Torino subtraction scheme for IR singularities at NNLO

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in collaboration with:
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based on:
Magnea et al., arXiv:1806.09570, arXiv:1809.05444

Motivations

- **Small deviations** from Standard Model predictions can provide important tests for New Physics models.
- Hunting for such deviations requires **high precision predictions** to compare with high precision experiments.
- **Next-to-next-to-leading** (NNLO) in **QCD** is the current accuracy standard.
- The automation of QCD computations needs a **fully general** and efficient **treatment of the IR singularities**.

Schemes and tricks to deal with the IR

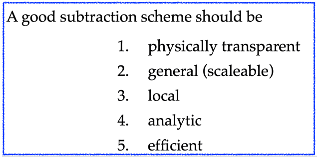
Few scheme available at NLO:

- Slicing: [Giele, Glover]
- Subtraction: dipole[Catani, Seymour 9602277], FKS [Frixione et al. 9512328], NS [Nagy, Soper 0308127]

Many schemes available at NNLO:

- Slicing: $q_\perp$ [Catani, Grazzini 0703012], N-Jettiness [Boughezal et al. 1505.03893, Gaunt et al. 1505.04794]
- Subtraction: Antenna [Gehrmann-DeRidder et al. 0505111], ColorfullNNLO [Del Duca et al. 1603.08927], Nested soft-collinear [Caola et al. 1702.01352], Geometric IR subtraction [Herzog 1804.07949], ϵ-prescription [Frixione, Grazzini 0411399], Sector decomposition [Bonoth et al. 0402265, Anastasiou et al. 0311311], residue subtraction [Czakon 1005.0274]
- New stategies: Unsubtraction [Sborlini et al. 1608.01584], FDR [Pittau 1208.5457]

→ Many options, but still there is room for improvement according to the *five criteria rule* [Melnikov, talk@Amplitude2019]



A good subtraction scheme should be

1. physically transparent
2. general (scaleable)
3. local
4. analytic
5. efficient

Torino Subtraction scheme at NLO

Subtraction pattern

IR-safe observable X at NLO: *[partons in the final state only, massless]*

$$\frac{d\sigma^{\rm NLO}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, V_n\,\delta_n + \int d\Phi_{n+1}\, R_{n+1}\delta_{n+1}\right\}$$

$\delta_i = \delta(X - X_i)$, X_i the i-particle conf., $V_n = 2\mathbf{Re}[\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}]$, $R_{n+1} = |\mathcal{A}_{n+1}^{(0)}|^2$.

Subtraction idea

make the real contribution finite before performing the PS integration by **adding and subtracting a counterterm**, which

- has the same singular limits as R, locally in phase space
- is analytically integrable in d dim

$$\frac{d\sigma_{ct}^{\rm NLO}}{dX} = \int \Phi_{n+1}\, K_{n+1}\,, \quad I_n = \int d\Phi_{\rm rad}\, K_{n+1}$$

$$\frac{d\sigma^{\rm NLO}}{dX} = \int \underbrace{d\Phi_n\left(V_n + I_n\right)\delta_n}_{\text{finite in } d=4} + \int \underbrace{d\Phi_{n+1}\left(R_{n+1}\,\delta_{n+1} - K_{n+1}\,\delta_n\right)}_{\text{finite in } d=4}$$

Implementation of the Subtraction method: the main ingredients

Ingredients of our method:

- **Fundamental limits $\mathbf{S}_i$, $\mathbf{C}_{ij}$** selecting the leading behaviour in terms of invariants $s_{ab} = 2k_a \cdot k_b$

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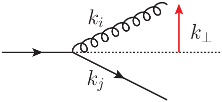

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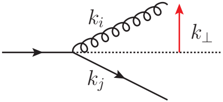

where the **singular structure of R** factorises

- universal soft and collinear NLO kernels
- Born matrix element

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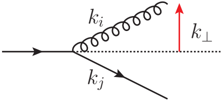

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$$\mathbf{S}_i R(\{k\}) = -\mathcal{N} \sum\nolimits_{c,d} \delta_{f_i g}\, \frac{s_{cd}}{s_{ic}\, s_{id}}\, B_{cd}(\{k\}_{\not{i}})$$

$$\mathbf{C}_{ij} R(\{k\}) = \mathcal{N}\, \frac{1}{s_{ij}}\, P_{ij}^{\mu\nu}(s_{ir}, s_{jr})\, B_{\mu\nu}(\{k\}_{\not{i}\not{j}}, k)$$

$$\mathbf{S}_i \mathbf{C}_{ij} R(\{k\}) = 2\,\mathcal{N}\, C_{f_j}\, \delta_{f_i g}\, \frac{s_{jr}}{s_{ij}\, s_{ir}}\, B(\{k\}_{\not{i}})$$

B_{cd}=color-correlated Born, $B_{\mu\nu}$=spin-correlated Born.

Born kinem.: mass-shell condition and momenta conservation just in the limits.

Implementation of the Subtraction method: the main ingredients

- **partition of the phase space** Φ_{n+1} with sector functions $\mathcal{W}_{ij}$, that satisfy two requirements *[Frixione, Kunszt, Signer 9512328]*:

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  - select the minimum number of singularities

$$\mathbf{S}_i \, \mathcal{W}_{ab} = 0 \,, \quad \forall i \neq a \qquad\qquad \mathbf{C}_{ij} \, \mathcal{W}_{ab} = 0 \,, \quad \forall a, b \notin \pi(i,j)$$

→ at most one soft and/or two collinear partons in a given sector.

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  - sum to unity

$$\sum_{i,j\neq i} \mathcal{W}_{ij} = 1\,, \quad \mathbf{S}_i \sum_{j\neq i} \mathcal{W}_{ij} = 1\,, \quad \mathbf{C}_{ij} \sum_{a,b\in \mathrm{perm}(ij)} \mathcal{W}_{ab} = 1$$

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- **momentum mapping**: $\{k_1, \ldots, k_{n+1}\} \rightarrow \{\bar{k}_1, \ldots, \bar{k}_n\}$ *[Catani, Seymour 9605323]*:

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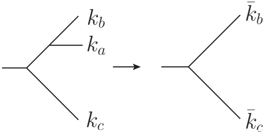
  - phase space factorisation $d\Phi_{n+1} = d\bar{\Phi}_n \, d\bar{\Phi}_{\mathrm{rad}}$
  - n on-shell particles conserving momentum.

$$\{\bar{k}\}^{(abc)} = \left\{ \{k\}_{\not a \not b \not c}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \right\}$$

$$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$$



Implementation of the Subtraction method: counterterm construction

Definition of the counterterm

Sector: $\mathcal{W}_{ij} \rightarrow$ minimal singularity structure $\mathbf{S}_i, \mathbf{C}_{ij}$

Candidate counterterm: $K_{ij} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1 - \mathbf{S}_i)\Big] R\mathcal{W}_{ij}$

→ $\mathbf{S}_i, \mathbf{C}_{ij}$ **commute** both on R and on sector function
→ **overlap** between $\mathbf{S}_i, \mathbf{C}_{ij}$ taken into account

Mapping $\{k_{n+1}\} \rightarrow \{k_n\}^{(abc)}$: *(abc)* chosen according to the **invariants in the kernels**

$$\overline{\mathbf{S}}_i R(\{k\}) = -\mathcal{N} \sum_{c,d \neq i} \delta_{f_i g} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}(\{\bar{k}\}^{(icd)})$$

$$\overline{\mathbf{C}}_{ij} R(\{k\}) = \mathcal{N} \frac{1}{s_{ij}} P_{ij}^{\mu\nu}(s_{ir}, s_{jr}) B_{\mu\nu}(\{\bar{k}\}^{(ijr)})$$

$$\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} R(\{k\}) = 2\mathcal{N} C_{f_j} \delta_{f_i g} \frac{s_{jr}}{s_{ij}\, s_{ir}} B(\{\bar{k}\}^{(ijr)})$$

- Collinear limit: single mapping → *dipole=(ijr)*
- Soft limit: different mapping for each contribution to $\mathbf{S}_i R(\{k\})$ → *dipole=(icd)*

Implementation of the Subtraction method: counterterm construction

Sector $\mathcal{W}_{ij}$: local counterterm in the remapped kinematic

$$\overline{K}_{ij} \equiv \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\right) R\,\mathcal{W}_{ij}$$

Sector function sum rules → summing over sectors $\overline{K}$ becomes independent of $\mathcal{W}_{ij}$

$$\begin{aligned}
\overline{K} &= \sum_{i,j\neq i}\overline{K}_{ij} = \sum_i \left(\overline{\mathbf{S}}_i R\right)\left[\,\overbrace{\mathbf{S}_i\sum_{j\neq i}\mathcal{W}_{ij}}^{=1}\,\right] + \sum_{i,j>i}\left(\mathbf{C}_{ij}R\right)\left[\,\overbrace{\mathbf{C}_{ij}\left(\mathcal{W}_{ij}+\mathcal{W}_{ji}\right)}^{=1}\,\right] \\
&\qquad - \sum_{i,j\neq i}\left(\overline{\mathbf{S}}_i\mathbf{C}_{ij}R\right)\left[\,\underbrace{\mathbf{S}_i\mathbf{C}_{ij}\mathcal{W}_{ij}}_{=1}\,\right] \\
&= \sum_i \overline{\mathbf{S}}_i\,R + \sum_{i,j>i}\overline{\mathbf{C}}_{ij}\left(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j\right)R
\end{aligned}$$

Remarks

- the integrated counterterm has to **match the poles of** V, which is **not** split into sectors.
- the sector functions would have made the integration much more involved.

→ this way **analytic integration** is feasible with **standard techniques**.

Implementation of the Subtraction method: counterterm integration

- **Parametrisation of the phase space** *[Catani, Seymour 9605323]*

$$\boxed{d\Phi_{n+1} = d\Phi_n^{(abc)}\, d\Phi_{\rm rad}^{(abc)} \equiv d\Phi_n^{(abc)} \times d\Phi_{\rm rad}\left(s_{bc}^{(abc)}; y, z, \phi\right)}$$

$$d\Phi_n^{(abc)} \propto \left(s_{bc}^{(abc)}\right)^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y)\left[(1-y)^2 y(1-z)z\right]^{-\epsilon}$$

$$s_{bc}^{(abc)} = s_{abc}\,, \quad s_{ab} = y s_{bc}^{(abc)}\,, \quad s_{ac} = z(1-y) s_{bc}^{(abc)}\,, \quad s_{bc} = (1-z)(1-y) s_{bc}^{(abc)}$$

- **Integration**
  1. we choose different parametrisation for the soft and the hard-collinear contr.
  2. soft kernel is parametrised differently for each term of the sum.

$$I^s = -\mathcal{N}\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_i \delta_{f_i g}\sum_{c,d\neq i}\int d\Phi_{\rm rad}\left(s_{cd}^{(icd)}; y, z, \phi\right)\frac{s_{cd}}{s_{ic}\, s_{id}}\, B_{cd}\left(\{\bar{k}\}^{(icd)}\right)$$

$$= -\mathcal{N}\,\frac{\varsigma_{n+1}}{\varsigma_n}\sum_i \delta_{f_i g}\sum_{c,d\neq i} B_{cd}\left(\{\bar{k}\}^{(icd)}\right)\left(s_{cd}^{(icd)}\right)^{-\epsilon}\frac{(4\pi)^{\epsilon-2}\,\Gamma(1-\epsilon)\,\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)}$$

Remark:

- freedom to **adapt the parametrisation** to the invariants appearing in the kernels.
- integrated counterterm exact in ϵ.

Subtraction pattern at NNLO

NNLO Subtraction pattern

- more configurations contribute

$$\frac{d\sigma^{\mathrm{NNLO}}}{dX} = \int d\Phi_n \; VV_n \, \delta_n(X) + \int d\Phi_{n+1} \, RV_{n+1} \; \delta_{n+1}(X) + \int d\Phi_{n+2} \, RR_{n+2} \; \delta_{n+2}(X)$$

$$RR_{n+2} = \left|\mathcal{A}^{(0)}_{n+2}\right|^2 \qquad VV_n = \left|\mathcal{A}^{(1)}_{n}\right|^2 + 2\mathbf{Re}\left[\mathcal{A}^{(0)\dagger}_{n}\mathcal{A}^{(2)}_{n}\right] \qquad RV_{n+1} = 2\mathbf{Re}\left[\mathcal{A}^{(0)\dagger}_{n+1}\mathcal{A}^{(1)}_{n+1}\right]$$

- more counterterms to add and subtract

$$\int d\Phi_{n+2} \, K^{(1)} \; \delta_{n+1} \; : \qquad K^{(1)} \quad \rightarrow$$ same 1-unr. singularities as RR

$$\int d\Phi_{n+2} \left(K^{(12)} + K^{(2)}\right) \delta_n \; : \qquad K^{(12)} + K^{(2)} \quad \rightarrow$$ same 2-unr. singularities as RR.
[1-unr.(2-unr.), pure 2-unr.]

$$\int d\Phi_{n+1} \, K^{(\mathrm{RV})} \; \delta_n \; : \qquad K^{(\mathrm{RV})} \quad \rightarrow$$ same 1-unr. singularities as RV

and integrate in the radiative phase space

$$I^{(i)} = \int d\Phi_{\mathrm{rad},i} \, K^{(i)} \, , \qquad I^{(12)} = \int d\Phi_{\mathrm{rad},1} \, K^{(12)} \, , \qquad I^{(\mathrm{RV})} = \int d\Phi_{\mathrm{rad}} \, K^{(\mathrm{RV})} \, ,$$

Subtraction pattern at NNLO

$$\frac{d\sigma^{\mathrm{NNLO}}}{dX} = \int d\Phi_n \Big[\underbrace{VV_n}_{\text{singular in d=4 , finite in } \Phi_n} \Big] \delta_n + \int d\Phi_{n+1} \Big[\underbrace{\big(RV_{n+1}\big)}_{\text{singular in d=4, singular in } \Phi_{n+1}} \delta_{n+1} \Big] + \int d\Phi_{n+2} \Big[\underbrace{RR_{n+2}}_{\text{finite in d=4, singular in } \Phi_{n+2}} \delta_{n+2} \Big]$$

Subtraction pattern at NNLO

$$\frac{d\sigma^{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left[\underbrace{VV_n}_{\text{singular in d=4 , finite in } \Phi_n} \right] \delta_n$$

$$+ \int d\Phi_{n+1} \left[\underbrace{\left(RV_{n+1}\right)}_{\text{singular in d=4, singular in } \Phi_{n+1}} \delta_{n+1} \right]$$

$$+ \int d\Phi_{n+2} \left[\underbrace{RR_{n+2}\,\delta_{n+2} - K^{(1)}\delta_{n+1} - \left(K^{(12)} + K^{(2)}\right)\delta_n}_{\text{finite in d=4 and in } \Phi_{n+2}} \right]$$

Subtraction pattern at NNLO

$$
\begin{aligned}
\frac{d\sigma^{\mathrm{NNLO}}}{dX} &= \int d\Phi_n \Big[\underbrace{VV_n}_{\text{singular in d=4 , finite in } \Phi_n} \Big] \delta_n \\
&+ \int d\Phi_{n+1} \Big[\underbrace{\Big(RV_{n+1}\Big)\, \delta_{n+1} - \Big(K^{(\mathrm{RV})}\Big)\, \delta_n}_{\text{singular in d=4, finite in } \Phi_{n+1}} \Big] \\
&+ \int d\Phi_{n+2} \Big[\underbrace{RR_{n+2}\, \delta_{n+2} - K^{(1)} \delta_{n+1} - \Big(K^{(12)} + K^{(2)}\Big) \delta_n}_{\text{finite in d=4 and in } \Phi_{n+2}} \Big]
\end{aligned}
$$

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& + \int d\Phi_{n+2} \Big[\underbrace{RR_{n+2}\, \delta_{n+2} - K^{(1)} \delta_{n+1} - \left(K^{(12)} + K^{(2)}\right) \delta_n}_{\text{finite in d=4 and in } \Phi_{n+2}} \Big]
\end{aligned}
$$

Subtraction pattern at NNLO

$$\frac{d\sigma^{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left[\underbrace{VV_n + I^{(2)} + I^{(\mathrm{RV})}}_{\text{finite in d=4 and in } \Phi_n} \right] \delta_n$$

$$+ \int d\Phi_{n+1} \left[\underbrace{\underbrace{\left(RV_{n+1} + I^{(1)} \right)}_{\text{finite in d=4, singular in } \Phi_{n+1}} \delta_{n+1} - \underbrace{\left(K^{(\mathrm{RV})} - I^{(12)} \right)}_{\text{finite in d=4, singular in } \Phi_{n+1}} \delta_n}_{\text{finite in d=4 and in } \Phi_{n+1}} \right]$$

$$+ \int d\Phi_{n+2} \left[\underbrace{RR_{n+2}\,\delta_{n+2} - K^{(1)}\delta_{n+1} - \left(K^{(12)} + K^{(2)} \right)\delta_n}_{\text{finite in d=4 and in } \Phi_{n+2}} \right]$$

Subtraction algorithm at NNLO: ingredients

Ingredients of our method:

- **new singular configurations of RR**

$\mathbf{S}_{ij} \to ij$ soft

$\mathbf{C}_{ijk} \to ijk$ collinear $\qquad \mathbf{C}_{ijkl} \to (ij), (kl)$ indep. collinear

$\mathbf{SC}_{ijk} \to i$ soft, jk collinear $\quad \mathbf{C}_{ij}\,[\mathbf{S}_k] \qquad \mathbf{CS}_{ijk} \to ij$ collinear, k soft $\quad \mathbf{S}_i\,[\mathbf{C}_{ij}]$

- **partition of Φ_{n+2}:** $s_{qi} = 2q_{\rm cm} \cdot k_i$, $e_i = s_{qi}/s$, $w_{ij} = (s\, s_{ij})/(s_{qi}\, s_{qj})$

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sum_{a,b\neq a} \sum_{c\neq a,d\neq a,c} \sigma_{abcd}}\,, \qquad \sigma_{ijkl} = \frac{1}{e_i^{\alpha}\, w_{ij}^{\beta}} \frac{1}{(e_k + \delta_{kj} e_i) w_{kl}} \qquad \alpha > \beta > 1$$

different topologies to select the minimum number of singularities:

$$\begin{array}{lcllllll} \mathcal{W}_{ijjk} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ij} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \\ \mathcal{W}_{ijkj} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ij} & \mathbf{C}_{ijk} & \mathbf{SC}_{ijk} & \mathbf{CS}_{ijk} \\ \mathcal{W}_{ijkl} & : & \mathbf{S}_i & \mathbf{C}_{ij} & \mathbf{S}_{ij} & \mathbf{C}_{ijkl} & \mathbf{SC}_{ikl} & \mathbf{CS}_{ijk} \end{array}$$

factorisation into NLO sector function under single-unresolved limits

$$\mathbf{S}_i \mathcal{W}_{ijkl} = \mathcal{W}_{kl}\, \mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} \quad \mathbf{C}_{ij} \mathcal{W}_{ijkl} = \mathcal{W}_{kl}\, \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \quad \mathbf{S}_i\mathbf{C}_{ij} \mathcal{W}_{ijkl} = \mathcal{W}_{kl}\, \mathbf{S}_i\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}$$

sum rules that reduce to 1 the sectors sharing the same singular limits

Subtraction algorithm at NNLO: ingredients

- **counterterm identification** [sector $\mathcal{W}_{ijjk}$]

$$\boxed{(1-\mathbf{S}_i)\,(1-\mathbf{C}_i)\left(1-\mathbf{C}_{ijk}\right)\left(1-\mathbf{SC}_{ijk}\right)\;RR\,\mathcal{W}_{ijjk}=\text{finite}}$$

according to the number of unresolved partons and the nature (democratic or hierarchical) of the limits, we define

$$RR\,\mathcal{W}_{ijjk}-K^{(\mathbf{1})}_{ijjk}-K^{(\mathbf{2})}_{ijjk}-K^{(\mathbf{12})}_{ijjk}=\text{finite}$$

$^{(1)}$ = one unres. , $^{(2)}$ = two unres. democratic , $^{(12)}$ = two unres. hierarchical

$$\begin{aligned}
K^{(\mathbf{1})}_{ijjk} &= \Big[\mathbf{S}_i+\mathbf{C}_{ij}\,(1-\mathbf{S}_i)\Big]\,RR\,\mathcal{W}_{ijjk}\\
K^{(\mathbf{2})}_{ijjk} &= \Big[\mathbf{S}_{ij}+\mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij})+\mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij})\left(1-\mathbf{C}_{ijk}\right)\Big]\,RR\,\mathcal{W}_{ijjk}\\
K^{(\mathbf{12})}_{ijjk} &= -\Big\{\Big[\mathbf{S}_i+\mathbf{C}_{ij}\,(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ij}+\mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij})\Big]\\
&\qquad +\mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij})\left(1-\mathbf{C}_{ijk}\right)\Big\}RR\,\mathcal{W}_{ijjk}
\end{aligned}$$

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$^{(1)}$ = one unres. , $^{(2)}$ = two unres. democratic , $^{(12)}$ = two unres. hierarchical

$$\begin{aligned}
K^{(\mathbf{1})}_{ijjk} &= \Big[\mathbf{S}_i + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\Big]\,RR\,\mathcal{W}_{ijjk}\\
K^{(\mathbf{2})}_{ijjk} &= \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij})\left(1-\mathbf{C}_{ijk}\right)\Big]\,RR\,\mathcal{W}_{ijjk}\\
K^{(\mathbf{12})}_{ijjk} &= -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}\,(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\,(1-\mathbf{S}_{ij})\Big]\\
&\qquad + \mathbf{SC}_{ijk}\,(1-\mathbf{S}_{ij})\left(1-\mathbf{C}_{ijk}\right)\Big\}\,RR\,\mathcal{W}_{ijjk}
\end{aligned}$$

Remarks:

- $\mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{S}_{ij}, \mathbf{C}_{ijk}, \mathbf{SC}_{ijk}$ commute
- soft-collinear contribution cancels in the sum $K^{(12)} + K^{(2)} \Longrightarrow$ minimal structure $(1-\mathbf{S}_i)\mathbf{SC}_{ijk}\,RR\mathcal{W}_{ijjk} = 0$

Subtraction algorithm at NNLO: ingredients

- **Singular structure of RR** under the fundamental limits
  - universal kernel *[Catani, Grazzini 9903516, 9810389] [Campbell, Glover 9710255]*
  - Born matrix element

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \Big[\sum_{e,f\neq i,j} \mathcal{I}^{(i)}_{cd}\, \mathcal{I}^{(j)}_{ef}\, B_{cdef}\Big(\{k\}_{\cancel{i}\cancel{j}}\Big) + \mathcal{I}^{(ij)}_{cd}\, B_{cd}\Big(\{k\}_{\cancel{i}\cancel{j}}\Big)\Big]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s^2_{ijk}}\, P^{\mu\nu}_{ijk}(s_{ir}, s_{jr}, s_{kr})\, B_{\mu\nu}\Big(\{k\}_{\cancel{i}\cancel{j}\cancel{k}}, k_{ijk}\Big)$$

$$\mathbf{C}_{ijkl}\, RR(\{k\}) \propto \frac{1}{s_{ij}\, s_{kl}}\, P^{\mu\nu}_{ij}(s_{ir}, s_{jr})\, P^{\rho\sigma}_{kl}(s_{kr'}, s_{lr'})\, B_{\mu\nu\rho\sigma}\Big(\{k\}_{\cancel{i}\cancel{j}\cancel{k}\cancel{l}}, k_{ij}, k_{kl}\Big)$$

$$\mathbf{SC}_{ijk}\, RR(\{k\}) = \mathbf{CS}_{jki} RR(\{k\}) \propto \frac{1}{s_{jk}} \sum_{c,d\neq i} P^{\mu\nu}_{jk}\, \mathcal{I}^{(i)}_{cd}\, B^{cd}_{\mu\nu}\Big(\{k\}_{\cancel{i}\cancel{j}\cancel{k}}, k_{jk}\Big)$$

$\mathcal{I}^{(i)}_{cd}$ = single eikonal current, $\mathcal{I}^{(ij)}_{cd}$ = double eikonal current.
$P^{\mu\nu}_{ijk}(s_{ir}, s_{jr}, s_{kr})$ = triple splitting function.

Born kinem.:
$K^{(1)}_{ijjk}, K^{(12)}_{ijjk}, K^{(2)}_{ijjk}$ do not satisfy **mass-shell condition** and **momenta conservation**

$\Longrightarrow$ momentum mapping needed!

Subtraction algorithm at NNLO: ingredients

- **double momentum mapping**: $\{k_1, \dots, k_{n+2}\} \to \{\bar{k}_1, \dots, \bar{k}_n\}$.

two kind of mapping to treat different kernels and simplify the integration.

1) two-steps mapping

$$\bar{k}_n^{(acd,bef)} = \bar{k}_n^{(acd)}\,, \qquad n \neq a, b, e, f$$

$$\bar{k}_e^{(acd,bef)} = \bar{k}_b^{(acd)} + \bar{k}_e^{(acd)} - \frac{\bar{s}_{be}^{(acd)}}{\bar{s}_{bf}^{(acd)} + \bar{s}_{ef}^{(acd)}}\,\bar{k}_f^{(acd)} \qquad \bar{k}_f^{(acd,bef)} = \frac{\bar{s}_{bef}^{(acd)}}{\bar{s}_{bf}^{(acd)} + \bar{s}_{ef}^{(acd)}}\,\bar{k}_f^{(acd)}$$

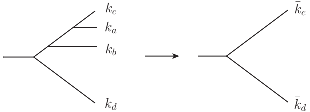
PS fact.: $d\Phi_{n+2} = d\Phi_n^{(acd,bef)} \cdot d\Phi_{\mathrm{rad},1}(\bar{s}_{bef}^{(acd)}; y', z', \phi') \cdot d\Phi_{\mathrm{rad},1}(s_{acd}; y, z, \phi)$

2) one-step mapping

$$\bar{k}_n^{(abcd)} = k_n\,, \qquad n \neq a, b, c, d$$

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad} + s_{bd} + s_{cd}}\,k_d$$

$$\bar{k}_d^{(abcd)} = \frac{s_{abcd}}{s_{ad} + s_{bd} + s_{cd}}\,k_d$$



PS fact.: $d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad},2}(\bar{s}_{cd}^{(abcd)}; y, z, \phi, y', z', x')$.

From the ingredients to the recipe

Example: **double unresolved counterterm** and its integral

Applying the sum rules to the sector functions we end up with

$$\begin{aligned}\overline{K}^{(2)} \;=\; \sum_{i}\Bigg\{ & \sum_{j>i}\overline{\mathbf{S}}_{ij} + \sum_{j>i}\sum_{k>j}\overline{\mathbf{C}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\right)\\ & + \sum_{j>i}\sum_{\substack{k>i\\k\neq j}}\sum_{\substack{l>k\\l\neq j}}\overline{\mathbf{C}}_{ijkl}\left(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}-\overline{\mathbf{S}}_{il}-\overline{\mathbf{S}}_{jl}\right)\\ & + \sum_{j\neq i}\sum_{\substack{k\neq i\\k>j}}\overline{\mathbf{SC}}_{ijk}\left(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}\right)\left(1-\overline{\mathbf{C}}_{ijk}-\sum_{l\neq i,j,k}\overline{\mathbf{C}}_{iljk}\right)\\ & + \sum_{j>i}\sum_{k\neq i,j}\overline{\mathbf{CS}}_{ijk}\left(1-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\right)\left(1-\overline{\mathbf{C}}_{ijk}-\sum_{l\neq i,j,k}\overline{\mathbf{C}}_{ijkl}\right)\Bigg\} RR\,,\end{aligned}$$

- **No sector functions left** as needed for matching the VV poles.
- Full freedom in defining the mapped terms.

From the ingredients to the recipe

Example: **double unresolved counterterm** and its integral

Applying the sum rules to the sector functions we end up with

$$
\begin{aligned}
\overline{K}^{(2)} \; = \; & \sum_{i} \Bigg\{ \sum_{j>i} \overline{\mathbf{S}}_{ij} + \sum_{j>i} \sum_{k>j} \overline{\mathbf{C}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) \\
& + \sum_{j>i} \sum_{\substack{k>i \\ k\neq j}} \sum_{\substack{l>k \\ l\neq j}} \overline{\mathbf{C}}_{ijkl} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk} - \overline{\mathbf{S}}_{il} - \overline{\mathbf{S}}_{jl}\right) \\
& + \sum_{j\neq i} \sum_{\substack{k\neq i \\ k>j}} \overline{\mathbf{SC}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik}\right) \Bigg(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{iljk}\Bigg) \\
& + \sum_{j>i} \sum_{k\neq i,j} \overline{\mathbf{CS}}_{ijk} \left(1 - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}\right) \Bigg(1 - \overline{\mathbf{C}}_{ijk} - \sum_{l\neq i,j,k} \overline{\mathbf{C}}_{ijkl}\Bigg) \Bigg\} RR \,,
\end{aligned}
$$

- **No sector functions left** as needed for matching the VV poles.
- Full freedom in defining the mapped terms.

Starting from the limit

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \Bigg[\sum_{e,f\neq i,j} \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{ef}^{(j)}\, B_{cdef}\Big(\{k\}_{\not{i}\,\not{j}}\Big) + \mathcal{I}_{cd}^{(ij)}\, B_{cd}\Big(\{k\}_{\not{i}\,\not{j}}\Big)\Bigg]$$

we are free to map each term separately, adapting the choice to the invariants appearing in the kernel

$$\begin{aligned}\overline{\mathbf{S}}_{ij}\, RR \;\propto\; \sum_{\substack{c\neq i,j\\ d\neq i,j,c}} \Bigg[& \sum_{\substack{e\neq i,j,c,d\\ f\neq i,j,c,d}} \mathcal{I}_{cd}^{(i)}\, \overline{\mathcal{I}}_{ef}^{(j)(icd)} B_{cdef}\Big(\{\bar{k}\}^{(icd,jef)}\Big) \\ & + 4 \sum_{e\neq i,j,c,d} \mathcal{I}_{cd}^{(i)}\, \overline{\mathcal{I}}_{ed}^{(j)(icd)} B_{cded}\Big(\{\bar{k}\}^{(icd,jed)}\Big) \\ & + 2\, \mathcal{I}_{cd}^{(i)}\, \mathcal{I}_{cd}^{(j)} B_{cdcd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \Big(\mathcal{I}_{cd}^{(ij)} - \frac{1}{2}\,\mathcal{I}_{cc}^{(ij)} - \frac{1}{2}\,\mathcal{I}_{dd}^{(ij)}\Big) B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big)\Bigg]\end{aligned}$$

The PS parametrisation follows the mapping structure to simplify the integral

Starting from the limit

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} \mathcal{I}^{(i)}_{cd}\, \mathcal{I}^{(j)}_{ef}\, B_{cdef}\Big(\{k\}_{\not{i}\,\not{j}}\Big) + \mathcal{I}^{(ij)}_{cd}\, B_{cd}\Big(\{k\}_{\not{i}\,\not{j}}\Big) \right]$$

we are free to map each term separately, adapting the choice to the invariants appearing in the kernel

$$\begin{aligned} \overline{\mathbf{S}}_{ij}\, RR \;\propto\; & \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d}} \mathcal{I}^{(i)}_{cd}\, \overline{\mathcal{I}}^{(j)(icd)}_{ef}\, B_{cdef}\Big(\{\bar{k}\}^{(icd,jef)}\Big) \\ & \quad + 4 \sum_{e\neq i,j,c,d} \mathcal{I}^{(i)}_{cd}\, \overline{\mathcal{I}}^{(j)(icd)}_{ed}\, B_{cded}\Big(\{\bar{k}\}^{(icd,jed)}\Big) \\ & \quad + 2\, \mathcal{I}^{(i)}_{cd}\, \mathcal{I}^{(j)}_{cd}\, B_{cdcd}\Big(\{\bar{k}\}^{(ijcd)}\Big) + \Big(\mathcal{I}^{(ij)}_{cd} - \frac{1}{2}\, \mathcal{I}^{(ij)}_{cc} - \frac{1}{2}\, \mathcal{I}^{(ij)}_{dd}\Big) B_{cd}\Big(\{\bar{k}\}^{(ijcd)}\Big) \Bigg] \end{aligned}$$

The PS parametrisation follows the mapping structure to simplify the integral

$$\begin{aligned} I^{(2)}_{\mathrm{SS},cdef} &= \int d\Phi_{\mathrm{rad},2}\, \mathcal{I}^{(i)}_{cd}\, \overline{\mathcal{I}}^{(j),(icd)}_{ef} = \int d\overline{\Phi}^{(icd,jef)}_{\mathrm{rad}}\, \overline{\mathcal{I}}^{(j),(icd)}_{ef} \int d\Phi^{(icd)}_{\mathrm{rad}}\, \mathcal{I}^{(i)}_{cd} \\ &= \delta_{f_i g}\delta_{f_j g} \left[\frac{(4\pi)^{\epsilon-2}}{(\bar{s}^{(icd,jef)}_{cd})^{\epsilon}}\, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \right] \left[\frac{(4\pi)^{\epsilon-2}}{(\bar{s}^{(icd,jef)}_{ef})^{\epsilon}}\, \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \right] \end{aligned}$$

All the contributions to $\overline{K}^{(2)}$ have been integrated

$$I^{(2)} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\, I^{(2)}_{\rm ss} + I^{(2)}_{\rm hcc} + I^{(2)}_{\rm cc4} + I^{(2)}_{\rm sc3} \,\right]$$

and organised according to the different colour structures

$$\begin{aligned} I^{(2)}_{\rm ss} = \Bigg[\, & 2\Big(\sum_{a,b} C_{f_a} C_{f_b}\Big) I^{\rm ss}_{C_f C_f} + 8\Big(\sum_{a} C_{f_a}^2\Big) I^{\rm ss}_{C_f^2} \\ & -\Big(\sum_{a} C_{f_a}\Big)\Big(N_f\, T_R\, I^{\rm ss}_{C_f T_R} - \frac{C_A}{2}\, I^{\rm ss}_{C_f C_A}\Big)\Bigg] B(\{\bar{k}\}) \\ & +2 \sum_{c,d\neq c}\Bigg[-2\Big(\sum_{a} C_{f_a}\Big) I^{\rm ss}_{C_f B_{cd}} - 2 C_{f_d} I^{\rm ss}_{C_d B_{cd}} + N_f\, T_R\, I^{\rm ss}_{T_R B_{cd}} - \frac{C_A}{2}\, I^{\rm ss}_{C_A B_{cd}}\Bigg] B_{cd}(\{\bar{k}\}) \\ & +2 \sum_{c,d\neq c} I^{\rm ss}_{B_{cdcd}}\, B_{cdcd}(\{\bar{k}\}) + 4 \sum_{\substack{c,d\neq c\\ e\neq d}} I^{\rm ss}_{B_{cded}}\, B_{cded}(\{\bar{k}\}) \\ & + \sum_{\substack{c,d\neq c\\ e,f\neq e}} I^{\rm ss}_{B_{cdef}}\, B_{cdef}(\{\bar{k}\}) + \mathcal{O}(\epsilon)\,. \end{aligned}$$

Remark: $I^{(2)}_{\rm cc4}, I^{(2)}_{\rm sc3}$ feature a NLO×NLO complexity.

$$
\begin{aligned}
I^{\rm ss}_{C_f C_f} &= \frac{1}{\epsilon^4}+\frac{4}{\epsilon^3}+\Big(16-\frac{7}{6}\,\pi^2\Big)\frac{1}{\epsilon^2}+\Big(60-\frac{14}{3}\,\pi^2-\frac{50}{3}\,\zeta(3)\Big)\frac{1}{\epsilon}+216-\frac{56}{3}\,\pi^2-\frac{200}{3}\,\zeta(3)+\frac{29}{120}\,\pi^4 \\
I^{\rm ss}_{C_f^2} &= \Big(1-\frac{\pi^2}{6}\Big)\frac{1}{\epsilon^2}+\Big(10-\frac{2}{3}\,\pi^2-6\,\zeta(3)\Big)\frac{1}{\epsilon}+68-4\,\pi^2-24\,\zeta(3)-\frac{7}{72}\,\pi^4 \\
I^{\rm ss}_{C_f T_R} &= \frac{2}{3}\,\frac{1}{\epsilon^3}+\frac{34}{9}\,\frac{1}{\epsilon^2}+\Big(\frac{464}{27}-\frac{7}{9}\,\pi^2\Big)\frac{1}{\epsilon}+\frac{5896}{81}-\frac{131}{27}\,\pi^2-\frac{76}{9}\,\zeta(3) \\
I^{\rm ss}_{C_f C_A} &= \frac{2}{\epsilon^4}+\frac{35}{3}\,\frac{1}{\epsilon^3}+\Big(\frac{487}{9}-\frac{8}{3}\,\pi^2\Big)\frac{1}{\epsilon^2}+\Big(\frac{6248}{27}-\frac{269}{18}\,\pi^2-\frac{154}{3}\,\zeta(3)\Big)\frac{1}{\epsilon}+\frac{77404}{81}-\frac{3829}{54}\,\pi^2-\frac{2050}{9}\,\zeta(3)-\frac{23}{60}\,\pi^4 \\
I^{\rm ss}_{C_f B_{cd}} &= \ln\frac{\bar{s}_{cd}}{\mu^2}\Big[-\frac{1}{\epsilon^3}-\frac{4}{\epsilon^2}-\Big(16-\frac{7}{6}\,\pi^2\Big)\frac{1}{\epsilon}-60+\frac{14}{3}\,\pi^2+\frac{50}{3}\,\zeta(3) \\
&\qquad +\frac{1}{2}\ln\frac{\bar{s}_{cd}}{\mu^2}\Big(\frac{1}{\epsilon^2}+\frac{4}{\epsilon}+16-\frac{7}{6}\,\pi^2\Big)-\frac{1}{6}\ln^2\frac{\bar{s}_{cd}}{\mu^2}\Big(\frac{1}{\epsilon}+4\Big)+\frac{1}{24}\ln^3\frac{\bar{s}_{cd}}{\mu^2}\Big] \\
I^{\rm ss}_{C_d B_{cd}} &= 4\ln\frac{\bar{s}_{cd}}{\mu^2}\Big[-\Big(1-\frac{\pi^2}{6}\Big)\frac{1}{\epsilon}-10+\frac{2}{3}\,\pi^2+6\,\zeta(3)+\frac{1}{2}\ln\frac{\bar{s}_{cd}}{\mu^2}\Big(1-\frac{\pi^2}{6}\Big)\Big] \\
I^{\rm ss}_{T_R B_{cd}} &= \ln\frac{\bar{s}_{cd}}{\mu^2}\Big[-\frac{2}{3}\,\frac{1}{\epsilon^2}-\frac{34}{9}\,\frac{1}{\epsilon}-\frac{464}{27}+\frac{7}{9}\,\pi^2+\ln\frac{\bar{s}_{cd}}{\mu^2}\Big(\frac{2}{3}\,\frac{1}{\epsilon}+\frac{34}{9}\Big)-\frac{4}{9}\ln^2\frac{\bar{s}_{cd}}{\mu^2}\Big] \\
I^{\rm ss}_{C_A B_{cd}} &= \ln\frac{\bar{s}_{cd}}{\mu^2}\Big[-\frac{2}{\epsilon^3}-\frac{35}{3}\,\frac{1}{\epsilon^2}-\Big(\frac{487}{9}-\frac{8}{3}\,\pi^2\Big)\frac{1}{\epsilon}-\frac{6248}{27}+\frac{269}{18}\,\pi^2+\frac{154}{3}\,\zeta(3) \\
&\qquad +\ln\frac{\bar{s}_{cd}}{\mu^2}\Big(\frac{2}{\epsilon^2}+\frac{35}{3}\,\frac{1}{\epsilon}+\frac{487}{9}-\frac{8}{3}\,\pi^2\Big)-\frac{2}{3}\ln^2\frac{\bar{s}_{cd}}{\mu^2}\Big(\frac{2}{\epsilon}+\frac{35}{3}\Big)+\frac{2}{3}\ln^3\frac{\bar{s}_{cd}}{\mu^2}\Big] \\
I^{\rm ss}_{B_{cdcd}} &= -4(1-\zeta(3))\Big(\frac{1}{\epsilon}-2\ln\frac{\bar{s}_{cd}}{\mu^2}\Big)-40-\frac{\pi^2}{3}+12\,\zeta(3)+\frac{13}{36}\,\pi^4 \\
I^{\rm ss}_{B_{cded}} &= \ln\frac{\bar{s}_{cd}}{\mu^2}\ln\frac{\bar{s}_{ed}}{\mu^2}\Big(1-\frac{\pi^2}{6}\Big) \\
I^{\rm ss}_{B_{cdef}} &= \ln\frac{\bar{s}_{cd}}{\mu^2}\ln\frac{\bar{s}_{ef}}{\mu^2}\Big[\frac{1}{\epsilon^2}+\frac{4}{\epsilon}+16-\frac{7}{6}\,\pi^2-\frac{1}{2}\Big(\ln\frac{\bar{s}_{cd}}{\mu^2}+\ln\frac{\bar{s}_{ef}}{\mu^2}\Big)\Big(\frac{1}{\epsilon}+4\Big)+\frac{1}{6}\Big(\ln^2\frac{\bar{s}_{cd}}{\mu^2}+\ln^2\frac{\bar{s}_{ef}}{\mu^2}\Big)+\frac{1}{4}\ln\frac{\bar{s}_{cd}}{\mu^2}\ln\frac{\bar{s}_{ef}}{\mu^2}\Big]
\end{aligned}
$$

Outlook

Outlook

Some work is done:

- General structure of a local, analytic sector subtraction has been proposed.
- All the integrals needed for $K^{(2)}$ and $K^{(RV)}$ are done.

Some work is in progress:

- Combining the results to check the cancellation of the IR poles for a generic process.

A lot of work remains to be done:

- Implementation in a differential code.
- Generalisation to initial state radiation.
- Extension to massive particles.

Backup

Sum rules and factorisation for NNLO sector functions

$$\mathbf{S}_{ik}\left(\sum_{b\neq i}\sum_{d\neq i,k}\mathcal{W}_{ibkd}+\sum_{b\neq k}\sum_{d\neq k,i}\mathcal{W}_{kbid}\right)=1\,,$$

$$\mathbf{C}_{ijk}\sum_{abc\,\in\,\mathrm{perm}(ijk)}\left(\mathcal{W}_{abbc}+\mathcal{W}_{abcb}\right)=1\,,$$

$$\mathbf{S}_{i}\,\mathbf{C}_{ijk}\left(\mathcal{W}_{ij}^{(\alpha\beta)}+\mathcal{W}_{ik}^{(\alpha\beta)}\right)=1\,,$$

$$\mathbf{S}_{ij}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\mathrm{perm}(ij)}\left(\mathcal{W}_{abbk}+\mathcal{W}_{akbk}\right)=1\,,\qquad \mathbf{S}_{ik}\,\mathbf{C}_{ijkl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{klij}\right)=1\,.$$

$$\mathbf{SC}_{ijk}\,\mathbf{S}_{ij}\sum_{b\neq i}\mathcal{W}_{ibjk}=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{S}_{ik}\sum_{d\neq i,k}\mathcal{W}_{ijkd}=1\,,$$

$$\mathbf{CS}_{ijk}\,\mathbf{C}_{ijk}\left(\mathcal{W}_{ijkj}+\mathcal{W}_{jiki}\right)=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{C}_{ijkl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{jikl}\right)=1\,,$$

$$\mathbf{CS}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkj}=1\,,\qquad \mathbf{CS}_{ijk}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkl}=1\,,$$

$$\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\sum_{ab\,\in\,\mathrm{perm}(jk)}\left(\mathcal{W}_{iaab}+\mathcal{W}_{iaba}\right)=1\,,\qquad \mathbf{SC}_{ikl}\,\mathbf{C}_{ijkl}\left(\mathcal{W}_{ijkl}+\mathcal{W}_{ijlk}\right)=1\,,$$

$$\mathbf{SC}_{ijk}\,\mathbf{C}_{ijk}\,\mathbf{S}_{ik}\left(\mathcal{W}_{ijkj}+\mathcal{W}_{ikkj}\right)=1\,,\qquad \mathbf{SC}_{ijk}\,\mathbf{C}_{ijkl}\,\mathbf{S}_{ik}\,\mathcal{W}_{ijkl}=1\,.$$

$$\mathbf{S}_i\,\mathcal{W}_{ijjk}=\mathcal{W}_{jk}\,\mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,\quad \mathbf{C}_{ij}\,\mathcal{W}_{ijjk}=\mathcal{W}_{[ij]k}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,\quad \mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ijjk}=\mathcal{W}_{jk}\,\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,$$

$$\mathbf{S}_i\,\mathcal{W}_{ijkj}=\mathcal{W}_{kj}\,\mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,\quad \mathbf{C}_{ij}\,\mathcal{W}_{ijkj}=\mathcal{W}_{k[ij]}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,\quad \mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ijkj}=\mathcal{W}_{kj}\,\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,$$

$$\mathbf{S}_i\,\mathcal{W}_{ijkl}=\mathcal{W}_{kl}\,\mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,\quad \mathbf{C}_{ij}\,\mathcal{W}_{ijkl}=\mathcal{W}_{kl}\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,\quad \mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ijkl}=\mathcal{W}_{kl}\,\mathbf{S}_i\,\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\,,$$

Example: **one unresolved counterterm** and its integral

$$K^{(1)} = \sum_{i,j\neq i}\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\, RR \sum_{k\neq i,j}\Big(\mathcal{W}_{ijjk} + \mathcal{W}_{ijkj} + \sum_{l\neq i,j,k}\mathcal{W}_{ijkl}\Big)$$

NNLO sectors factorise into NLO sectors and mapping is applied

$$\begin{aligned}\overline{K} \;&=\; \sum_{i,j\neq i}\sum_{\substack{k\neq i\\ l\neq i,k}}\Big[\Big(\mathbf{S}_i\,\mathcal{W}_{ij}^{(\alpha\beta)}\Big)\,(\overline{\mathbf{S}}_i\,RR)\,\overline{\mathcal{W}}_{kl} + \Big(\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\Big)\,(\overline{\mathbf{C}}_{ij}\,RR)\,\overline{\mathcal{W}}_{kl}\\ &\qquad\qquad - \Big(\mathbf{S}_i\mathbf{C}_{ij}\,\mathcal{W}_{ij}^{(\alpha\beta)}\Big)\,(\overline{\mathbf{S}}_i\overline{\mathbf{C}}_{ij}\,RR)\,\overline{\mathcal{W}}_{kl}\Big]\\ &=\; \sum_{\substack{k\neq i\\ l\neq i,k}} \underbrace{\overline{\mathcal{W}}_{kl}}_{\text{NLO sector}}\;\underbrace{\Big[\sum_i \overline{\mathbf{S}}_i\,RR + \sum_{i,j>i}\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j)\,RR\Big]}_{\text{1-unresolved structure}}\end{aligned}$$

Kinematic mapping of sector functions allows to factorise the structure of **NLO sectors out of the radiation phase space**, and integrate only single-unresolved kernels.

$$I^{(1)} \propto \sum_{k,l}\overline{\mathcal{W}}_{kl}\Big[\sum_{i,j>i}\int d\Phi^{(ijr)}_{\mathrm{rad},1}\,\overline{\mathbf{C}}_{ij}(1-\overline{\mathbf{S}}_i-\overline{\mathbf{S}}_j)\,RR(\{k\}) + \sum_i\int d\Phi_{\mathrm{rad},1}\,\overline{\mathbf{S}}_i\,RR(\{k\})\Big]$$

Real-virtual limits

Collinear, soft and soft-collinear limits of the real-virtual matrix element read

[Bern et al. 9903516] [Catani, Grazzini 0007142]

$$
\begin{aligned}
\mathbf{C}_{ij}\, RV &= \mathcal{N}_1 \frac{1}{s_{ij}} \left[P_{ij}^{\mu\nu} V_{\mu\nu} + \mathcal{N}_1 c_\Gamma \,(s_{ij})^{-\epsilon} \cos(\pi\epsilon)\, P_{ij}^{(1)\mu\nu} B_{\mu\nu} \right] \\
\mathbf{S}_i\, RV &= -\mathcal{N}_1 \sum_{k,\, l\neq k} \mathcal{I}_{kl}^{(i)} \Bigg\{ V_{kl} - \mathcal{N}_1 \Bigg[\left(\frac{C_A}{\epsilon^2} \frac{\pi\epsilon\, c_\Gamma}{\tan(\pi\epsilon)} \left(\mathcal{I}_{kl}^{(i)}\right)^{\epsilon} + \frac{\beta_0 S_\epsilon}{2\epsilon(4\pi)^2\mu^{2\epsilon}} \right) B_{kl} \\
&\qquad - \frac{2\pi}{\epsilon} \sum_{p\neq k,l} \left(\mathcal{I}_{lp}^{(i)}\right)^{\epsilon} B_{klp} \Bigg] \Bigg\} \\
\mathbf{S}_i\mathbf{C}_{ij}\, RV &= \mathcal{N}_1\, 2\, C_{f_j} \left[\mathcal{I}_{jr}^{(i)} V - \mathcal{N}_1 \left(\frac{C_A}{\epsilon^2} \frac{\pi\epsilon\, c_\Gamma}{\tan(\pi\epsilon)} \left(\mathcal{I}_{jr}^{(i)}\right)^{1+\epsilon} + \frac{\beta_0 S_\epsilon}{2\epsilon(4\pi)^2\mu^{2\epsilon}} \mathcal{I}_{jr}^{(i)} \right) B \right]
\end{aligned}
$$

$\mathcal{N}_1 = \frac{8\pi\alpha_s\mu^{2\epsilon}}{S_\epsilon}$, $S_\epsilon = (4\pi e^{-\gamma_E})^{\epsilon}$, $B_{klp} = \sum_{a,\, b,\, c} f_{abc} \langle \mathcal{M}_B | T_k^a T_l^b T_p^c | \mathcal{M}_B \rangle \rightarrow$ tripole

$$
V_{\mu\nu} = \frac{\alpha_s}{\pi} \left[-\frac{1}{2\epsilon^2} \left(\sum_i C_{f_i} \right) B_{\mu\nu} + \frac{1}{\epsilon} \left(\sum_i \gamma_i^{(1)} \right) B_{\mu\nu} - \frac{1}{2\epsilon} \sum_{i,j\neq i} \ln \frac{s_{ij}}{\mu^2} B_{\mu\nu,ij} + H_{\mu\nu} \right]
$$

<u>Remark</u>: $\mathbf{S}_i\mathbf{C}_{ij} RV$ is independent of tripoles thank to the symmetry properties of B_{klp}.

Double virtual poles

$$
\begin{aligned}
VV\Big|_{1/\epsilon} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ & -\frac{1}{\epsilon^4}\frac{1}{8}\left(\sum_i C_{f_i}\right)^2 B \\
& +\frac{1}{\epsilon^3}\,\frac{1}{4}\left(\sum_i C_{f_i}\right)\left[\left(\frac{3}{8}\,b_0 + 2\sum_i \gamma_i^{(1)}\right) B - \sum_{i,j\neq i} \ln\frac{s_{ij}}{\mu^2}\; B_{ij}\right] \\
& +\frac{1}{\epsilon^2}\,\frac{1}{4}\Bigg[\left(-\frac{b_0}{2}\sum_i \gamma_i^{(1)} - \frac{\widehat{\gamma}_K^{(2)}}{4}\sum_i C_{f_i} - 2\left(\sum_i \gamma_i^{(1)}\right)^2\right) B \\
& \qquad +\left(\frac{b_0}{4} + 2\sum_i \gamma_i^{(1)}\right)\sum_{i,j\neq i} \ln\frac{s_{ij}}{\mu^2}\; B_{ij} - \frac{1}{4}\sum_{\substack{i,j\neq i\\ k,l\neq k}} \ln\frac{s_{ij}}{\mu^2}\;\ln\frac{s_{kl}}{\mu^2}\; B_{ijkl}\Bigg] \\
& +\frac{1}{\epsilon}\,\frac{1}{8}\left[4\sum_i \gamma_i^{(2)}\, B - \widehat{\gamma}_K^{(2)}\sum_{i,j\neq i} \ln\frac{s_{ij}}{\mu^2}\; B_{ij}\right]\Bigg\} \\
+\left(\frac{\alpha_s}{\pi}\right)\Bigg\{ & -\frac{1}{\epsilon^2}\,\frac{1}{2}\left(\sum_i C_{f_i}\right) V + \frac{1}{\epsilon}\left(\sum_i \gamma_i^{(1)}\right) V - \frac{1}{\epsilon}\,\frac{1}{2}\sum_{i,j\neq i} \ln\frac{s_{ij}}{\mu^2}\; V_{ij}\Bigg\}\,.
\end{aligned}
$$

$$b_0 = \frac{11C_A - 4T_R\,N_f}{3}\,, \quad \hat{\gamma}_K^{(1)} = 2, \quad \gamma_q^{(1)} = -\tfrac{3}{4}\,C_F\,, \quad \gamma_g^{(1)} = -\tfrac{1}{4}\,b_0\,, \quad \hat{\gamma}_K^{(2)} = \left(\tfrac{67}{18} - \zeta(2)\right) C_A - \tfrac{5}{9}\,N_f$$

$$\gamma_q^{(2)} = \left(-\tfrac{3}{32} + \tfrac{3}{4}\,\zeta(2) - \tfrac{3}{2}\,\zeta(3)\right) C_F^2 + \left(-\tfrac{961}{864} - \tfrac{11}{16}\,\zeta(2) + \tfrac{13}{8}\,\zeta(3)\right) C_A\,C_F + \left(\tfrac{65}{432} + \tfrac{1}{8}\,\zeta(2)\right) N_f\,C_F$$

$$\gamma_g^{(2)} = \left(-\tfrac{173}{108} + \tfrac{11}{48}\,\zeta(2) + \tfrac{1}{8}\,\zeta(3)\right) C_A^2 + \left(\tfrac{8}{27} - \tfrac{1}{24}\,\zeta(2)\right) N_f\,C_A + \tfrac{1}{8}\,N_f\,C_F$$

Cancellation of poles proportional to V

$$VV\Big|_{1/\epsilon}^{V} = -\left(\frac{\alpha_s}{\pi}\right)\left\{\frac{1}{2\epsilon^2}\left(\sum_i C_{f_i}\right)V + \frac{1}{\epsilon}\sum_i\left[\delta_{f_i\{q,\bar{q}\}}\frac{3}{4}C_F + \delta_{f_i g}\frac{11C_A - 4\,T_R\,N_f}{12}\right]V + \frac{1}{2\epsilon}\sum_{i,j\neq i}\ln\frac{s_{ij}}{\mu^2}\,V_{ij}\right\}.$$

The hard-collinear and the soft contributions to $I^{(\mathbf{RV})}$ are

$$I_{\mathrm{HC}}^{(\mathbf{RV})}\Big|_{1/\epsilon}^{V} = \left[I_{\mathrm{C}}^{(\mathbf{RV})} - I_{\mathrm{SC}}^{(\mathbf{RV})}\right]\Big|_{1/\epsilon}^{V} = -\left(\frac{\alpha_s}{\pi}\right)\sum_p\left\{\delta_{f_p g}\frac{C_A + 4\,T_R\,N_f}{12}\,\frac{1}{\epsilon} + \delta_{f_p\{q,\bar{q}\}}\frac{C_F}{4}\frac{1}{\epsilon}\right\}V$$

$$I_{\mathrm{S}}^{(\mathbf{RV})}\Big|_{1/\epsilon}^{V} = \left(\frac{\alpha_s}{\pi}\right)\left[\left(\frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\right)\sum_p\left(\delta_{f_p\{q,\bar{q}\}}\,C_F + \delta_{f_p g}\,C_A\right)V + \frac{1}{2\epsilon}\sum_{k,\,l\neq k}\log\frac{s_{kl}}{\mu^2}V_{kl}\right]$$

The contribution $\left[I_{\mathrm{HC}}^{(\mathbf{RV})} - I_{\mathrm{S}}^{(\mathbf{RV})}\right]\Big|_{1/\epsilon}^{V}$ cancels all the poles of VV proportional to V.

→ $VV + I^{(\mathbf{RV})}$: only "*finite* $\times V$" coming from the finite part of $I^{(\mathbf{RV})}$.

NLO counterterms: factorisation approach

Factorised **virtual** amplitude

A *n*-particle massless virtual amplitude factorises in regions according to
[Sterman 9606312] [Gardi, Magnea 0908.3273]

Factorisation formula

$$\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i\Big((p_i\cdot n_i)^2/(n_i^2\mu^2)\Big)}{\mathcal{J}_{i,\mathrm{E}}\Big((\beta_i\cdot n_i)^2/n_i^2\Big)}\right]\mathcal{S}_n(\beta_i\cdot\beta_j)\,\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right)$$

where $p_i^\mu = Q\beta_i^\mu$, $\beta_i^2 = 0$, and $n_i^2 \neq 0$ auxiliary vector, μ renormalisation scale.

Functions' properties:

- universality
- gauge invariance
- simple operator definition

Remarks: $\mathcal{J}_\mathrm{E}$ avoids soft-collinear double counting;
$n_i^2 \neq 0$ avoids spurious collinear singularities (in practice $n_i^2 = 0$).

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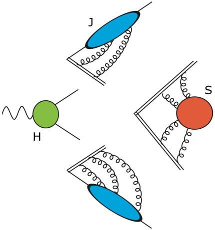
where $p_i^\mu = Q\beta_i^\mu$, $\beta_i^2 = 0$, and $n_i^2 \neq 0$ auxiliary vector, μ renormalisation scale.

- Collinear function: $\bar{u}_s(p)\mathcal{J}_q = \langle p, s|\,\overline{\psi}(0)\Phi_n(0,\infty)\,|0\rangle$
- Soft-Collinear function: $\mathcal{J}_\mathrm{E} = \langle 0|\,\Phi_\beta(\infty,0)\Phi_n(0,\infty)\,|0\rangle$
- Soft function: $\mathcal{S}_n(\beta_i\cdot\beta_j) = \langle 0|\prod_{k=1}^{n}\Phi_{\beta_k}(\infty,0)\,|0\rangle$
- Hard region: $\mathcal{H}_n$ colour vector, finite reminder



Wilson line operator: $\Phi_v(\lambda_2,\lambda_1)$

Remarks: $\mathcal{J}_\mathrm{E}$ avoids soft-collinear double counting;
$n_i^2 \neq 0$ avoids spurious collinear singularities (in practice $n_i^2 = 0$).

Counterterms at NLO

Recall:
$$\frac{d\sigma^{\mathrm{NLO}}}{dX} = \int \underbrace{d\Phi_n \left[V_n + I_n\right] \delta_n}_{\text{finite in } d=4} + \int \underbrace{d\Phi_{n+1} \left[R_{n+1}\,\delta_{n+1} - K_{n+1}\,\delta_n\right]}_{\text{finite in } d=4}$$

Factorisation → virtual contribution decomposed in functions

$$V_n = \mathcal{H}_n^{(0)\dagger}(p_i)\; S_{n,0}^{(1)}\; \mathcal{H}_n^{(0)}(p_i) + \sum_{i=1}^{n} \mathcal{H}_n^{(0)\dagger}(p_i) \left[J_{i,0}^{(1)}(p_i) - J_{i,\mathrm{E},0}^{(1)}(\beta_i)\right] \mathcal{H}_n^{(0)}(p_i)\;.$$

Completeness → virtual functions linked to real functions

$$J_{i,0}^{(1)}(l,p,n) + \int d\Phi_1\; J_{i,1}^{(0)}(k;l,p,n) = \mathrm{fin.} \qquad S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1\; S_{n,1}^{(0)}(k,\beta_i) = \mathrm{fin.}$$

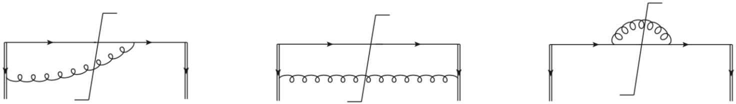
⟹ Starting from the virtual structure we can **identify real emission counterterms**.

Counterterm list at NLO

$$K_{n+1}^{s} = \mathcal{H}_n^{(0)\dagger}(p_i)\; S_{n,1}^{(0)}\; \mathcal{H}_n^{(0)}(p_i)$$
$$K_{n+1}^{c} = \sum_{i=1}^{n} \mathcal{H}_n^{(0)\dagger}(\ldots p_{i-1}, l, p_{i+1}\ldots)\, J_{i,1}^{(0)}(k_i; l, p_i, n_i)\, \mathcal{H}_n^{(0)}(\ldots p_{i-1}, l, p_{i+1}\ldots)$$
$$K_{n+1}^{sc} = \sum_{i=1}^{n} \mathcal{H}_n^{(0)\dagger}(\ldots p_{i-1}, l, p_{i+1}\ldots)\, J_{i,\mathrm{E},1}^{(0)}(k_i; l, p_i, n_i)\, \mathcal{H}_n^{(0)}(\ldots p_{i-1}, l, p_{i+1}\ldots)$$

Collinear splitting at NLO

Test: splitting function for quark-induced process



Single-radiative jet function at the lowest perturbative order, Feynman gauge

$$\sum_s J_{q,1}(k;l,p,n) = \frac{4\pi\alpha_s C_F}{(l^2)^2}(2\pi)^d\delta^d(l-p-k)\left[-\not{l}\gamma_\mu\not{p}\gamma^\mu\not{l} + \frac{l^2}{k\cdot n}\left(\not{l}\not{n}\not{p}+\not{p}\not{n}\not{l}\right)\right]$$

$\Longrightarrow$ **Sudakov parametrisation** for momenta p^μ and k^μ

$$p^\mu = zl^\mu + \mathcal{O}(l_\perp)\,, \qquad k^\mu = (1-z)l^\mu + \mathcal{O}(l_\perp)\,, \qquad n^2 = 0\,.$$

$\Longrightarrow$ **Leading behaviour** in the $l_\perp \to 0$ limit

$$\sum_s J_{q,1}(k;l,p,n) = \frac{8\pi\alpha_s C_F}{l^2}(2\pi)^d\,\delta^d(l-p-k)\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right],$$

The leading order DGLAP splitting function $P_{q\to qg}$.

NNLO counterterms for the purely soft contribution

Factorisation @NNLO → pure soft contribution

$$VV_n\Big|_{\text{soft}} = (VV)_n^{(2s)} + (VV)_n^{(1s)} = \mathcal{H}_n^{(0)\,\dagger}\, S_{n,0}^{(2)}\, \mathcal{H}_n^{(0)} + \left(\mathcal{H}_n^{(0)\,\dagger}\, S_{n,0}^{(1)}\, \mathcal{H}_n^{(1)} + \text{h.c.}\right)$$

$$RV_{n+1}\Big|_{\text{soft}} = \mathcal{H}_{n+1}^{(0)\,\dagger} S_{n+1,0}^{(1)} \mathcal{H}_{n+1}^{(0)}$$

Completeness → virtual functions linked to real functions

$$S_{n,0}^{(2)}(\beta_i) + \int d\Phi_1\ \underline{\underline{S_{n,1}^{(1)}(k,\beta_i)}} + \int d\Phi_2\ \underline{S_{n,2}^{(0)}(k_1,k_2,\beta_i)} = \text{finite}$$

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1\ \underline{\underline{S_{n,1}^{(0)}(k,\beta_i)}} = \text{finite}$$

$$S_{n+1,0}^{(1)}(\beta_i) + \int d\Phi_1\ S_{n+1,1}^{(0)}(k,\beta_i) = \text{finite}$$

$\Longrightarrow$ from the virtual structure we can identify **real emission soft counterterms** according to their kinematics

Purely Soft counterterms at NNLO

$$\mathcal{H}_n^{(0)\,\dagger}\ \underline{\underline{S_{n,1}^{(1)}}}\ \mathcal{H}_n^{(0)} + \left(\mathcal{H}_n^{(0)\,\dagger}\ \underline{\underline{S_{n,1}^{(0)}}}\ \mathcal{H}_n^{(1)} + \text{h.c.}\right) \ \rightarrow\ K^{(\text{RV})}$$

$$\mathcal{H}_n^{(0)\,\dagger}\ \underline{S_{n,2}^{(0)}}\ \mathcal{H}_n^{(0)} \ \rightarrow\ K^{(12)} + K^{(2)} \qquad \mathcal{H}_{n+1}^{(0)\,\dagger}\ S_{n+1,1}^{(0)}\ \mathcal{H}_{n+1}^{(0)} \ \rightarrow\ K^{(1)}$$