Recent developments and comparisons of regularization schemes

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WorkStop/ThinkStart 3.0
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back in 2016 . . .

goals:

- review status of regularization schemes
- give comprehensive and practical introduction for the 'outside world'
- investigate differences/similarities

in arXiv:1705.01827 (hopefully) achieved by:

- finding common language/nomenclature/notation
- providing proper and unique definitions of the schemes
- homework: computing benchmark processes @ NLO (electron self-energy, $e^+e^- \to \gamma^* \to q\bar{q}$)
- explicitly highlighting differences and similarities (29 footnotes in the publication)

To d . . . . . . or not to d ?

traditional dimensional schemes

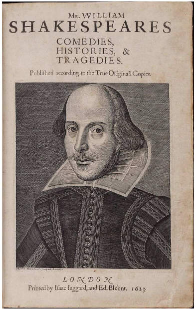
- 't Hooft / Veltman (HV) '72
- conventional dim. reg. (CDR) '73
- dim. reduction (DRED) '79
- four-dim. helicity (FDH) '92

reformulations of dimensional schemes

- six-dim. formalism (SDF) '09
- four-dim. formalism (FDF) '14

non-dimensional schemes

- implicit reg. (IREG) '98
- loop regularization (LORE) '03
- four-dim. reg. / ren. (FDR) '12
- four-dim. unsubtraction (FDU) '16



mathematical consistency ✓

unitarity, causality ✓

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symmetries?

computational efficiency? (analytical/numerical automation)

$$\int \frac{d^d k_{[d]}}{(2\pi)^d}$$

Dimensional schemes - unified framework

- loop/phase-space integration shifted to $d\!=\!4\!-\!2\epsilon$

$$\int \frac{d^4 k_{[4]}}{(2\pi)^4} \cdots \;\rightarrow\; \mu_{\text{DS}}^{4-d} \int \frac{d^d k_{[d]}}{(2\pi)^d} \cdots$$

- freedom for the metric:

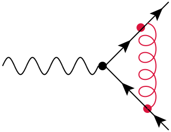
$S_{[4]}$	$\subset$	$QS_{[d]}$	$\subset$	$QS_{[d_s]}$
strictly four-dim.		quasi d-dim.		quasi d_s-dim.
(unregularized)		(integration)		(usually $d_s = 4$)

	CDR	HV	FDH	DRED
'singular' vector fields (related to divergences)	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[d_s]}$	$g^{\mu\nu}_{[d_s]}$
'regular' vector fields (all other VFs)	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[4]}$	$g^{\mu\nu}_{[4]}$	$g^{\mu\nu}_{[d_s]}$

DRED covers all elements of the other schemes.

Dimensional schemes - unified framework

- example: form factor



$$\sim g^{\mu\nu}_{[d_s]}$$

$$\sim \int \frac{d^d k_{[d]}}{(2\pi)^d} \frac{\left[\gamma^\alpha \gamma^\rho \gamma^\mu \gamma^\sigma \gamma_\alpha\right]_{[d_s]}}{[\dots][\dots][\dots]} \times \left[k_\rho k_\sigma\right]_{[d]}$$

$$\sim -2\left[\gamma^\sigma \gamma^\mu \gamma^\rho\right]_{[d_s]} + (4 - d_s)\left[\gamma^\rho \gamma^\mu \gamma^\sigma\right]_{[d_s]}$$

CDR / HV: $d_s \to d = 4 - 2\epsilon$
second term contributes

FDH / DRED: $d_s = 4$
second term vanishes

FDH / DRED:

Do algebra in $d_s = 4$ from the beginning, even though nothing is *strictly* 4-dim!

$\rightarrow$ simplified algebra

Dimensional schemes - renormalization

$$S_{[4]} \quad \subset \quad QS_{[d]} \quad \subset \quad QS_{[d_s]} \quad \equiv \quad QS_{[d]} \oplus QS_{[n_\epsilon]}$$

'evanescent' space

- vector fields in FDH/DRED: $A^{\mu}_{[d_s]} = \underbrace{A^{\mu}_{[d]}}_{\text{gauge field}} + \underbrace{A^{\mu}_{[n_\epsilon]}}_{\epsilon\text{-scalar}}$

QED: $$D^{\mu}_{[d_s]}\,\psi_i = \partial^{\mu}_{[d]}\,\psi_i + i\Big(e\,A^{\mu}_{[d]} + e_e\,A^{\mu}_{[n_\epsilon]}\Big)\,Q\,\psi_i$$

QCD: $$D^{\mu}_{[d_s]}\,\psi_i = \partial^{\mu}_{[d]}\,\psi_i + i\Big(g_s\,A^{\mu,a}_{[d]} + g_e\,A^{\mu,a}_{[n_\epsilon]}\Big)\,T^a_{ij}\,\psi_j$$

- evanescent couplings $e_e,\ g_e$: independent renormalization (non-SUSY theories) e. g.

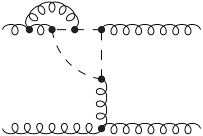
$$\beta = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\Big(\frac{e}{4\pi}\Big)^2 = -\Big(\frac{e}{4\pi}\Big)^4\Big[-\frac{4}{3}N_F\Big] + \dots$$

$$\beta_e = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\Big(\frac{e_e}{4\pi}\Big)^2 = -\Big(\frac{e_e}{4\pi}\Big)^4\big[-4-2\,N_F\big] - \Big(\frac{e}{4\pi}\Big)^2\Big(\frac{e_e}{4\pi}\Big)^2\big[+6\big] + \dots$$

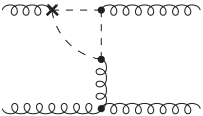
Dimensional schemes - renormalization

- When do we need ϵ-scalars??

$gg \to gg$

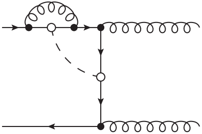


$\mathcal{O}(\alpha_s^2)$

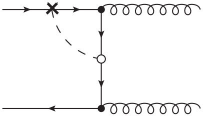


$\mathcal{O}(\alpha_s\,\delta Z_{\alpha_s})$

$q\bar{q} \to gg$



$\mathcal{O}(\alpha_e \alpha_s)$



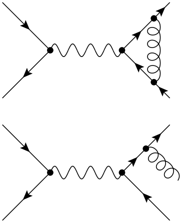
$\mathcal{O}(\alpha_e\,\delta Z_{\alpha_e})$

$\alpha_e = \alpha_s$ (ren.) but $\delta Z_{\alpha_e} \neq \delta Z_{\alpha_s}$

> l-loop calculation in FDH/DRED:
>
> Evanescent couplings have to be distinguished up to $(l-1)$ loops.
> (except you are lucky, e. g. $gg \to gg$ in FDH)

Dimensional schemes

- one-loop example: $e^+e^- \to 2\,jets$



$$\sigma^{(v)}_{\text{FDH/DRED}} \sim \left(\frac{\alpha_s}{\pi}\right)\left[-\frac{1}{\epsilon^2}-\frac{3}{2\epsilon}-\frac{7}{2}\right]$$

$$\sigma^{(v)}_{\text{HV}} \sim \left(\frac{\alpha_s}{\pi}\right)\left[-\frac{1}{\epsilon^2}-\frac{3}{2\epsilon}-\frac{8}{2}\right]$$

$$\sigma^{(r)}_{\text{FDH/DRED}} \sim \left(\frac{\alpha_s}{\pi}\right)\left[+\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+\frac{17}{4}\right]$$

$$\sigma^{(r)}_{\text{HV}} \sim \left(\frac{\alpha_s}{\pi}\right)\left[+\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}+\frac{19}{4}\right]$$

$$\sigma^{(v)}_{\text{DS}} + \sigma^{(r)}_{\text{DS}}\Big|_{\epsilon\to 0} = \left(\frac{\alpha_s}{\pi}\right)\frac{3}{4}C_F$$

CDR, HV, FDH, DRED are perfectly consistent (if done properly) and yield the same results for physical observables.

Dimensional schemes

→ The (quasi) four-dimensional algebra of DRED/FDH simplifies the evaluation of the Lorentz algebra.

→ The UV renormalization of the evanescent couplings is tedious, but not always necessary, done in the standard way, and well-known to high orders.

→ The virtual corrections well understood in all schemes at least up to NNLO. Moreover, transition rules for virtual amplitudes are available.

- NNLO massless [Broggio, CG, Signer, Stöckinger, Visconti '15]
- NNLO massive [CG, Signer, Visconti '16]

> The regularization can be chosen independently for the virtual and the real part.

→ The real corrections are so far understood up to NLO in all schemes, and currently investigated also at NNLO.

- to be published this month/year [CG, Signer]

Real corrections at NNLO - sneak preview

- computed $e^+e^- \to 2jets$ at NNLO in all dimensional schemes, including double-virtual, double-real, and real-virtual contributions

  the good news:

> All dimensional schemes can be used at NNLO and yield same finite physical result. ✓

- double-real contributions in DRED simply obtained by evaluating
  (a) the algebra in four(!) dimensions ($d_s = 4$)
  (b) the ps integrals in d dimensions (as usual)

  order ϵ terms in the matrix element can be neglected,
  $\Rightarrow$ easier integral evaluation

> The evaluation of the double-real contributions is in DRED easier than in CDR.

$$\int \frac{d^d k_{[d]}}{(2\pi)^d} \cdots \gamma_5 \cdots$$

Dimensional schemes and γ_5

$$\int \frac{d^4 k_{[4]}}{(2\pi)^4} \cdots \quad \to \quad \mu_{\mathrm{DS}}^{4-d} \int \frac{d^d k_{[d]}}{(2\pi)^d} \cdots$$

essentially two options for treatment of γ_5:

(i) construction prescription:

['t Hooft/Veltman '72, Breitenlohner/Maison '77]

$$\gamma_5^{\mathrm{BM}} \equiv \frac{i}{4!} \left(\varepsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \right)_{[4]} \equiv \frac{i}{4!} \varepsilon_{[4]}^{\mu\nu\rho\sigma} \left(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \right)_{[d]}$$

(ii) algebraic definition:

(same properties as for $d = 4$)

$$\{\gamma_5^{\mathrm{AC}}, \gamma_{[d]}^{\mu}\} \equiv 0$$

Dimensional schemes and γ_5

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(same properties as for $d = 4$)

$$\{\gamma_5^{\text{AC}}, \gamma_{[d]}^\mu\} \equiv 0$$

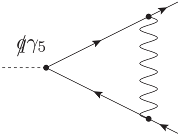
$d \neq 4$:

$$\{\gamma_5^{\text{BM}}, \gamma_{[d]}^\mu\} = 2\,\gamma_{[d-4]}^\mu \gamma_5^{\text{BM}} \neq \{\gamma_5^{\text{AC}}, \gamma_{[d]}^\mu\}$$

Both definitions yield different (intermediate) results!

Example 1

- pseudoscalar form factor in FDH:



$$\sim g^{\mu\nu}_{[d_s]} = g^{\mu\nu}_{[d]} + g^{\mu\nu}_{[n_\epsilon]}$$

(unrenormalized) result for γ_5^{BM}:

$$\sim \varepsilon^{\mu\nu\rho\sigma}_{[4]} \int \ldots \left(\not{q}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\not{q}\right)_{[d]} \cdots$$

$$\sim \frac{1 - \frac{n_\epsilon}{2}}{\epsilon} + \frac{9}{2} + \mathcal{O}(\epsilon)$$

(unrenormalized) result for γ_5^{AC}:

$$\sim \int \ldots \left(\not{q}\gamma_5^{\text{AC}}\right)_{[d]} \cdots$$

$$\sim \frac{1 + \frac{n_\epsilon}{2}}{\epsilon} + \frac{1}{2} + \mathcal{O}(\epsilon)$$

as in CDR: chiral and Lorentz invariance broken for γ_5^{BM}
$\Rightarrow$ additional counterterm(s)

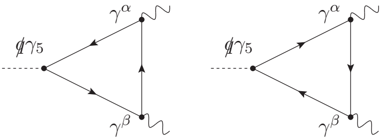
$$\delta Z^{\text{BM}} = \frac{n_\epsilon}{\epsilon} - 4$$

here γ_5^{BM} more complicated:

- strictly(!) 4-dim. $\varepsilon^{\mu\nu\rho\sigma}_{[4]}$
- complicated algebra
- additional counterterms

Example 2

- AVV correlator in FDH:



(unrenormalized) result for $\gamma_5^{\rm BM}$:

$$\sim \varepsilon_{[4]}^{\mu\nu\rho\sigma} \int \mathrm{Tr}\big[(\not{q}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\not{q})\ldots\big]_{[d]}$$

$$\sim \varepsilon_{[4]}^{\alpha\beta\mu\nu}(p_{1,\mu}p_{2,\nu})_{[d]} + \mathcal{O}(\epsilon)$$

(unrenormalized) result for $\gamma_5^{\rm AC}$:

$$\sim \int \mathrm{Tr}\big[\not{q}\gamma_5^{\rm AC}\ldots\big]_{[d]} = 0$$

for standard cyclic trace

$\gamma_5^{\rm BM}$: correct result for (anomalous) axial Ward-identity ✓

here $\gamma_5^{\rm AC}$ more complicated:

- $\gamma_5^{\rm AC}$-odd traces vanish
- breaking of vector Ward-identity and therefore of gauge invariance

Dimensional schemes and γ_5

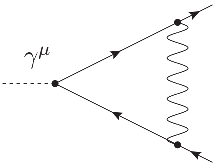
Even though FDH is somehow 'more 4-dimensional' than CDR and HV,
it has similar features regarding γ_5.

Try to go even more towards strictly(!) 4 dimensions
but stay *within* the dimensional framework.

FDF [Fazio,Mastrolia,Mirabella,Torres Bobadilla '14]

inpose abstract algebra:

- start from FDH-regularized quantities, e. g. $\not{k}_{[d]} = \not{k}_{[4]} + \not{k}_{[d-4]} \equiv \not{k}_{[4]} + i\,\mu\,\gamma_5$
- neglect *odd* powers of μ $\not{k}_{[d]}\not{k}_{[d]} = k^2_{[d]} \equiv k^2_{[4]} - \mu^2$



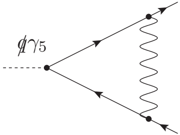
$$\sim \int d^d k_{[d]} \, \frac{\left[\gamma^\alpha \, (\not{k}_1 + i\,\mu\,\gamma_5)\, \gamma^\mu \, (\not{k}_2 + i\,\mu\,\gamma_5)\, \gamma_\alpha\right]_{[4]}}{\left[\, k_1^2 \;\, k_2^2 \;\, k^2 \,\right]_{[d]}}$$

$$= \int d^d k_{[d]} \, \frac{f_1([4]) \; + \; f_2(\mu^2)}{g([d])} \qquad \equiv \qquad \mathcal{M}^{(1)}_{\text{FDH}}\Big|_{n_\epsilon = 2\epsilon}$$

FDF recovers FDH results by using a strictly(!) 4-dimensional numerator algebra.

γ_5 in FDF

- FDF algebra realized in strictly(!) 4 dimensions $\Rightarrow \gamma_5^{\text{BM}} = \gamma_5^{\text{AC}} \equiv \gamma_5$
- example 1:



FDF:

$$\sim \int \ldots \left[(\not{k}_1 + i\,\mu\,\gamma_5)\ \not{q}\gamma_5\ (\not{k}_2 + i\,\mu\,\gamma_5)\right]_{[4]} \ldots$$

$$\sim \frac{1}{\epsilon} + \frac{7}{2} + \mathcal{O}(\epsilon)$$

FDH+ γ_5^{BM}:

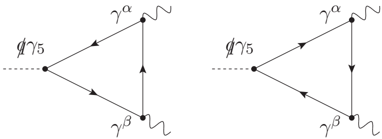
$$\sim \varepsilon_{[4]}^{\mu\nu\rho\sigma} \int \ldots \left(\not{q}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\not{q}\right)_{[d]} \ldots$$

$$\sim \frac{1 - \frac{n_\epsilon}{2}}{\epsilon} + \frac{9}{2} + \mathcal{O}(\epsilon) \overset{n_\epsilon = 2\epsilon}{=} \frac{1}{\epsilon} + \frac{7}{2} + \mathcal{O}(\epsilon)$$

FDF yields the same results as FDH+ γ_5^{BM} (same counterterms).
However, the result is obtained in a much simpler way.

γ_5 in FDF

- example 2:



FDF:

$$\sim \int \mathrm{Tr}\Big[(\not{k}_1 + i\,\mu\,\gamma_5)\ \not{q}\gamma_5\ (\not{k}_2 + i\,\mu\,\gamma_5) \ldots\Big]_{[4]}$$

$$\sim \varepsilon^{\alpha\beta\mu\nu}_{[4]}(p_{1,\mu}p_{2,\nu})_{[4]} + \mathcal{O}(\epsilon)$$

(result stems from μ^2 terms only)

FDH + γ_5^{BM}:

$$\sim \varepsilon^{\mu\nu\rho\sigma}_{[4]} \int \mathrm{Tr}\Big[(\not{q}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\not{q}) \ldots\Big]_{[d]}$$

$$\sim \varepsilon^{\alpha\beta\mu\nu}_{[4]}(p_{1,\mu}p_{2,\nu})_{[d]} + \mathcal{O}(\epsilon)$$

FDF yields correct result for the axial Ward-identity.
(strictly 4-dim. algebra $\Rightarrow$ no breaking of gauge invariance)

Again, the algebra the simpler than FDH+ γ_5^{BM}.

That is (still) the question.

Dimensional schemes

- FDH / DRED perfectly consistent schemes at least up to NNLO
- quasi 4-dim. algebra offers computational advantages (even in non-SUSY theories)
- transition rules for virtual corrections available up to NNLO
- important for comparisons with non-dimensional schemes

FDF

- strictly 4-dim. algebra to recover FDH
- offers advantages in the treatment of γ_5
- (so far) only NLO?

Non-dimensional schemes

- strictly 4-dim. algebra and phase-space integration
- IR divergences regularized in terms of $\ln(\mu_0)$
- transition rules $1/\epsilon \leftrightarrow \ln(\mu_0)$ available, (so far) only NLO?