Fock Quantization of the Dirac Field in Hybrid Quantum Cosmology

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21/10/19

What physical properties should one ask of a Fock representation in a generally curved spacetime?

-Respecting the spacetimes symmetries
-Avoid divergences on physical quantities

- Our model will have a homogeneous sector; a flat FRW cosmology with scale factor a and an inflaton ϕ, and an inhomogeneous sector; a Dirac field Ψ treated as perturbations.
- We describe the system in a canonical fashion, so that constraints naturally arise.
- We truncate the action up to quadratic terms in the perturbations
- We see that the hybrid formalism allows us both to study different Fock representation of the Dirac and to split the degrees of freedom in both sectors in ways to define one with desirable physical properties and compare the results to the adiabatic approach.

One may choose a set of creation and annihilation variables by decomposing the Dirac fields in modes of the Dirac operator, which has eigenvalues ω_k

$$\Psi \xrightarrow{\textit{decompose}} \begin{pmatrix} x_k^+, & x_k^- \\ \bar{y}_k^+, & \bar{y}_k^- \end{pmatrix} \equiv \boldsymbol{x}^\pm$$

$$\begin{aligned} a_k^\pm &= f_1^{k,\pm}(\alpha, \pi_\alpha, \phi, \pi_\phi) x_k^\pm + f_2^{k,\pm}(\alpha, \pi_\alpha, \phi, \pi_\phi) \bar{y}_{-k}^\pm, \\ \bar{b}_k^\pm &= g_1^{k,\pm}(\alpha, \pi_\alpha, \phi, \pi_\phi) x_k^\pm + g_2^{k,\pm}(\alpha, \pi_\alpha, \phi, \pi_\phi) \bar{y}_{-\vec{k}}^\pm, \quad \text{where} \\ g_1^{k,\pm} &= e^{iJ_k^\pm} \bar{f}_2^{k,\pm}, \qquad g_2^{k,\pm} = -e^{iJ_k^\pm} \bar{f}_1^{k,\pm}, \\ f_2^{k,\pm} &= e^{iF_2^{k,\pm}} \sqrt{1 - \left| f_1^{k,\pm} \right|^2}. \end{aligned}$$

Which leads to a contribution to the Hamiltonian constraint

$$\tilde{H}_k = \sum_{\pm} \Big[h_D^k \Big(\bar{a}_k^\pm a_k^\pm - a_k^\pm \bar{a}_k^\pm + \bar{b}_k^\pm b_k^{(x,y)} - b_k^\pm \bar{b}_k^\pm \Big) + h_J^k \Big(\bar{b}_k^\pm b_{\vec{k}}^\pm - b_k^\pm \bar{b}_k^\pm \Big) + h_I^{k,\pm} a_k^\pm b_k^\pm - \bar{h}_I^{k,\pm} \bar{a}_k^\pm \bar{b}_k^\pm \Big],$$

where the interaction term is

$$\begin{aligned} \bar{h}_I^k &= e^{-iJ_k} \left\{ i \left(f_2^k \partial f_1^k - f_1^k \partial f_2^k \right) + \frac{2\omega_k}{a} f_1^k f_2^k + \tilde{M} \left[\left(f_1^k \right)^2 - \left(f_2^k \right)^2 \right] \right\}, \\ \partial &= \{ H_{|0}, \cdot \} \end{aligned}$$

$$[g_{\mu\nu}] \xrightarrow{LQG} \left[\{A_i^a, E_a^i\} + \mathcal{H}, \mathcal{G}_a, \mathcal{H}_i\right] \xrightarrow{LQC} \left[\{v, b\} + \mathcal{H}\right]$$

v is proportional to the physical volume of the spatial sections and b is related to the Hubble parameter

The total Hilbert space is

$$\mathcal{H} = \mathcal{H}_{\text{kin}}^{\text{matt}} \otimes \mathcal{H}_{\text{kin}}^{\text{grav}} \otimes \mathcal{F}_D,$$

where $\mathcal{F}_D$ is the associated Fock space, with states $|\mathcal{N}_D\rangle$. In this way, the tot.

Quantum evolution:

$$\begin{pmatrix} \hat{a}_k(\eta,\eta_0) \\ \hat{b}_k^\dagger(\eta,\eta_0) \end{pmatrix} = \begin{pmatrix} \alpha_k(\eta,\eta_0) & \beta_k(\eta,\eta_0) \\ -\tilde{\beta}_k(\eta,\eta_0) & \tilde{\alpha}_k(\eta,\eta_0) \end{pmatrix} \begin{pmatrix} \hat{a}_k(\eta_0) \\ \hat{b}_k^\dagger(\eta_0) \end{pmatrix}$$

where $\beta_k = O(h_l^k \omega_k^{-1})$

$$f_1^k = e^{iF_2^k} \frac{Ma}{2\omega_k} + \mathcal{O}\left(\frac{1}{\omega_k^2}\right)$$

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Backreaction and well-defined quantum Hamiltonian constraint:

$$\langle \hat{\mathcal{H}}_0^2 - \hat{\mathcal{H}}_{pert}^2 \rangle = \mathcal{O}\left(\left(h_l^k \right)^2 \right)$$

$$f_1^k = e^{iF_2^k} \frac{Ma}{2\omega_k} - i e^{iF_2^k} \frac{\pi M \pi_a}{3 l_0^3 \omega_k^2} + \mathcal{O}\left(\frac{1}{\omega_k^3} \right)$$

One can then choose pick subsequent terms of f_1 to continue lowering the order of the interaction part of the Hamiltonian with the ansatz

$$f_1^{k,(x,y)} = e^{iF_2^{k,(x,y)}} \sum_{n=1}^{\infty} \frac{(-i)^{n+1}\gamma_n}{\omega_k^n}, \quad f_2^{k,(x,y)} = e^{iF_2^{k,(x,y)}} \sum_{n=0}^{\infty} \frac{(-i)^{n}\tilde{\gamma}_n}{\omega_k^n}.$$

Where the γ coeficients are real and diagonalize the Hamiltonian if

$$\tilde{\gamma}_0 = 1$$

$$\gamma_{n+1} = \frac{-Ma}{2}\tilde{\gamma}_n + \frac{a}{2}\sum_{l=1}^{n} \gamma_{n-l}\partial\gamma_l - \gamma_l\partial\tilde{\gamma}_{n-l} - \frac{2}{a}\tilde{\gamma}_l\gamma_{n+1-l} - M(\gamma_l\gamma_{n-l} - \tilde{\gamma}_l\tilde{\gamma}_{n-l}).$$

$$i\partial_\eta \boldsymbol{x} = \boldsymbol{H}(\eta)\boldsymbol{x}, \qquad \boldsymbol{H} = \pm \begin{pmatrix} -\omega_k & Ma \\ Ma & \omega_k \end{pmatrix}.$$

One diagonalizes the time-dependent Schrödinger Hamiltonian $\boldsymbol{H}(\eta)$ by means of a unitary matrix $\boldsymbol{U}_0$, such that the new variables $\boldsymbol{x}_0 = \boldsymbol{U}_0^\dagger \boldsymbol{x}$ satisfy

$$i\partial_\eta \boldsymbol{x}_0 = \boldsymbol{H}_0 \boldsymbol{x}_0, \qquad \boldsymbol{H}_0 = \boldsymbol{D}_0 - i\boldsymbol{U}_0^\dagger \partial_\eta \boldsymbol{U}_0.$$

This process can be repeated iteratively. At each step one gets the following new variables and Hamiltonian:

$$\boldsymbol{x}_{j+1} = \boldsymbol{U}_{j+1}^\dagger \boldsymbol{x}_j, \qquad \boldsymbol{H}_{j+1} = \boldsymbol{D}_{j+1} - i\boldsymbol{U}_{j+1}^\dagger \partial_\eta \boldsymbol{U}_{j+1}.$$

The diagonal matrix $\boldsymbol{D}_{j+1}$ and the unitary matrix $\boldsymbol{U}_{j+1}$ are found diagonalizing $\boldsymbol{H}_j$, and then $i\partial_\eta \boldsymbol{x}_{j+1} = \boldsymbol{H}_{j+1}\boldsymbol{x}_{j+1}$.
The following approximation improves for every step

$$\tilde{\boldsymbol{x}}_n(\eta) = \tilde{\boldsymbol{U}}_n(\eta, \tilde{\eta}_0)\mathfrak{h}(\tilde{\eta}_0), \quad \mathfrak{h}(\tilde{\eta}_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\tilde{\boldsymbol{U}}_n = \operatorname{diag}\left(\exp\left(-i\int_{\tilde{\eta}_0}^{\eta}\Omega_n\right), \exp\left(i\int_{\tilde{\eta}_0}^{\eta}\Omega_n\right)\right),$$

One obtains a choice for creation and annihilation variables after evolving and undoing all the unitary changes of variables

$$\boldsymbol{x}_{|n}(\eta_0) = \left(\prod_{i=0}^{n}\boldsymbol{U}_i(\eta_0)\right)\tilde{\boldsymbol{U}}_n(\eta_0, \tilde{\eta}_0)\mathfrak{h}(\tilde{\eta}_0).$$

Ultraviolet properties of adiabatic states

$$f_{1|0}^{k,\pm}(\eta) = \frac{Ma(\eta)}{2\omega_k} + \mathcal{O}(\omega_k^{-2}),$$

$$f_{1|1}^{k,\pm}(\eta) = \frac{Ma(\eta)}{2\omega_k} + \frac{iMa'(\eta)}{4\omega_k^2} + \mathcal{O}(\omega_k^{-3}) = \frac{Ma(\eta)}{2\omega_k} - i\frac{\pi M \pi_a(\eta)}{3 l_0^3 \omega_k^2} + \mathcal{O}(\omega_k^{-3}).$$

- Summarizing, we used the available freedom in hybrid Quantum Cosmology to separate the homogeneous background and the inhomogeneous perturbations to find a formalism in which the fundamental tool (the Hamiltonian constraint) is well defined.
- We further restrict our choice, decreasing the asymptotic order of the interaction part Hamiltonian getting an asymptotically diagonal evolution.
- We compare this with the other vacua in the literature, namely adiabatic states and study their ultraviolet properties
- One may study specific cosmologies, like de Sitter, to see if an exact diagonalization exists