, these points just lead to coordinate singularities. As in the Schwarzschild case, this indicates the presence of an event horizon!

3.2 Principal null congruences and extension across the event horizon.

Since Boyer-Lindsquest coordinates are singular at points where $\Delta(r) = 0$, we need another coordinate system to extend the Kerr metric beyond $\Delta = 0$. As for the Schwarzschild black hole, we can tailor this coordinate system by analyzing the behaviour of ingoing null geodesics.

Let $u^a = \dot{x}^a = \frac{dx^a}{d\lambda} = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ represent the tangent vector field of geodesics in the Kerr spacetime, with affine parameter λ. The geodesic equation for null curves produces (see exercise 19 of Exercise List)

$$\Sigma^2\dot{t} = -a(aE\sin^2\theta - L) + \frac{r^2 + a^2}{\Delta}\left[E(r^2 + a^2) - aL\right], \tag{179}$$

$$\Sigma^2\dot{r} = \pm\sqrt{[E(r^2 + a^2) - aL]^2 - \Delta[(L - aE)^2 + Q]}, \tag{180}$$

$$\Sigma^2\dot{\theta} = \pm\sqrt{Q + \cos^2\theta\left(a^2E^2 - \frac{L^2}{\sin^2\theta}\right)}, \tag{181}$$

$$\Sigma^2\dot{\phi} = -\left[aE - \frac{L}{\sin^2\theta}\right] + \frac{a}{\Delta}\left[E(r^2 + a^2) - aL\right]. \tag{182}$$

In this system of equations there are 3 constants of motion: (i) E, which is a measure of the energy of the test particle, derived from the stationary KVF $k = \frac{\partial}{\partial t}$, (ii) L, which measures its angular momentum, derived from the axial KVF $m = \frac{\partial}{\partial \phi}$, and (iii) Q, called the Carter constant, which is related to the existence of a Killing tensor (see exercise 19 of Exercise List for details).

To solve these equations, let us choose $L = aE\sin^2\theta$, $Q = -(L - aE)^2$. Then

$$\dot{t} = \frac{E(r^2+a^2)}{\Delta}\,, \quad \dot{r} = \pm E\,, \quad \dot{\theta} = 0\,, \quad \dot{\phi} = a\frac{E}{\Delta}\,. \tag{183}$$

We can now redefine $\lambda \to \lambda/E$ to get rid of the constant E. The tangent vector fields for ingoing/outgoing principal congruence are then

$$\ell = \frac{\partial t}{\partial\lambda}\frac{\partial}{\partial t} + \frac{\partial r}{\partial\lambda}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial\lambda}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial\lambda}\frac{\partial}{\partial\phi} = \frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t} - \frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial\phi}\,, \tag{184}$$

$$k = \frac{\partial t}{\partial\lambda}\frac{\partial}{\partial t} + \frac{\partial r}{\partial\lambda}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial\lambda}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial\lambda}\frac{\partial}{\partial\phi} = \frac{r^2+a^2}{\Delta}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial\phi}\,. \tag{185}$$

Motivated by this result, let us introduce ingoing null coordinates $\{v, r, \theta, \chi\}$ by

$$v = t + r_*\,, \quad \chi = \phi + r_\#\,, \tag{186}$$

as well as outgoing null coordinates $\{u, r, \theta, \psi\}$ by

$$u = t - r_*\,, \quad \psi = \phi - r_\#\,, \tag{187}$$

where $dr_* = \frac{r^2+a^2}{\Delta}dr$ and $dr_\# = \frac{a}{\Delta}dr$. It is easy to see that in the Schwarzschild limit $a = 0$ these coordinates reduce to ingoing and outgoing Eddington-Finkelstein coordinates, respectively.

Proposition. In the coordinate system $\{v, r, \theta, \chi\}$ the Kerr metric (172) takes the form

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}dv^2 + \Sigma d\theta^2 + \frac{\left[(r^2+a^2)^2 - \Delta a^2\sin^2\theta\right]}{\Sigma}\sin^2\theta d\chi^2 \\ &+2dvdr - \frac{2a\sin^2\theta\left(r^2+a^2-\Delta\right)}{\Sigma}dvd\chi - 2a\sin^2\theta dr d\chi\,. \end{aligned} \tag{188}$$

This metric is not singular for points that satisfy $\Delta(r) = 0$, therefore $r = r_\pm$ are mere coordinate singularities in Boyer-Lindquist coordinates (analogous to the coordinate singularity $r = 2M$ of the Schwarzschild metric). Consequently, the Kerr metric can be extended across the (future) event horizon.

Proof. Using $dv = dt + \frac{r^2+a^2}{\Delta}dr$ and $d\chi = d\phi + \frac{a}{\Delta}dr$ we find

$$dt^2 = dv^2 + \left(\frac{r^2+a^2}{\Delta}\right)^2 dr^2 - \frac{2\left(r^2+a^2\right)}{\Delta}dvdr\,, \tag{189}$$

$$d\phi^2 = d\chi^2 + \frac{a^2}{\Delta^2}dr^2 - \frac{2a}{\Delta}d\chi dr\,, \tag{190}$$

$$dtd\phi = dvd\chi + \frac{a\left(r^2+a^2\right)}{\Delta^2}dr^2 - \frac{a}{\Delta}dvdr - \frac{r^2+a^2}{\Delta}drd\chi\,. \tag{191}$$

Therefore, the metric (172) reads

$$\begin{aligned} ds^2 = &\frac{a^2\sin^2\theta - \Delta}{\Sigma}\left(dv^2 + \left(\frac{r^2+a^2}{\Delta}\right)^2 dr^2 - \frac{2\left(r^2+a^2\right)}{\Delta}dvdr\right) + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &-2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}\left(dvd\chi + \frac{a\left(r^2+a^2\right)}{\Delta^2}dr^2 - \frac{a}{\Delta}dvdr - \frac{r^2+a^2}{\Delta}drd\chi\right) \end{aligned}$$

$$+\sin^2\theta\frac{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}{\Sigma}\left(d\chi^2+\frac{a^2}{\Delta^2}dr^2-\frac{2a}{\Delta}d\chi dr\right). \quad (192)$$

The terms in red simplify as

$$\begin{aligned}2\frac{\Delta-a^2\sin^2\theta}{\Delta\Sigma}\left(r^2+a^2\right)+2\frac{a^2}{\Delta\Sigma}\left[r^2+a^2-\Delta\right]\sin^2\theta &= \frac{2\left(r^2+a^2\right)}{\Sigma}-2\frac{a^2\sin^2\theta}{\Sigma}\\ &= \frac{2\left(r^2+a^2\cos^2\theta\right)}{\Sigma}=2\,.\end{aligned}$$

The terms in blue simplify as

$$\begin{aligned}&2a\sin^2\theta\frac{\left(r^2+a^2-\Delta\right)\left(r^2+a^2\right)}{\Delta\Sigma}-\frac{2a\sin^2\theta\left[\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta\right]}{\Delta\Sigma}\\ &\qquad=\frac{2a\sin^2\theta}{\Delta\Sigma}\left[-\Delta\left(r^2+a^2\right)+\Delta a^2\sin^2\theta\right]=-2a\sin^2\theta\,.\end{aligned}$$

and finally the terms in green provide

$$\begin{aligned}&\frac{\left(a^2\sin^2\theta-\Delta\right)\left(r^2+a^2\right)^2-2a^2\sin^2\theta\left(r^2+a^2\right)\left[\left(r^2+a^2\right)-\Delta\right]+a^2\sin^2\theta\left[(r^2+a^2)^2-\Delta a^2\sin^2\theta\right]}{\Delta^2\Sigma}\\ &\qquad=\frac{-\Delta\left(r^2+a^2\right)^2+2a^2\sin^2\theta\left(r^2+a^2\right)\Delta-\Delta a^4\sin^4\theta}{\Delta^2\Sigma}\\ &\qquad=\frac{-\Delta\left(r^2+a^2\right)\left(r^2+a^2-a^2\sin^2\theta\right)+a^2\Delta\sin^2\theta\left(-a^2\sin^2\theta+r^2+a^2\right)}{\Delta^2\Sigma}\\ &\qquad=-\frac{\Sigma}{\Delta}=-\frac{1}{\Delta}\left(r^2+a^2-a^2\sin^2\theta\right)\,.\end{aligned}$$

Plugging all these results in (192) we obtain (208). □

3.3 Killing horizons and surface gravity

Given the isometries of the Kerr metric mentioned above, a notion of killing horizon is available.

Proposition. The hypersurfaces $\{r=r_\pm=M\pm\sqrt{M^2-a^2}\}$ are killing horizons of the killing vector fields $\xi_\pm=k+\frac{a}{r_\pm^2+a^2}m=\frac{\partial}{\partial t}+\frac{a}{r_\pm^2+a^2}\frac{\partial}{\partial\phi}$, with surface gravities $\kappa_\pm=\frac{r_\pm-r_\mp}{2(r_\pm^2+a^2)}$

Proof. See exercise 20 of Exercise List.

Corollary. By Hawking's Rigidity theorem, $\{r=r_+\}$ is an event horizon.

In the last part of this section we will see that $\{r=r_-\}$ is another type of horizon, called Cauchy horizon.

3.4 Maximal extension of the Kerr spacetime

Because the Kerr spacetime has no spherical symmetry, Carter-Penrose diagrams are not particularly useful. To study the causal structure we would actually need a 3-dimensional diagram. Nevertheless, a 2-dimensional Carter-Penrose diagram can still be useful for the $\{\theta=0\}$ or $\{\theta=\frac{\pi}{2}\}$ submanifolds, because these submanifolds are totally geodesic (meaning that geodesics always remain tangent to them).

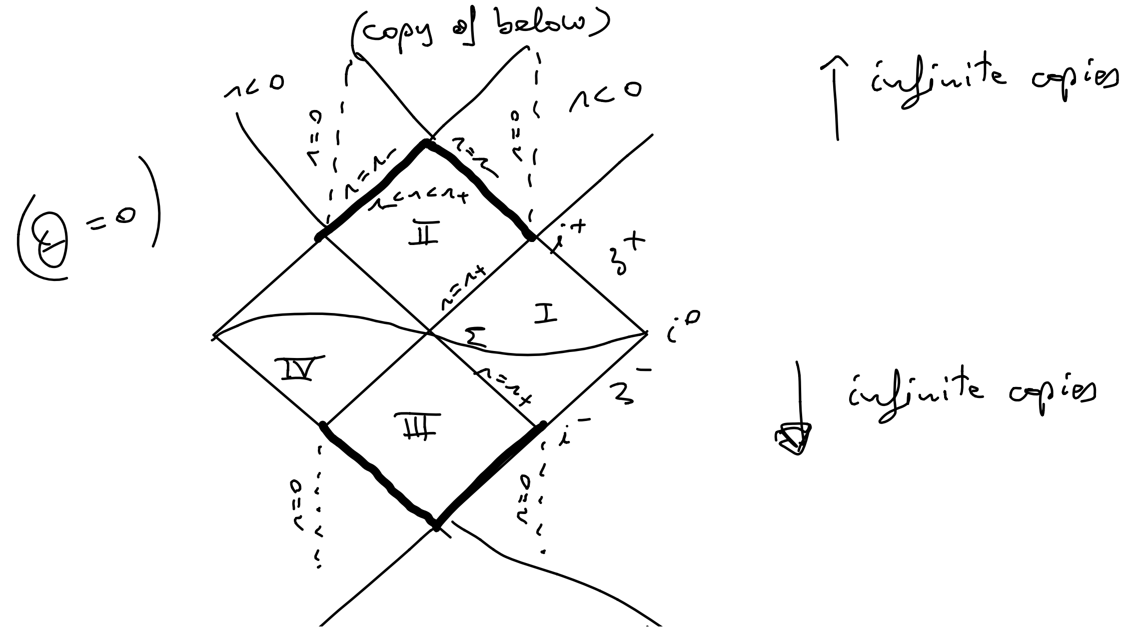

Figure 18: Carter-Penrose diagram for the totally geodesic submanifold $\{\theta = 0\}$ of the maximally extended Kerr spacetime.

A Carter-Penrose diagram of the maximally analytical extension when $M^2 > a^2$ for the submanifolds $\{\theta = 0\}$ and $\{\theta = \frac{\pi}{2}\}$ can be found, but details will be omitted. The result can be consulted in Figure 18. The diagram can be infinitely extended in both time directions. The hypersurface $\{r = r_+\}$ denotes the event horizon, while the hypersurface $\{r = r_-\}$ indicates the location of the Cauchy horizon (to be explained in later sections).

The case $M^2 = a^2$ is called the extreme Kerr black hole, and it is unstable. It has degenerate killing horizons at $r_\pm = M$ (the surface gravity vanishes, $\kappa_\pm = 0$, as can easily verified), with KVF given by $\xi_\pm = k + \Omega_H m$, $\Omega_H = \frac{1}{2M}$. This case is unphysical: the third law of black hole mechanics shows that is impossible to spin up a black hole to the extreme Kerr configuration.

3.5 Dragging of inertial frames and ergoregion

The fact that the Kerr black hole is rotating introduces a whole series of new phenomena that were absent in the non-rotating, Schwarzschild black hole. One of these effects is the dragging of inertial frames by the rotating body sufficiently close to the horizon, in the strong-field regime. We will explore this topic and its physical implications in this and next sections.

Let $\gamma_1(t)$ represent the integral curves of the vector field $k = \frac{\partial}{\partial t}$, where t is the time coordinate in Boyer-Lindquist coordinates. These curves represent the orbits of static observers. In particular, their azimutal variable remains constant along these curves:

$$\dot{\phi} = k^a \nabla_a \phi = 0 \longrightarrow \phi(\gamma_1(t)) = \phi_0 = \text{const}\,. \tag{193}$$

These observers must be held fixed by an external agent, otherwise they would fall into the black hole by the gravitational attraction. Consequently, the motion of static observers in the Kerr black hole cannot be geodesic (i.e. they are not in free fall). The parametrization

t is the time measured by the static observers at spatial infinity.

$$\lim_{r\to\infty}\int_{t_1}^{t_2} dt'\sqrt{-g_{ab}(\gamma(t'))k^a(\gamma(t'))k^b(\gamma(t'))} = \lim_{r\to\infty}\int_{t_1}^{t_2} dt'\sqrt{1-\frac{2Mr}{\Sigma}} = t_2 - t_1\,. \quad (194)$$

Let us switch now to observers moving with 4-velocity $\xi^a = k^a + \Omega m^a$, with Ω a constant real number. One can check that ξ^a is indeed timelike, $\xi^a\xi_a < 0$, at least in the exterior of the horizon $r > r_+$. From a physical viewpoint, what class of observers do the integral curves of ξ^a represent? On orbits $\gamma_2(\lambda)$ of $\xi^a = \frac{\partial}{\partial\lambda}$ we have

$$\xi^a\nabla_a(\phi - \Omega t) = -\Omega + \Omega\times 1 = 0 \longrightarrow \phi(\gamma_2(\lambda)) = \Omega\, t(\lambda) + \phi_0\,. \quad (195)$$

Therefore, particles following the orbits of the vector field ξ^a rotate with uniform angular velocity Ω around the black hole as measured by static particles located at spatial infinity (those on orbits of k^a which measure in time t), and hence relative to a static frame at spatial infinity. Because the radial coordinate function of these particles still remains constant, $r(\gamma_2(\lambda)) = \text{const}$, we call them stationary observers, as they follow periodic motions in a coordinate plane with constant angular velocity Ω. A stationary observer with $\Omega = 0$ reduces to a static observer.

It is easy to see that the null (geodesic) generators of the Killing Horizon $\{r = r_+\}$ follow the orbits of ξ^a with $\Omega = \Omega_H \equiv \frac{a}{r_\pm^2 + a^2}$. See exercise 20 of Exercise List for details. Consequently, the Kerr black hole horizon is rotating with angular velocity Ω_H as measured by static observers at spatial infinity. We call Ω_H the angular velocity of the horizon.

Static observers in the Kerr spacetime have a 4-velocity given by $u^a = \frac{k^a}{\sqrt{-k^a k_a}}$, normalized to $u^a u_a = -1$. However, similarly to what happed in the Schwarzschild case, they cannot exist everywhere as timelike vector fields in the spacetime since a static limit exists when $k^a k_a = 0$:

$$k^a k_a = g_{tt} = -\frac{\Delta - ar^2\sin^2\theta}{\Sigma}\,. \quad (196)$$

Thus, the vector field k^a is timelike iff $\Delta - ar^2\sin^2\theta > 0$ (notice that $\Sigma > 0$), which is satisfied iff $(r - r_+^e)(r - r_-^e) = r^2 - 2Mr + a^2\cos^2\theta > 0$, where

$$r_\pm^e = \frac{2M \pm \sqrt{4M^2 - 4a^2\cos^2\theta}}{2} = M \pm \sqrt{M^2 - a^2\cos^2\theta}\,. \quad (197)$$

Since $r_+^e > r_-^e$, then the vector field k^a is timelike iff $r > r_+^e$ (or $r < r_-^e$, but this case is physically irrelevant and will not be considered here). The hypersurface $\{r = r_+^e\}$ where the static vector field k^a becomes null is called the ergosphere, and it represents the "static limit surface". The ergosphere lies outside the event horizon:

$$r_+^e = M \pm \sqrt{M^2 - a^2\cos^2\theta} \geq M \pm \sqrt{M^2 - a^2} = r_+\,. \quad (198)$$

Notice the sharp contrast with respect to the Schwarzschild spacetime: the KVF k^a is timelike at spatial infinity but can become spacelike in a region outside the event horizon!

Definition. The spacetime bounded by the event horizon $\{r = r_+\}$ and the ergosphere $\{r = r_+^e\}$ is called the ergoregion, $\{r_+ < r < r_+^e\}$. See Figure 20.

What is the physical meaning of this ergoregion? Does the ergosphere represent a notion of horizon, in some sense? To answer these questions, let us consider circular, null curves,

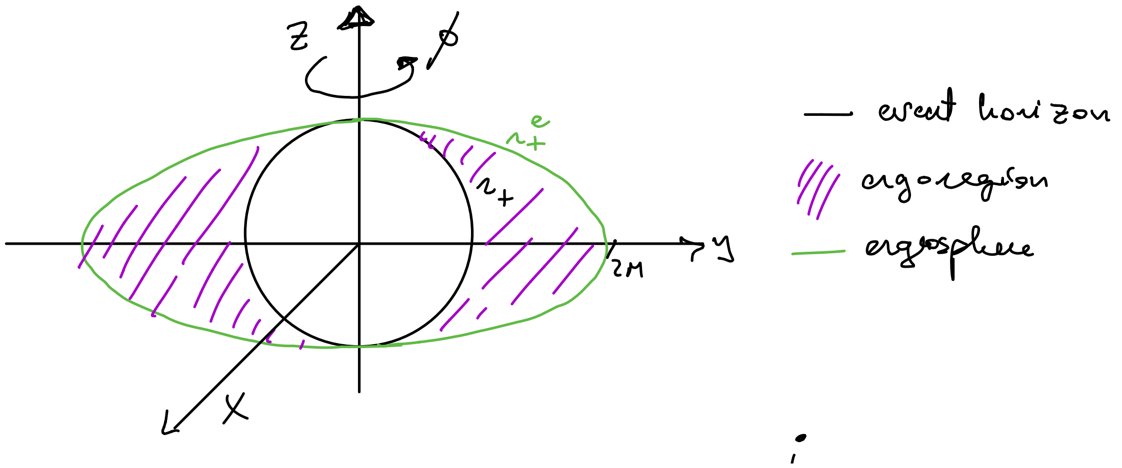

Figure 19: Illustration of the ergoregion in the Kerr spaceetime. The ergosphere (in green) lies outside the event horizon (in black), and the spacetime in between defines a region where dragging of inertial frames is unavoidable.

parametrized with the time t measured by static observers at infinity. In coordinates $x^a(t) = (t, r(t), \theta(t), \phi(t))$, with 4-velocity $u^a = \frac{x^a}{dt}$, we require $\dot{r} = \dot{\theta} = ds^2 = 0$. The null condition $u^a u_a = 0$ gives

$$0 = g_{tt} dt^2 + 2 g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 \,, \tag{199}$$

or, equivalently,

$$0 = g_{tt} + 2 g_{t\phi} \frac{d\phi}{dt} + g_{\phi\phi} \left(\frac{d\phi}{dt} \right)^2 \,. \tag{200}$$

Let us define $\Omega := \frac{d\phi}{dt}$. Then the solutions of this equation are

$$\Omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}} \,. \tag{201}$$

Given that

$$g_{t\phi} = -a \sin^2\theta \frac{2Mr}{\Sigma} < 0 \,, \tag{202}$$

$$\begin{aligned} g_{t\phi} &= \frac{\left(\left(r^2 + a^2 \right) - a^2 \sin^2\theta \right) \left(r^2 + a^2 \right) + 2Mra^2 \sin^2\theta}{\sum} \sin^2\theta \\ &= \left[\left(r^2 + a^2 \right) + \frac{2Mra^2 \sin^2\theta}{\Sigma} \right] \sin^2\theta \geq 0 \,, \end{aligned} \tag{203}$$

we conclude that $\Omega_+ > 0$. In other words, a photon orbits the black hole in the same direction of rotation as the horizon does.

Proposition. The angular velocity $\Omega_- = \frac{-g_{t\phi} - \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}}$ of circular null curves satisfies $\Omega_- \leq 0$ iff $r \geq r_+^e$.

Proof. We have $\Omega_- \leq 0$ iff $-g_{t\phi} \leq \sqrt{g_{t\phi}^2 - g_{\phi\phi} g_{tt}}$ iff $g_{t\phi}^2 \leq g_{t\phi}^2 - g_{\phi\phi} g_{tt}$ iff $g_{\phi\phi} g_{tt} \leq 0$ iff $g_{tt} \leq 0$ iff $r \geq r_+^e$.

Corollary. If $r < r_+^e$ (inside ergosphere), then the angular velocity of circular null curves satisfies $\Omega_\pm \geq 0$.

Consequently, particles are forced to orbit the black hole in the direction of rotation of the horizon as measured by static observers at infinity. This phenomenon is called "dragging of inertial frames". Particles at $r = r_+^e$ attempting to orbit against the black hole direction of rotation must travel at the speed of light $v = c$, and they can only manage to remain static (relative to a static frame at infinity) at most. Notice that observers with mass (not traveling at the speed of light) cannot remain static inside the ergoregion, even if an arbitrarily large force is applied to them, because the dragging of inertial frames compels them to rotate with the black hole.

We have seen that inside the ergosphere observers must move in the same azimutal direction $\dot{\phi}$ as the black hole horizon does as seen by static observers at infinity. However, it is important to keep in mind that these observers are still free to move radially towards or away from the event horizon. In other words, they are not "radially trapped", they are only "azimutally trapped".

On the other hand, stationary observers have 4-velocity $u^a = \frac{1}{\sqrt{-\xi^a \xi_a}} \xi^a$, normalized to $u^a u_a = -1$. They cannot exist everywhere in the Kerr spacetime as timelike vector fields either. A stationary limit exists when $\xi^a \xi_a = 0$, which imposes a constraint on the allowed values of Ω. This is, precisely,

$$\Omega_-(r,\theta) < \Omega < \Omega_+(r,\theta)\,. \tag{204}$$

As r decreases, $\Omega_-(r,\theta)$ increases while $\Omega_+(r,\theta)$ decreases. A critical point is reached when $\Omega_- = \Omega = \Omega_+$, which is satisfied iff $r = r_+$, which we identify with the event horizon. In conclusion, stationary observers cannot exist inside the black hole region, as expected since they must necessarily fall inwards radially. In fact, any arbitrary observer/curve inside must satisfy $\dot{r} < 0$ (exercise).

3.6 Energy extraction: Penrose process and superradiance

The existence of an ergoregion leads to rich and interesting phenomenology of physical interest. In particular, when there is an ergosphere, energy can be extracted from a black hole in different ways. In this subsection will study two mechanisms that illustrate this effect.

Let us consider a particle that propagates from infinity towards a Kerr black hole. The initial energy of the particle is $E_0 = -p^a k_a$, where $k = \frac{\partial}{\partial t}$ is the timelike KVF of static observers at infinity, and p^a is the causal curve representing the particle's trajectory. Since both k and p are future-directed (around spatial infinity), $E_0 > 0$. Furthermore, since k is killing and the particle follows a geodesic (free fall), E_0 is constant along the trajectory.

Suppose that when the particle gets in the ergoregion it decays into two particles, one of which falls inside the horizon with energy E_1, while the other escapes to infinity with energy E_2. See Figure 20. By local conservation of energy, $E_0 = E_1 + E_2$, so $E_2 = E_0 - E_1$. Typically, in most physical situations one has $E_1 > 0$ so $E_2 < E_0$, i.e. the outgoing particle has always less energy than the ingoing one (because part of the energy is carried away by the second particle of the decay). However, $E_1 = -p_1^a k_a$ is not necessarily positive when $r < r_+^e$ since k becomes spacelike. Therefore, $E_1 < 0$ is allowed, which leads to amplification of energy: $E_2 > E_0$. This is called the Penrose process: energy can be

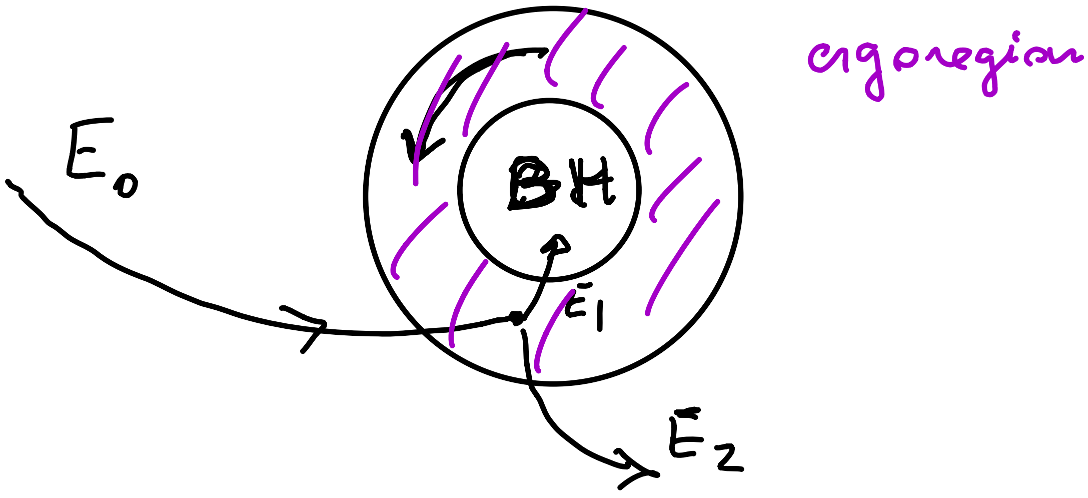

Figure 20: Penrose process. A particle enters the ergoregion and decays into two particles, one falling through the horizon with negative energy E_1 and another one escaping to infinity with energy E_2 greater than the energy E_0 of the original particle.

extracted from the black hole.[6]

How much energy can be extracted from a Kerr black hole? Are there any limitations? Particles with $E < 0$ also have negative angular momentum $L := p^a m_a < 0$. Given that (i) $\xi^a = k^a + \Omega_H m^a$ is a future-directed null vector field on the horizon $r = r_+$ (see previous subsection), and (ii) p^a is a future-directed timelike or null vector field, then we must have $-p_a\xi^a \geq 0$, which gives $E - \Omega_H L \geq 0$, or[7]

$$L \leq \frac{E}{\Omega_H}\,. \tag{205}$$

If the particle inside the ergoregion falls into the horizon with negative energy, $E < 0$, then it must also carry negative angular momentum $L < 0$. Therefore, the black hole's angular momentum is also reduced in the Penrose process. More precisely, an initial black hole in the state (M, J), with mass M and angular momentum J, transitions to a state $(M + \delta M, J + \delta J)$ after absorbing the particle, with $\delta M = E < 0$ and $\delta J = L < 0$. In particular, $\delta J \leq \frac{\delta M}{\Omega_H}$ during the Penrose process. Once the black hole reaches the limit $J = 0$, there is no ergosphere anymore, and there is no further energy extraction. This is, energy can be extracted only until all the angular momentum of the black hole is gone. In other words, in the Penrose process the outgoing particle extracts only rotational kinetic energy of the black hole, not its mass.

The constraint $\delta J \leq \frac{\delta M}{\Omega_H}$ mentioned above during the Penrose process has a simple geometrical interpretation. Let us perturb the quantity $M^2 + \sqrt{M^4 - J^2}$:

$$\delta(M^2 + \sqrt{M^4 - J^2}) = 2M\delta M + \frac{1}{2\sqrt{M^4 - J^2}}(4M^3\delta M - 2J\delta J)\,. \tag{206}$$

If we demand this result to be equal or greater than 0, we obtain

$$\frac{\delta M}{\sqrt{M^4 - J^2}}(2M\sqrt{M^4 - J^2}) + 2M^3) \geq \frac{J\delta J}{\sqrt{M^4 - J^2})} \longrightarrow \delta J \leq \frac{2M}{J}\delta M(\sqrt{M^4 - J^2}) + M^2) \tag{207}$$

[6]Note that both E_1 and E_2 cannot be negative at the same time since $E_0 > 0$.

[7]$\Omega_H \geq 0$ without loss of generality.

This last equation shows that the inequality $\delta J \leq \frac{\delta M}{\Omega_H}$ is equivalent to $\delta(M^2+\sqrt{M^4-J^2}) \geq 0$. Consequently, the particular combination $M^2+\sqrt{M^4-J^2}$ must increase in a Penrose process!

Proposition. The quantity $A := 4\pi(r_+^2+a^2) = 8\pi(M^2+\sqrt{M^4-J^2})$ is the area of the event horizon for a Kerr black hole.

Proof. The area of a 2-dimensional surface in the spacetime is given by the integral $A = \int_{\mathbb{S}^2} \sqrt{\det h}\, d\theta d\phi$, where h_{ab} is the induced 2-dimensional metric on the sphere $\mathbb{S}^2$. The sphere of interest is the intersection $\mathcal{H}^+ \cap \{v = v_0\}$ for any constant $v_0 \in \mathbb{R}$, where $\mathcal{H}^+ = \{r = r_+\}$ is the null hypersurface corresponding to the event horizon. The induced metric can be read off from the full Kerr metric in ingoing coordinates (208):

$$ds^2|_{\{v=v_0\}\cap\{r=r_H\}} = -\frac{0-a^2\sin^2\theta}{\Sigma}0 + \Sigma d\theta^2 + \frac{\left[(r^2+a^2)^2 - 0a^2\sin^2\theta\right]}{\Sigma}\sin^2\theta(d\phi + \frac{a}{\Delta}0)^2 +20dr - \frac{2a\sin^2\theta\left(r^2+a^2-0\right)}{\Sigma}0d\chi - 2a\sin^2\theta 0 d\chi \quad (208)$$

Then,

$$\det h = \Sigma\frac{(r_+^2+a^2)^2}{\Sigma}\sin^2\theta \to \sqrt{\det h} = (r_+^2+a^2)\sin\theta\,, \quad (209)$$

and the area formula gives

$$A = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{\det h} = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta (r_+^2+a^2) = 4\pi(r_+^2+a^2)\,. \quad \square \quad (210)$$

Corollary. Energy extraction by Penrose process requires $\delta A \geq 0$.

This last result shows that the Penrose process is just a particular case of what is known as the 2nd Law of black hole mechanics [8]. More precisely:

Hawking's area theorem. Under certain reasonable assumptions (weak energy condition plus cosmic censorship), the area of the future event horizon of an asymptotically flat spacetime is always non-decreasing.

Although the Penrose process is physically possible and plausible, it is actually not an efficient extraction mechanism of rotational energy from astrophysical black holes [10]. Fortunately, there is an analogue mechanism using classical waves instead of particles: this is known as super-radiant scattering. This is a simpler energy extraction mechanism. We will illustrate this idea here with a simple example.

Superradiance of a massless scalar field by a Kerr black hole. The stress-energy tensor of a minimally coupled, massless scalar field reads

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla_c\phi\nabla^c\phi\,, \quad \nabla_a T^{ab} = 0\,. \quad (211)$$

[8] Black holes in general relativity satisfy a mechanical analogue of the 4 laws of (equilibrium) thermodynamics if one interprets the surface gravity κ and horizon area A_H as a notion of temperature and entropy for them, respectively [9]. The zero law is just the statement that the surface gravity (temperature) is constant on each point of the event horizon. For a proof in a Kerr black hole, see exercise 20 of Exercise List. The first law claims that a small perturbation of a black hole of mass M and angular momentum J satisfies $\frac{\kappa}{8\pi}\delta A_H = \delta M - \Omega_H \delta J$. For a proof in a Kerr black hole, see exercise 21 of Exercise List. The second law is the statement that area (entropy) cannot decrease in any physical dynamical process. Finally, the third law is the statement that it is impossible, by any procedure, to reduce the surface gravity κ (temperature) to zero by a finite sequence of operations.

Given the timelike stationary KVF $k = \frac{\partial}{\partial t}$, we can construct a conserved energy current given by the formula (see exercise 9 of Exercise List)

$$j^a = -T^a_{\ b} k^b = -\nabla^a \phi \, k^b \nabla_b \phi + \frac{1}{2} k^a \nabla^c \phi \nabla_c \phi \,. \tag{212}$$

It is easy to check that this current is future-directed:

$$-k_a j^a = (k^a \nabla_a \phi)^2 - \frac{k^a k_a}{2} \nabla^c \phi \nabla_c \phi = (k^a \nabla_a \phi)^2 + \frac{k^a k_a}{2} T^a_{\ a} > 0 \,, \tag{213}$$

where the last inequality follows from the timelike nature of k^a, $k^a k_a < 0$, and from the classical energy conditions applied to the trace $T^a_{\ a}$.

Let Σ_1, Σ_2 be two spacelike hypersurfaces within a foliation of the exterior spacetime of the Kerr black hole, as illustrated in Figure 21, and let N denote the portion of the event horizon bounded by Σ_1 and Σ_2. Let n represent the timelike normal to this foliation. The null vector normal to the horizon is $\xi^a = k^a + \Omega_H m^a$. If the spacetime volume bounded by these three hypersurfaces is denoted by S, the energy-flux conservation $\nabla_a j^a = 0$ implies,

$$\begin{aligned} 0 &= \int_S d^4x \sqrt{-g} \nabla_a j^a = \int_{\partial S} dS^a j_a = \int_{\Sigma_2} dS_a j^a - \int_{\Sigma_1} dS_a j^a + \int_N dS_a j^a \\ &= E_2 - E_1 + \int_N dS_a j^a \,, \end{aligned} \tag{214}$$

where in the second equality we used Stokes theorem and the identity $\nabla_a j^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} j^a)$. Above, E_i is the energy of the scalar field on the spatial hypersurface Σ_i. The energy flux across the event horizon is calculated as

$$\Delta E = E_2 - E_1 = \int_N dS_a j^a = -\int_N dA dv \xi_a j^a \,, \tag{215}$$

where in the last equality we noted that $-\xi^a$ is the outward-directed normal to the hypersurface N (this is determined by continuity of the normal vector $-\frac{1}{\sqrt{g_{rr}}} g^{ab} \nabla_b r$, see Figure 21). The energy per unit time v gives

$$P = \frac{dE}{dv} = -\int_{\mathbb{S}^2} dA \, \xi_a j^a = \int_{\mathbb{S}^2} dA \, (\xi^a \nabla_a \phi)(k^b \nabla_b \phi) = \int_{\mathbb{S}^2} dA \left[\frac{\partial \phi}{\partial v} + \Omega_H \frac{\partial \phi}{\partial \chi} \right] \frac{\partial \phi}{\partial v} \,, \tag{216}$$

where we used $\xi^a k_a|_{\mathcal{N}} = 0$ in the third equality, and $k|_{\mathcal{N}} = \frac{\partial}{\partial v}|_{\mathcal{N}}$, $\xi|_{\mathcal{N}} = \frac{\partial}{\partial v} + \Omega_H \frac{\partial}{\partial \chi}|_{\mathcal{N}}$ in the last equality. To obtain a more explicit result, let us assume a monochromatic scalar wave of constant frequency ω, azimutal number m, and constant amplitude ϕ_0, $\phi = \phi_0 \cos(\omega v - m\chi)$. The integral above is trivial and we obtain:

$$P = \frac{\phi_0^2}{2} A_H \omega (\omega - m\Omega_H) \,. \tag{217}$$

For field modes with $\omega > m\Omega_H$ the energy flux crossing the event horizon is positive, and therefore $E_2 < E_1$, i.e. the monochromatic scalar wave loses energy after the scattering off by the Kerr black hole. However, if $\omega < m\Omega_H$ then the energy flux across the horizon is actually negative, $P < 0$. This is, a wave mode (ω, m) that satisfies this condition gets amplified by the black hole. This can only occur when there is rotation, Ω_H, so the scalar wave extracts rotational energy from the black hole in this process.

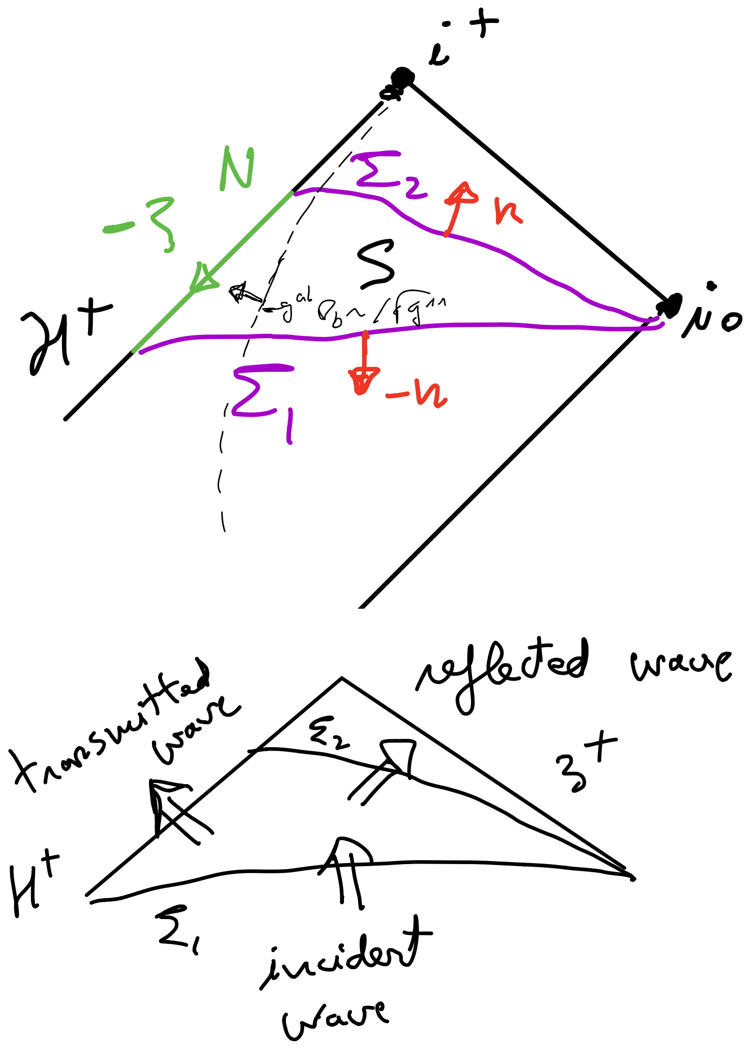

Figure 21: Analysis of the energy flux conservation of a classical field in the exterior spacetime of a Kerr black hole. The presence of an ergoregion lying outside the event horizon can make some waves to produce a negative energy flux across the event horizon, leading to the phenomenon of superradiance. If the transmitted energy flux is negative, then the reflected wave amplitude is greater than the incident wave amplitude.

Similar conclusions apply for electromagnetic and gravitational waves, but the calculation is technically more involved. The scalar case provides a simple example to illustrate the main point of superradiance.

In conclusion, if a scalar, electromagnetic or gravitational wave is incident upon a black hole, the transmitted wave will be absorbed by the black hole and the reflected wave will escape back to infinity. Normally the transmitted wave carries positive energy and the reflected wave has, as a consequence of energy conservation, less energy than the incident wave mode. However, if the black hole is rotating, and if the wave mode satisfies the superradiant condition $0 < \omega < m\Omega$, then the transmitted wave will carry negative energy to the black hole horizon, and therefore the reflected wave will return with more energy!

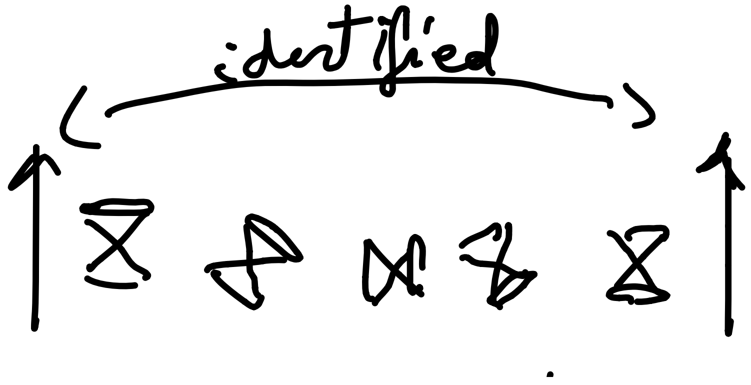

Figure 22: Example of a non time-orientable spacetime.

3.7 Predictability and Cauchy horizons

In previous sections we identified the hypersurface $\{r = r_+\}$ as the event horizon of the Kerr black hole. What is the physical or mathematical significance of the hypersurface $\{r = r_-\}$? In this section we will see that it is connected, from a PDE viewpoint, with the limit of predictability of the theory. In this sense, it represents another type of "horizon", one beyond which the physics cannot be determined by fixing initial data on a spatial hypersurface.

Before getting into specific details on the topic of predictability, it will be convenient to review basic aspects on the causal structure of spacetimes.

Basic definitions on causal structure

Definition. A spacetime (M, g_{ab}) is time-orientable if a continuous assignment of future lightcones can be made across M. An example of a non-orientable spacetime is displayed in Figure 22 (although the underlying manifold M is orientable).

Proposition. A spacetime (M, g_{ab}) is time-orientable iff there exists a smooth non-vanishing timelike vector field t^a on M. For a proof, see [3].

Definition. Let (M, g_{ab}, t^a) be a time-orientable spacetime. A timelike/null vector v^a lying in the future/past light cone of t^a will be called future/past directed. This happens whenever $g_{ab}t^a v^b < 0$ / $g_{ab}t^a v^b > 0$.

Definition. Let (M, g_{ab}, t^a) be a time-orientable spacetime. A curve $\gamma(t)$ is a future-directed causal curve if $\forall t \in I$ the tangent vector $v^a(t)$ is future-directed timelike or null.

Definition. Let (M, g_{ab}, t^a) be a time-orientable spacetime. We define the chronological past(-)/future(+) of a point $p \in M$ or of a subset $S \subset M$ as the following subsets:

$$\begin{aligned}
I^+(p) &= \{q \in M \,/\, \exists \gamma : [0,1] \to M \,\text{future-directed timelike curve with}\, \gamma(0) = p, \gamma(1) = q\} \\
I^-(p) &= \{q \in M \,/\, \exists \gamma : [0,1] \to M \,\text{past-directed timelike curve with}\, \gamma(0) = p, \gamma(1) = q\} \\
I^\pm(S) &= \bigcup_{p \in S} I^\pm(p)
\end{aligned}$$

Definition. Let (M, g_{ab}, t^a) be a time-orientable spacetime. We define the causal past(-

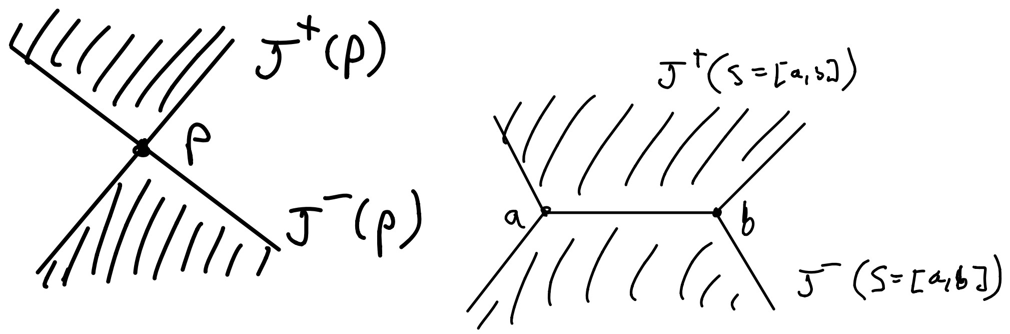

Figure 23: Causal past (J^-) and future (J^+) of an arbitrary point $p \in (\mathbb{R}^2, \eta_{ab})$ (left) and of a set $[a, b] \in \mathbb{R}$ (right).

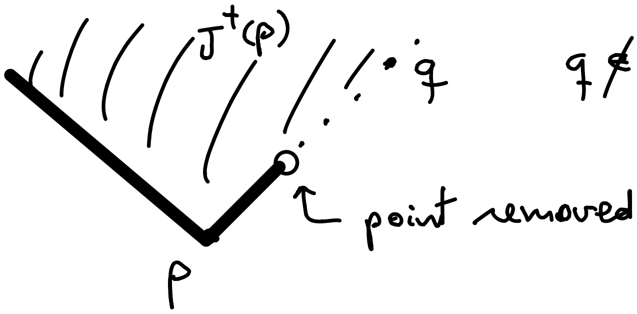

Figure 24: An example in $(\mathbb{R}^2 - \{0\}, \eta_{ab})$ where $\bar{J}^{\pm}(S) \neq J^{\pm}(S)$. The point removed in the figure represents "0". Although $q \notin J^+(p)$, we do really have $q \in \bar{J}^+(p)$.

)/future(+) of a point $p \in M$ or of a subset $S \subset M$ as the following subsets:

$$
\begin{aligned}
J^+(p) &= \{q \in M \,/\, \exists \gamma : [0,1] \to M \text{ future-directed causal curve with } \gamma(0) = p, \gamma(1) = q\} \\
J^-(p) &= \{q \in M \,/\, \exists \gamma : [0,1] \to M \text{ past-directed causal curve with } \gamma(0) = p, \gamma(1) = q\} \\
J^{\pm}(S) &= \bigcup_{p \in S} J^{\pm}(p)
\end{aligned}
$$

Example. In $(\mathbb{R}^2, \eta_{ab})$, $J^{\pm}(p)$ is the interior and surface of the future/past lightcone of p, including the point p itself. See Figure 23.

Proposition. Let $\bar{J}^{\pm}(S)$ be the union of $J^{\pm}(S)$ and the 'limiting points of S" (more precisely, the topological closure of the set $J^{\pm}(S)$). $\bar{J}^{\pm}(S) = J^{\pm}(S)$ iff $J^{\pm}(S)$ is closed (as happens in the full Minkowski space).

For a general spacetime $(M,\, g_{ab})$, $J^{\pm}(S)$ might not be closed. An example is given in Figure 24.

Proposition. The following properties hold:

(a) $I^+(p) \subset J^+(p)$

(b) If $q \in J^+(p) - I^+(p)$, any causal curve connecting p to q must be a null geodesic.

(c) $J^+(S) \subset \bar{I}^+(S)$. This, together with $I^+(S) \subset J^+(S)$, implies $\bar{I}^+(S) \subset \bar{J}^+(S) \subset \bar{I}^+(S)$, or $\bar{I}^+(S) = \bar{J}^+(S)$.

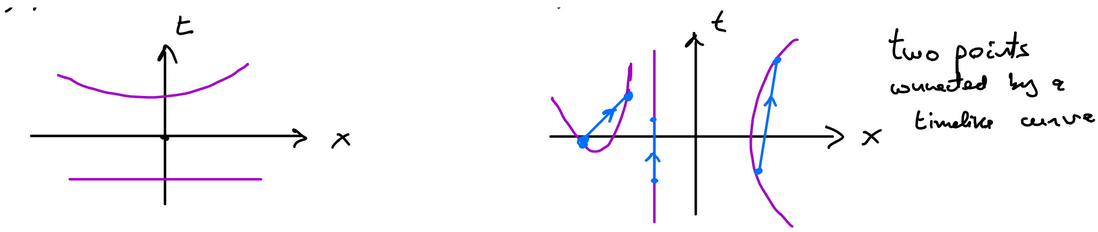

Figure 25: Examples of achronal sets (left) and non-achronal sets (right) in 1+1 dimensional Minkowski space.

(d) $I^+(S) = \text{interior}[J^+(S)]$ (topological notion of interior).

(e) $\partial I^+(S) = \partial J^+(S)$.

Basic definitions on predictability

Roughly speaking, we can say that predictability is the notion by which a collection of physical events can be "entirely determined" from a set of events Σ located "in the past". This is basically the initial value problem in the theory of PDEs. To define all these concepts in precise terms, we need the following:

Definition. A subset $\Sigma \subset M$ is called an achronal set if no two points are connected by a timelike curve in M, i.e. $I^+(\Sigma) \cap \Sigma = \emptyset$ (it is a hypersurface if edge(Σ)$= \emptyset$).

Examples. The left panel of Figure 25 shows two achoral sets in Minkowski space $(\mathbb{R}^2, \eta_{ab})$. The right panel of Figure 25 shows three sets in $(\mathbb{R}^2, \eta_{ab})$ that are not achronal because two points can be connected by timelike curves.

Definition. The future ($+$) and past ($-$) domain of dependence of an achronal set Σ are the sets

$$\begin{aligned} D^+(\Sigma) &= \{p \in M / \text{every past-inextendible causal curve through } p \text{ intersects } \Sigma\}\,, \\ D^-(\Sigma) &= \{p \in M / \text{every future-inextendible causal curve through } p \text{ intersects } \Sigma\}\,. \end{aligned}$$

The full domain of dependence of Σ is $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$. Note that $\Sigma \subset D^\pm(\Sigma) \subset J^\pm(\Sigma)$.

A picture that illustrates the significance of the future domain of dependence is given in Figure 26. Given initial conditions on some achronal set Σ, we should be able to predict what happens at $p \in D^+(\Sigma)$. If $p \in I^+(\Sigma)$ but $p \notin D^+(\Sigma)$, it is still possible to send a signal to p without influencing the data on Σ, so initial conditions on Σ do not suffice to determine what happens at that p. In other words, $D(\Sigma)$ represents the complete set of events for which all information can be entirely determined by a knowledge of conditions on Σ. The behaviour of solutions of hyperbolic PDEs outside $D^+(\Sigma)$ is not determined only by initial data on Σ.

Example. Let us consider Minkowski space $(\mathbb{R}^2, \eta_{ab})$ and the achronal set $\Sigma = \{(0, x) \in \mathbb{R}^2 / x > 0\}$. It is easy to see that the future and past domain of dependence of Σ are given by

$$\begin{aligned} D^+(\Sigma) &= \{(t, x) \in \mathbb{R}^2 / 0 \leq t < x\}\,, \\ D^-(\Sigma) &= \{(t, x) \in \mathbb{R}^2 / 0 \leq -t < x\}\,. \end{aligned}$$

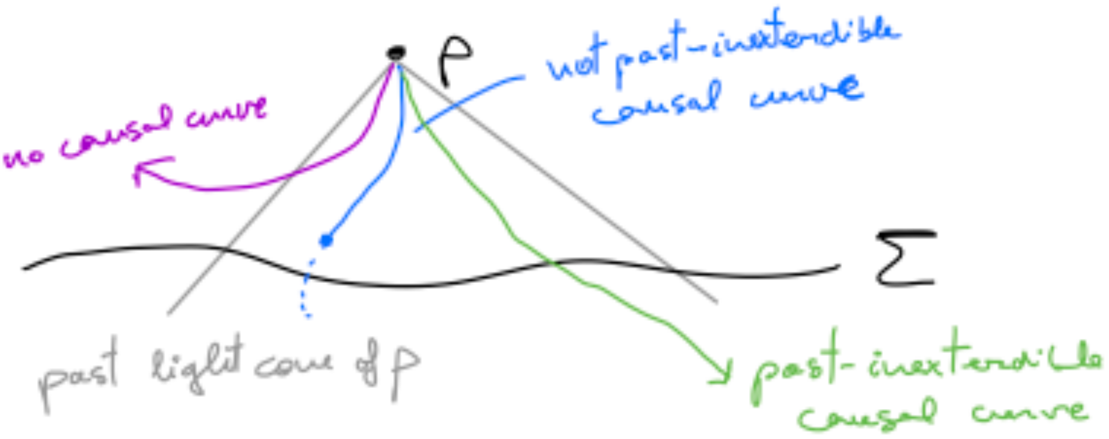

Figure 26: Example of a domain of dependence of the achronal set Σ.

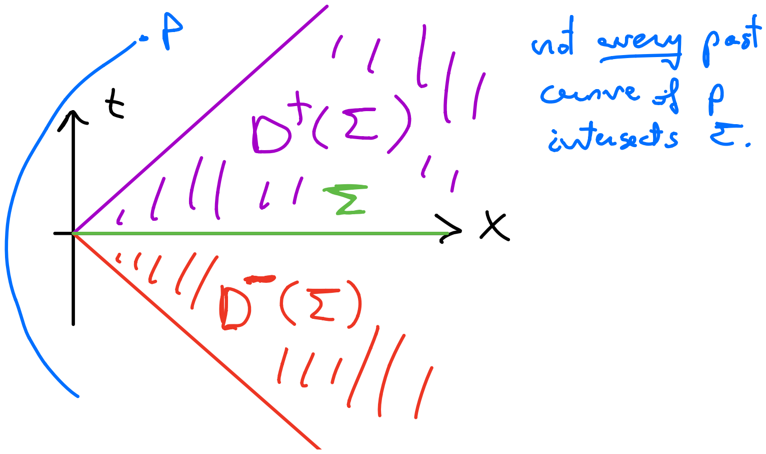

Figure 27: Future (+) and past (-) domain of dependence of $\Sigma = \{(0, x) \in \mathbb{R}^2 / x > 0\}$ in Minkowski space $(\mathbb{R}^2, \eta_{ab})$.

These sets are drawn in Figure 29.

Definition. A Cauchy surface Σ is a (topologically closed) achronal set with $D(\Sigma) = M$. Since Σ is achronal, we may informally think of Σ as representing an 'instant' of time in M.

Definition. A spacetime (M, g_{ab}) is called globally hyperbolic if it contains/admits a Cauchy surface.

In a globally hyperbolic spacetime the entire future (past) history of the Universe can be predicted (retrodicted) from conditions at the "instant of time" Σ. In a non-globally hyperbolic spacetime, we would have a breakdown of predictability, i.e. knowledge of conditions at a single "instant of time" would never suffice to determine the complete history of the Universe. Physically realistic spacetimes are expected to be globally hyperbolic, to ensure a notion of predictability (incidentally, global hyperbolicity also ensures that no closed timelike curves exist, which constitute a clear violation of causality).

Examples. See Figure 28.

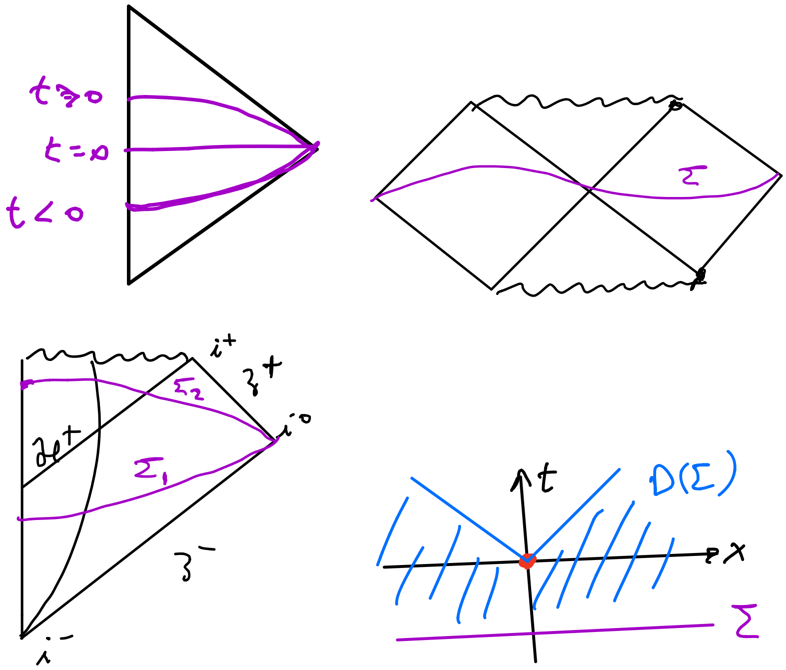

Figure 28: Top left: $\{t = 0\}$ is a Cauchy surface for Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$, therefore it is globally hyperbolic. Top right: $\Sigma = \{U + V = 0\}$ is a Cauchy surface for Kruskal spacetime, therefore it is globally hyperbolic. Bottom left: a spherical, pressure-free gravitational collapse is globally hyperbolic, as either Σ_1 or Σ_2 are Cauchy surfaces. Bottom right: $(\mathbb{R}^2 - \{(0,0)\}, \eta_{ab})$ is not globally hyperbolic, because $D(\Sigma) \neq \mathbb{R}^2$.

Theorem. Suppose (M, g_{ab}) is a globally hyperbolic spacetime. Then:

(a) There exists a global time function, $\hat{t} : M \to \mathbb{R}$, such that $-g^{ab}\nabla_b \hat{t}$ is a future-directed timelike vector field.

(b) $\Sigma_t = \{p \in M \,/\, \hat{t}(p) = t\}$ are Cauchy hypersurfaces and $\Sigma_t \simeq \Sigma_{t'} \equiv \Sigma$ for all $t, t' \in \mathbb{R}$.

(c) $M \simeq \mathbb{R} \times \Sigma$.

In particular, notice that $\Sigma_t \cup \Sigma_{t'} = \emptyset$ for $t \neq t'$, and $M = \bigcup_{t \in \mathbb{R}} \Sigma_t$. Because of this, we say that the spacetime M is foliated by $t =$ const Cauchy hypersurfaces.

Proof. See Theorem 8.2.2, Theorem 8.3.14, and Proposition 8.3.13 of [3].

Example 1. Let us consider Minkowski space $(\mathbb{R}^4, \eta_{ab})$ covered by global cartesian coordinates $\{t, x, y, z\}$. The coordinate function $t : \mathbb{R}^4 \to \mathbb{R}$ is such that $t^a = -\eta^{ab}\nabla_b t$ is timelike ($t^a t_a = \eta^{tt} = -1 < 0$) and future-directed ($t^t = -\eta^{tt} = 1$). Furthermore, $\{t = \text{const}\} = \mathbb{R}^3$ are Cauchy surfaces and $\mathbb{R}^4 \simeq \mathbb{R} \times \mathbb{R}^3$. See Figure **??**.

Example 2. Let us consider Kruskal spacetime (M, g_{ab}) covered by Kruskal coordinates $\{U, V, \theta, \phi\}$. There exists the function $T : M \to \mathbb{R}$, defined by $T(p) = U(p) + V(p)$, which allows to obtain the vector field $T^a = -g^{ab}\nabla_b T$, which is timelike ($T^a T_a = 2g^{UV} < 0$) and future-directed. Then $\{U + V = \text{const}\} = \mathbb{R} \times \mathbb{S}^2$ are Cauchy surfaces and $M \simeq \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2) \simeq \mathbb{R}^2 \times \mathbb{S}^2$.

If a spacetime (M, g_{ab}) is not globally hyperbolic, then $D^+(\Sigma)$ and/or $D^-(\Sigma)$ will have a boundary in M for any $\Sigma \subset M$, that we call the Cauchy horizon.

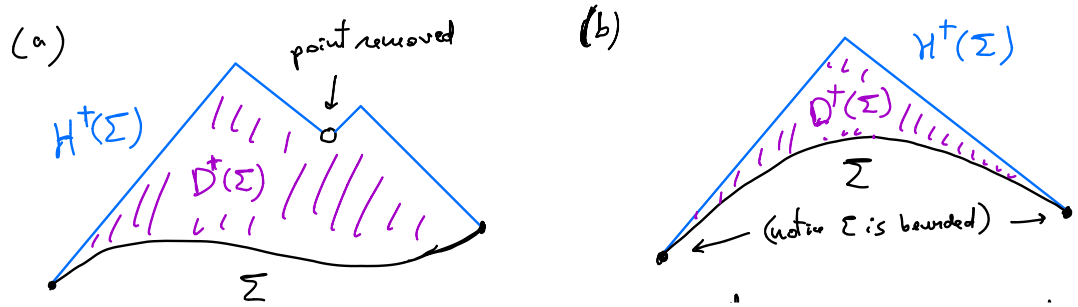

Figure 29: Examples of future Cauchy horizons $H^+(\Sigma)$ in 1+1 dimensional Minkowski space $(\mathbb{R}^2, \eta_{ab})$.

Definition. The future (+) and past (-) Cauchy horizons $H^{\pm}(\Sigma)$ of a (closed) achronal set Σ are

$$H^+(\Sigma) = \overline{D^+(\Sigma)} - I^-[D^+(\Sigma)]\,,$$
$$H^-(\Sigma) = \overline{D^-(\Sigma)} - I^+[D^-(\Sigma)]\,.$$

The full Cauchy horizon is $H(\Sigma) = H^+(\Sigma) \cup H^-(\Sigma)$.

Proposition. The Cauchy horizon is given by the boundary of the domain of dependence: $H(\Sigma) = \text{Front}[D(\Sigma)]$.

Proof. See proposition 8.3.6 of [3].

Examples. See Figure 29.

Corollary. If M is connected, a (closed) achronal set Σ is a Cauchy surface for (M, g_{ab}) iff $H(\Sigma) = \emptyset$.

In other words, $H(\Sigma)$ measures the failure of Σ to be a Cauchy surface.

Example. The maximal analytical extension of the Kerr spacetime is not globally hyperbolic. See Figure 30.

We are now familiarized with the domain of predictability of a given achronal set. The remaining question is: how can we formulate an initial value problem for General Relativity? We will summarize the discussion in [3] (page 255).

Einstein's equations $G_{ab} = 8\pi T_{ab}$ can be viewed as an initial value problem. Consider a 3 dimensional Riemannian submanifold $\Sigma \subset M$ with metric h_{ab}, unit normal n^a and extrinsic curvature $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab} \equiv \frac{1}{2}\dot{h}_{ab}$, with trace $K = g^{ab}K_{ab}$. Let D_a denote the Levi-Civita connection of h_{ab}. Then:

$$\begin{aligned}
G_{ab}n^a n^b &= 8\pi T_{ab}n^a n^b \to R^3 - K^{ab}K_{ab} + K^2 = 16\pi T_{ab}n^a n^b\,, \quad \text{(Hamiltonian constraint)}\\
G_{a'b}h^{a'}_a n^b &= 8\pi T_{a'b}h^{a'}_a n^b \to D_b K^b_a - D_a K = 8\pi T_{a'b}h^{a'}_a n^b\,, \quad \text{(Momentum constraint)}\\
G_{a'b'}h^{a'}_a h^{b'}_b &= 8\pi T_{a'b'}h^{a'}_a h^{b'}_b \to \mathcal{L}_{nN}K_{ab} = -D_a D_b N + N\{{}^3R + KK_{ab} - 2K_{ac}K^c{}_b + 4\pi(\dots)\}\,.\\
&\qquad \text{(evolution equation)}
\end{aligned}$$

Theorem (Choquet-Bruhat, Geroch 1969). Given initial data (Σ, h_{ab}, K_{ab}) satisfying the vacuum constraints, there exists a unique (up to diffeomorphism) spacetime (M, g_{ab}), known as the maximal Cauchy development of (Σ, h_{ab}, K_{ab}), such that:

(a) (M, g_{ab}) obeys the vacuum Einstein's field equations, $R_{ab}[g] = 0$.

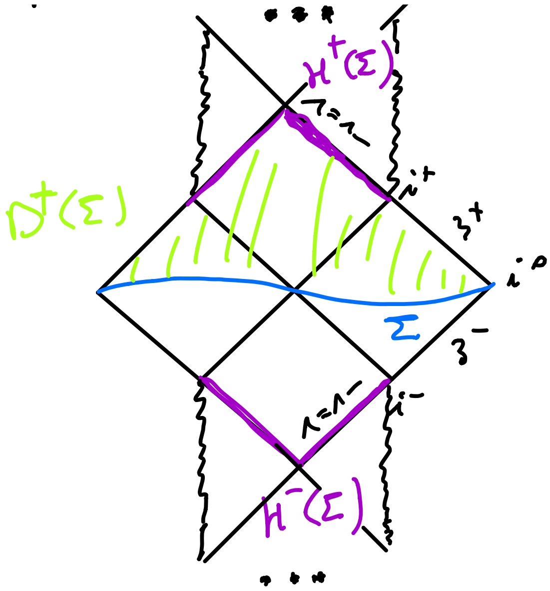

Figure 30: Carter-Penrose diagram for the totally geodesic submanifold $\{\theta = 0\}$ of the maximally extended Kerr spacetime. The achronal surface Σ is not of Cauchy surface for the maximally extended Kerr manifold because the domain of dependence has the boundary $H^+(\Sigma) \cup H^-(\Sigma)$. Even if we fix initial conditions on Σ, there occurs a breakdown of predictability beyond $H^+(\Sigma)$.

(b) (M, g_{ab}) is globally hyperbolic with Cauchy surface Σ, $D(\Sigma) = M$.

(c) The induced metric and extrinsic curvature of Σ are, precisely, h_{ab} and K_{ab}, respectively.

(d) Any other spacetime obeying (a)-(c) is isometric to a subset of (M, g_{ab}).

Furthermore, the solution for g_{ab} on M depends continuously on the initial data (h_{ab}, K_{ab}) on Σ.

Proof. See Theorem 10.2.2 of [3].

This theorem ensures that general relativity has a well-posed initial value formulation.

3.8 Inextendibility and Strong Cosmic Censorship

The theorem by Choquet-Bruhat and Geroch ensures that there always exists a maximal Cauchy development (M, g_{ab}) of some given initial data (Σ, h_{ab}, K_{ab}). However, it does not guarantee that (M, g_{ab}) is inextendible, so that a well-defined notion of predictability exists. In other words, the maximal Cauchy development given by the theorem may still be extendible, in the sense that it may contain a Cauchy horizon. In that case, the extension is not unique outside $D(\Sigma) = M$, as those points are not determined by the initial data on Σ (i.e. the extension cannot have Σ as a Cauchy surface) and the resulting spacetime fails to be predictable beyond the Cauchy horizon. What conditions should the initial data (Σ, h_{ab}, K_{ab}) satisfy in order to get an inextendible maximal Cauchy development?

To begin with, we will first give a simple example where, indeed, the maximal Cauchy development given by the Theorem of Choquet-Bruhat and Geroch fails to be inextendible.

Example. Let us consider the initial data (Σ, h_{ab}, K_{ab}) given by

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 \,/\, x > 0\}\,, \tag{218}$$
$$h_{ab} = \delta_{ab}\,, \tag{219}$$
$$K_{ab} = 0\,. \tag{220}$$

The maximal Cauchy development (M, g_{ab}) obtained from this initial data is

$$M = \{(t,x,y,z) \in \mathbb{R}^4 / |t| < x\}\,, \tag{221}$$
$$g_{ab} = \eta_{ab}\,. \tag{222}$$

This M can be isometrically embedded in Minkowski space $(\mathbb{R}^4, \eta_{ab})$, therefore M is extendible to all Minkowski space.

In this previous example (M, g_{ab}) was extendible because (Σ, h_{ab}) is extendible. Is the maximal Cauchy development always inextendible if we restrict to inextendible initial data? Let us see a counter-example:

Example. Let us consider the Schwarzschild spacetime with negative mass $M < 0$. The manifold is $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$, and the metric reads

$$ds^2 = -\left(1 + \frac{2|M|}{r}\right) dt^2 + \left(1 + \frac{2|M|}{r}\right)^{-1} dr^2 + r^2 d\Omega^2\,. \tag{223}$$

This spacetime contains a naked curvature singularity at $r = 0$ (there is no event horizon anymore). Take the following initial data

$$\Sigma = \{(r,\theta,\phi) \in \mathbb{R} \times \mathbb{S}^2 / r > 0\} = \{t = 0\}\,, \tag{224}$$
$$h_{ab} = \left(1 + \frac{2|M|}{r}\right) \nabla_a r \nabla_b r + r^2 d\Omega\,, \tag{225}$$
$$K_{ab} = \mathcal{L}_n h_{ab}, \quad \text{with} \quad n^a = -g^{ab}\nabla_b t\,. \tag{226}$$

This initial data (Σ, h_{ab}) is inextendible. Does the maximal Cauchy development cover the full manifold $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$? Let us study the domain of dependence of Σ. For this, consider null outgoing and radial geodesics:

$$0 = g_{ab}u^a u^b = g_{tt}\left(\frac{dt}{d\lambda}\right)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 + g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 \tag{227}$$
$$= -\left(1 + \frac{2|M|}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 + \frac{2|M|}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2, \tag{228}$$

where u^a denotes the tangent vector field of these geodesics and λ the affine parameter. If we change the parametrization of the geodesics from λ to r, this result implies

$$\frac{dt}{dr} = +\left(1 + \frac{2|M|}{r}\right)^{-1} = \frac{r}{2|M|} + O(r^2)\,, \tag{229}$$

which can be integrated to give

$$t(r) = t_0 + \frac{r^2}{4|M|} + O(r^3)\,. \tag{230}$$

We see from this that if $t_0 > 0$, these null geodesics never intersect Σ. Therefore, $\exists p \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$ such that $p \notin D(\Sigma)$, as illustrated in Figure 31. Consequently, $D(\Sigma) \neq$

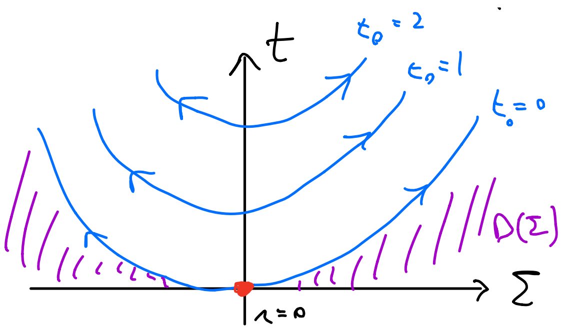

Figure 31: Null outgoing, radial geodesics (in blue) in the negative-mass Schwarzschild spacetime. It is clear that Σ, although inextendible, is not a Cauchy surface of this spacetime.

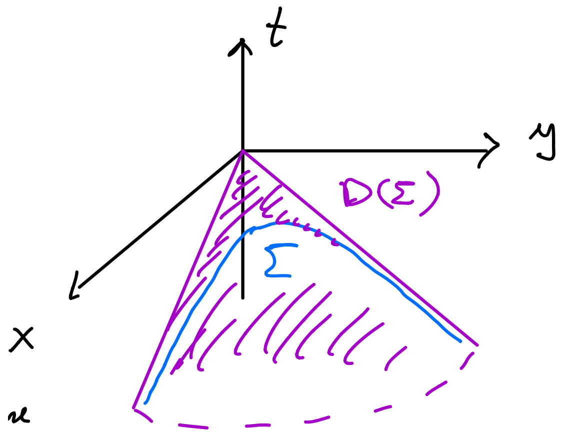

Figure 32: Maximal Cauchy development of the initial data specified in Example X. The presence of a Cauchy horizon indicates that the maximal Cauchy development is extendible.

$\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$, i.e. Σ is not a Cauchy surface of $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$. The Cauchy horizon is $H(\Sigma) = \{t_0 = 0 \quad \text{geodesics}\}$. The solution outside $D(\Sigma)$ is not determined by the data on Σ (in particular, it could differ from the negative-mass Schwarzschild spacetime).

In the example above the initial data (Σ, h_{ab}) was inextendible, but it was not geodesically complete (because of the $r = 0$ singularity). The maximal Cauchy development was extendible because the initial data was singular. Is the problem finally solved if we restrict to non-singular initial data? Again, the answer is no:

Example. Let us consider Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$, and the initial data

$$\Sigma = \{(t, \vec{x}) \in \mathbb{R}^4 / -t^2 + x^2 + y^2 + z^2 = -1, t < 0\}\,, \tag{231}$$

$$h_{ab} = \text{induced metric on } \Sigma\,, \tag{232}$$

$$K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}, \quad n^a = \text{normal vector to } \Sigma\,. \tag{233}$$

The maximal Cauchy development of this initial data is illustrated in Figure 32, and corresponds to the interior of the past light-cone of the origin. This is clearly extendable, as indicated in the figure.

Notice how in the previous example the initial data (Σ, h_{ab}) is inextendible and geodesically

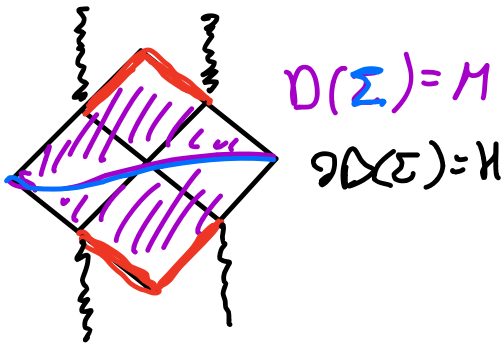

Figure 33: Cauchy horizon (in red) of the spatial hypersurface Σ (in blue) in the Kerr spacetime.

complete, but (M, g) is still extendable because the initial data set is "asymptotically null". To avoid this problem, we will further restrict to asymptotically flat initial data.

Definition. We say that (Σ, h_{ab}, K_{ab}) is an asymptotically flat end if:

(a) Σ is diffeomorphic to $\mathbb{R}^3 - B$, where B is a closed ball centered at the origin of $\mathbb{R}^3$.

(b) $h_{ab} = \delta_{ab} + O(r^{-1})$, $K_{ab} = O(r^{-2})$, in cartesian coordinantes $(x, y, z) \in \mathbb{R}^3$.

(c) $\partial_c h_{ab} = O(r^{-2})$.

Definition. A set of initial data is called asymptotically flat with N ends if it is the union of a compact set with N asymptotically flat ends

$$(\Sigma, h_{ab}, K_{ab}) = \bigcup_{i \in I} (\Sigma^{(i)}, h_{ab}^{(i)}, K_{ab}^{(i)}) \,. \tag{234}$$

Example. In the Schwarzschild spacetime with $M > 0$, $\Sigma = \{t = t_0, r > 2M\}$ is asymptotically flat with 2 ends (connected by the Einstein-Rosen bridge), for any $t_0 \in \mathbb{R}$.

Now, will the asymptotically flat initial data lead to an inextendible maximal Cauchy development, as desired? Let us see the following:

Example. Let us recall the Carter-Penrose diagram for the totally geodesic submanifold of the maximally extended Kerr spacetime. This is given in Figure 33. It turns out that the (Σ, h_{ab}, K_{ab}) given in the Figure is an inextendible, geodesically complete, and asymptotically flat data. Still, the maximal Cauchy development (M, g_{ab}) is extendible beyond the Cauchy horizon $H = \{r = r_-\}$, where the metric is perfectly regular. Is this result indicating the end of physical validity of General Relativity? Physics cannot be predicted inside black holes?

Despite this result, it turns out that the Cauchy horizon usually becomes singular when subjected to any perturbation, no matter how small. Gravitational perturbations of the exterior region will generate gravitational waves. Some of these waves will cross the horizon either directly or after secondary scattering by the curvature of the background spacetime. As a result, one expects an infinite number of scattered waves between an observer outside the event horizon, and future null infinity. These scattered waves will accumulate near future null infinity and, by continuity, also near the Cauchy horizon. Therefore, an infinite density of wave-fronts will accumulate around the Cauchy horizon. An observer attempting to cross the Cauchy horizon will experience the impact of an infinite flux of gravitational

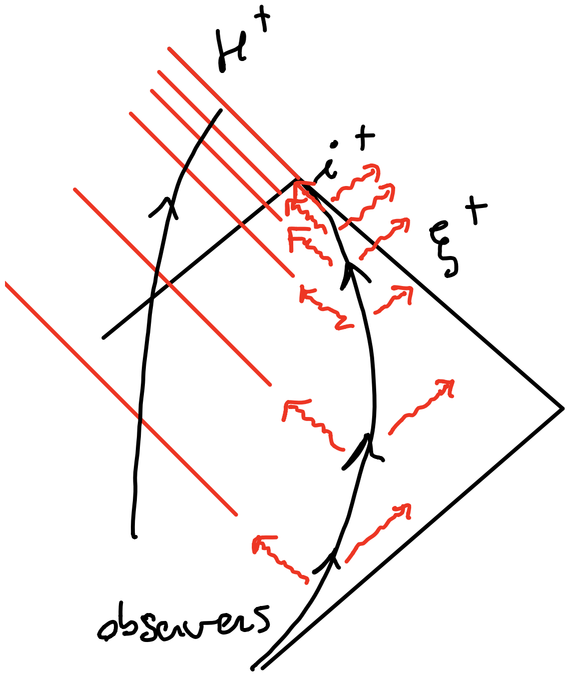

Figure 34: Instability of the Cauchy horizon of the Kerr spacetime.

radiation, and so it should see the Cauchy horizon as an effective physical barrier beyond which it cannot cross. See Figure 34.

Although a violation of global hyperbolicity is known to occur in the Kerr spacetime, it appears that small perturbations of initial data (h_{ab}, K_{ab}) on an achronal surface Σ may destroy the extendibility of the maximal Cauchy development, so that no "generic" violations are possible. This motivated Penrose to conjecture that all physically reasonable spacetimes are globally hyperbolic and lack Cauchy horizons:

Strong Cosmic Censorship conjecture. Given vacuum initial data (Σ, h_{ab}, K_{ab}) that is geodesically complete and asymptotically flat, then, "generically", the maximal Cauchy development is inextendible (thus, globally hyperbolic).

A general proof of the conjecture is still lacking. The conjecture has been shown to be true for nearly flat data. However, without the "generic" assumption, which is left open and needs to be rigorously stated, the conjecture is known to fail in some examples. In other words, with a carefully fine-tuned choice of initial data, it is indeed possible to find counterexamples of the conjecture, such as the Kerr black hole found above, or even the Kerr-Newman spacetime.

References

[1] P. K. Townsend. Black holes. 1997.

[2] Eric Poisson. A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics. Cambridge University Press, 2004.

[3] Robert M. Wald. General Relativity. Chicago Univ. Pr., Chicago, USA, 1984.

[4] Sean M. Carroll. Spacetime and Geometry: An Introduction to General Relativity. Cambridge University Press, 2019.

[5] Eric Poisson and Clifford M. Will. Gravity: Newtonian, Post-Newtonian, Relativistic. Cambridge University Press, 2014.

[6] Erik Curiel. The many definitions of a black hole. Nature Astronomy, 3(1):27?34, January 2019.

[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler. Gravitation. 1973.

[8] Demetrios Christodoulou. The formation of black holes in general relativity, 2008.

[9] James M. Bardeen, B. Carter, and S. W. Hawking. The Four laws of black hole mechanics. Commun. Math. Phys., 31:161–170, 1973.

[10] James M. Bardeen, William H. Press, and Saul A. Teukolsky. Rotating Black Holes: Locally Nonrotating Frames, Energy Extraction, and Scalar Synchrotron Radiation. apj, 178:347–370, December 1972.