
3rd workstop

“paving the way to alternative NNLO strategies”

The FKS² subtraction scheme: status and future

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what this is and is not

- this is us asking for help, not us telling you the solution
- ⇒ this is **not** a conference talk, everything is in flux
- in the spirit of a workstop

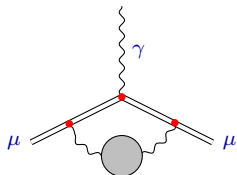
what I will talk about

- explain massification
- explain FKS and FKS^2
- state the problems
 - numerical finite matrix elements in $d = 4$
 - collinear stabilisation / massification

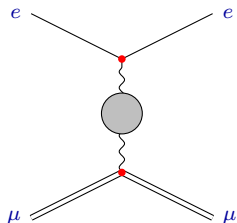
- measure HVP using t -channel μ - e scattering at 10ppm

$$a_{\mu}^{\text{HLO}} = \frac{\alpha}{\pi} \int_0^1 dx (1-x) \underbrace{\Delta\alpha_{\text{had}}\left(\frac{x^2}{x-1}m_{\mu}^2\right)}_{\propto d\sigma/dt}$$

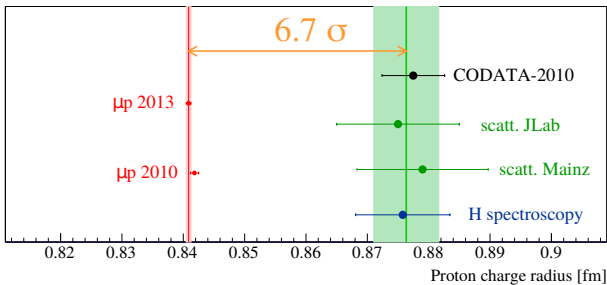
[Carloni Calame et al. 2015, Abbiendi et al 2016]



- proposed experiment @ CERN's M2
[Matteuzzi, YU et al 2019 (LOI)]
- huge theory effort [Alacevich et al 2018, Mastrolia et al 2017 and 2018, Fael, Passera 2019]



⇒ NNLO emission-from- e -line-only (largest part) [Banerjee, Engel, Signer, YU soon] using [Bernreuther et al. 2004] with FKS²

Proton radius puzzle: 6σ -discrepancy between μp and $e p$


[A. Antognini, PSI 2019]

- re-measure $e p$ scattering @ JLab (PRad)

$$\left(\frac{d\sigma}{d\Omega}\right)_{ep} = \epsilon_{\text{exp}} \times \frac{N(ep \rightarrow ep \text{ in } \theta)}{N(ee \rightarrow ee)} \times \underbrace{\left(\frac{d\sigma}{d\Omega}\right)_{ee}}_{\text{main uncertainty}}$$

what we want

- parton-level, NNLO (and beyond) (fully) massive calculations
- ⇒ fully differential, no analytic integration !
- MEG, MUONE, MUSE, MESA, Prad, ...

what we have

- a way to obtain leading mass effects in any two-loop amplitude (massification)
- an NNLO subtraction scheme for massive QED (FKS^2)
- the framework to implement many processes (MCMULE)

massification /'masɪ'fɪkeɪʃ(ə)n/

(noun)

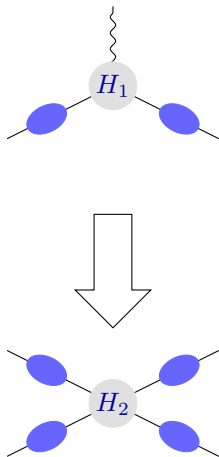
Methods to add leading mass effects
to amplitudes in perturbation theory

Derivatives:

- **massify** verb
- **massified** adjective

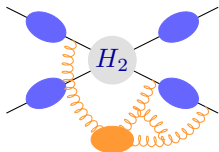
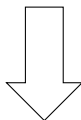
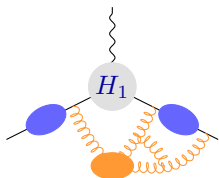
one external mass $m \ll Q^2 = s$

- SCET inspired \sim fragmentation fct.
- Bhabha scattering (photonic) [Penin 2005]
matching $1/\epsilon \rightarrow \ln m_e, \ln m_\gamma$
- QCD with $n_f = n_m = n_h = 0$
[Mitov, Moch 2006]
- matching
 $F_{\gamma^*}(m) = \sqrt{Z_J} \times \sqrt{Z_J} \times F(0)$
- Z_J : jet fct., independent of hard scale s , $\supset \ln(m^2/\mu^2)$

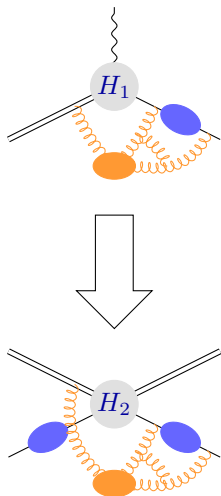


light fermion loops

- n_m terms [Becher, Melnikov 07]
- $F_{\gamma^*}(m) = \mathcal{S} \times \sqrt{Z_J} \times \sqrt{Z_J} \times F(0)$
- $\mathcal{S}(s, m)$: soft function, only contributions from vacuum polarization diagrams with massive fermions, $\supset \ln(m^2/s)$
- $\mathcal{A}_{ee \rightarrow ee}(m) = \mathcal{S}' \times Z_J^{A/2} \times \mathcal{A}(0)$
- factorisation \leftrightarrow resummation via RG equations



- two different masses $M \gg m \gg 0$
[Engel, Gneidiger, Signer, YU 18]
- $F_\mu(m) = \mathcal{S} \times \sqrt{Z_q} \times F_\mu(0)$
- $\mathcal{A}_{\mu e}(m) = \mathcal{S}' \times \sqrt{Z_q \times \bar{Z}_q} \times \mathcal{A}(0)$



The FKS² scheme: status

- originally developed for QCD [Frixione, Kunszt, Signer 1995]
 - everything massive \rightarrow no collinear singularities
- \Rightarrow most complexity drops out

credo at NLO and beyond

- everything that vegas sees needs to be finite (explicitly and implicitly)
- \Rightarrow keep subtracting stuff until it is

- example: $\mu(p) \rightarrow \nu\bar{\nu}e(q) + \gamma(k)$

$$\lim_{\xi \rightarrow 0} \mathcal{M}_{n+1}^{(0)} = \mathcal{M}_n^{(0)} \mathcal{E} \propto \frac{1}{(p \cdot k)(q \cdot k)} \propto \frac{1}{\xi^2} \frac{1}{1 - y\beta}$$

- with $\xi \propto k^0$, $\cos \angle(q, k) = y$, and $\beta = |\vec{q}|/q^0$
- write $d\sigma_r$

$$d\sigma_r = \underbrace{d\phi_{n+1}}_{\propto \xi^{1-2\epsilon} d\xi} \underbrace{\mathcal{M}_{n+1}^{(0)}}_{\propto \xi^{-2}} \propto d\phi_n \times \frac{dy d\Omega^{(2-2\epsilon)}}{(1-y^2)^\epsilon}$$

$$d\xi \xi^{-1-2\epsilon} \underbrace{(\xi^2 \mathcal{M}_{n+1}^{(0)})}_{\text{finite for } \xi \rightarrow 0}$$

- introduce unphysical ξ_c and expand as ϵ -distribution

$$\xi^{-1-2\epsilon} = \frac{\xi_c^{-2\epsilon}}{2\epsilon} \delta(\xi) + \left(\xi^{-1-2\epsilon} \right)_c$$

$$\int d\xi \left(\frac{1}{\xi^n} \right)_c f(\xi) = \int d\xi \frac{f(\xi) - f(0)\Theta(\xi - \xi_c)}{\xi^n}$$

- $d\sigma_r = d\sigma^{(s)} + d\sigma^{(h)}$

$$d\sigma^{(s)} \propto d\phi_n \mathcal{M}_n^{(0)} \times \underbrace{\frac{\xi_c^{-2\epsilon}}{2\epsilon} \int \frac{dy d\Omega^{(2-2\epsilon)}}{(1-y^2)^\epsilon}}_{\hat{\mathcal{E}}(\xi_c)} \mathcal{E}$$

$$d\sigma^{(h)} \propto d\phi_n^{d=4} d\xi dy d\Omega^{(2)} \times \left(\frac{1}{\xi} \right)_c (\xi^2 \mathcal{M}_{n+1}^{(0)})$$

- $d\sigma^{(h)}$ is finite $\Rightarrow \epsilon \rightarrow 0$ numerically
- $d\sigma^{(s)}$ is 'trivial' because of $\delta(\xi)$

$$d\sigma^{(s)} \propto \frac{\xi_c^{-2\epsilon}}{2\epsilon} \mathcal{M}_n^{(0)} \int d\Omega \mathcal{E} = \hat{\mathcal{E}}(\xi_c) \mathcal{M}_n^{(0)}$$

- eikonal \mathcal{E} and $\hat{\mathcal{E}}$ is build from building blocks
- $\hat{\mathcal{E}}\mathcal{M}_n^{(0)} + \mathcal{M}_n^{(1)} = \text{finite}$ (KLN)
- use ξ_c to test implementation (any IR safe $\frac{d^n\sigma}{dx_1 \dots dx_n}$ can't depend on ξ_c)

problems at NNLO

- matrix elements may not finite any more (explicit $1/\epsilon$)
- more ways photons can become soft

recipe to extend this to NNLO:

1st step everything vegas sees needs to be finite!

2nd step keep subtracting until it is

3rd step hope for the best and deal with fall-out later

term 1: $\mathcal{M}_{n+1}^{(1)}$ (real \times virtual)

- like NLO, but $d\sigma^{(h)} \propto \mathcal{M}_{n+1}^{(1)}$ is not finite \Rightarrow split further
 - chop the pole (and induced terms): $\overline{\text{MS}}$ -like subtraction
 - $\mathcal{M}_{n+1}^{(1)} = \underbrace{\mathcal{M}_{n+1}^{(1)f}}_{\Rightarrow d\sigma^{(fin)}} - \underbrace{\hat{\mathcal{E}}(\xi_{cB})\mathcal{M}_{n+1}^{(0)}}_{\Rightarrow d\sigma^{(sin)}} : \text{eikonal subtraction}$

$\mathcal{M}_{n+1}^{(1)f}$ is finite (KLN of the $n + 1$ process)

- $d\sigma^{(fin)}$ and $d\sigma^{(sin)}$ depend on two a-priori different ξ_{c_i}
- $d\sigma^{(fin)}$ is now finite $\Rightarrow \epsilon \rightarrow 0$
- we hope for the best and deal with \mathcal{I} later

$$d\sigma^{(sin)} \equiv -\mathcal{I}(\xi_{cA}, \xi_{cB}) \propto \int_{\xi_{cA}} \hat{\mathcal{E}}(\xi_{cB}) (\xi^2 \mathcal{M}_{n+1}^{(0)})$$

term 2: $\mathcal{M}_{n+2}^{(0)}$ (real \times real)

- do normal FKS twice with two ξ_c , but (h) mixes with (s)

$$d\sigma_{n+2} = \underbrace{d\sigma^{(hh)}}_{\epsilon \rightarrow 0} + \underbrace{d\sigma^{(ss)}}_{\hat{\mathcal{E}}(\xi_{c_1}) \hat{\mathcal{E}}(\xi_{c_2})} + d\sigma^{(hs)} + d\sigma^{(sh)}$$

- mixed terms $d\sigma^{(hs)}$ are troublesome

$$d\sigma^{(hs)} \propto \frac{1}{2!} \int_{\xi_{c_1}} \hat{\mathcal{E}}(\xi_{c_2}) (\xi^2 \mathcal{M}_{n+1}^{(0)}) = \frac{1}{2!} \mathcal{I}(\xi_{c_1}, \xi_{c_2})$$

- **luckily** $d\sigma^{(aux)} \equiv d\sigma^{(sin)} + d\sigma^{(hs)} + d\sigma^{(sh)} = 0$ if all ξ_{c_i} are equal

we now have

$$d\sigma_n = d\phi_n^{d=4} \underbrace{\left(\mathcal{M}_n^{(2)} + \hat{\mathcal{E}}(\xi_c) \mathcal{M}_n^{(1)} + \frac{1}{2!} \hat{\mathcal{E}}(\xi_c)^2 \mathcal{M}_n^{(0)} \right)}_{\mathcal{M}_n^{(2)f}(\xi_c)}$$

$$d\sigma_{n+1} = \frac{1}{1!} d\varphi_{n+1}^{d=4} \left(\frac{1}{\xi} \right)_c \left(\xi^2 \mathcal{M}_{n+1}^{(1)f}(\xi_c) \right)$$

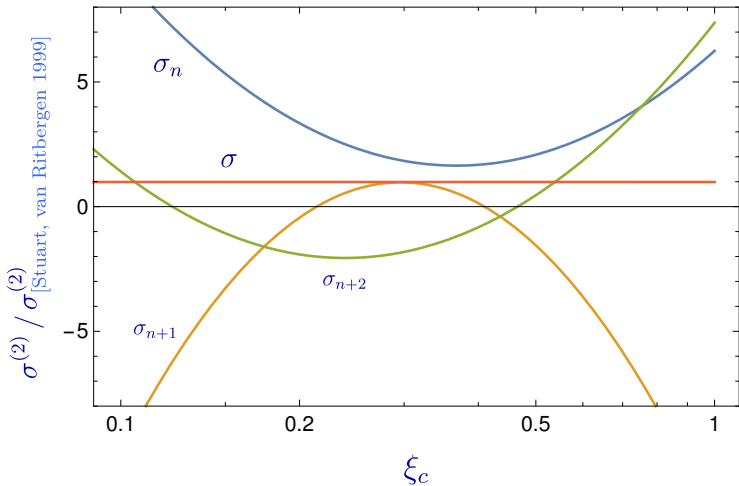
$$d\sigma_{n+2} = \frac{1}{2!} d\varphi_{n+2}^{d=4} \left(\frac{1}{\xi_1} \right)_c \left(\frac{1}{\xi_2} \right)_c \left(\xi_1^2 \xi_2^2 \mathcal{M}_{n+2}^{(0)f} \right)$$

where $d\varphi_n$ is the n -particle phasespace without the $\xi^{1-2\epsilon}$ factors

very simple NNLO scheme!

- no complicated analytic integrals
- no ϵ/ϵ -terms
- $\mathcal{M}_n^{(\ell)f}$ is scale- and regularisation-scheme independent
- obtain $\mathcal{M}_n^{(2)}$ using massification
- $\mathcal{O}(\epsilon)$ terms of $\hat{\mathcal{E}}$ vanish exactly
- **down-side**: phasespace parametrisation is non-trivial

ξ_c independent!



many more terms at N³LO!

$$\begin{aligned}
 d\sigma_n^{(3)}(\xi_c) = & d\Phi_n \left(\mathcal{M}_n^{(3)} + \underbrace{\hat{\mathcal{E}}(\xi_c) \mathcal{M}_n^{(2)}}_{d\sigma_s^{(3)}} + \underbrace{\frac{1}{2!} \hat{\mathcal{E}}(\xi_c)^2 \mathcal{M}_n^{(1)}}_{d\sigma_{ss}^{(3)}} + 1 \times \underbrace{\frac{1}{3!} \hat{\mathcal{E}}(\xi_c)^3 \mathcal{M}_n^{(0)}}_{d\sigma_{sss}^{(3)}} \right) \\
 & \underbrace{\left(-\mathcal{I}(\xi_c) - \frac{1}{2!} \mathcal{J}(\xi_c) + \frac{1}{2!} \mathcal{I}(\xi_c) + \frac{1}{2!} \mathcal{I}(\xi_c) - \frac{1}{2!} \mathcal{K}(\xi_c) + 3 \times \frac{1}{3!} \mathcal{J}(\xi_c) + 3 \times \frac{1}{3!} \mathcal{K}(\xi_c) \right)}_0 \\
 & \underbrace{\left(-\mathcal{I}(\xi_c) - \frac{1}{2!} \mathcal{J}(\xi_c) \right)}_{d\sigma_d^{(3)}} \underbrace{\left(\frac{1}{2!} \mathcal{I}(\xi_c) + \frac{1}{2!} \mathcal{I}(\xi_c) \right)}_{d\sigma_{hs}^{(3)} + d\sigma_{sh}^{(3)}} \underbrace{\left(-\frac{1}{2!} \mathcal{K}(\xi_c) \right)}_{d\sigma_{hd}^{(3)}} \underbrace{\left(3 \times \frac{1}{3!} \mathcal{J}(\xi_c) \right)}_{d\sigma_{hss}^{(3)} + \dots} \underbrace{\left(3 \times \frac{1}{3!} \mathcal{K}(\xi_c) \right)}_{d\sigma_{hhs}^{(3)} + \dots}
 \end{aligned}$$

$$d\sigma_{n+1}^{(3)}(\xi_c) = d\sigma_f^{(3)} = \frac{1}{1!} d\varphi_{n+1} \left(\frac{1}{\xi} \right)_c \left(\xi^2 \mathcal{M}_{n+1}^{(2)f}(\xi_c) \right)$$

$$d\sigma_{n+2}^{(3)}(\xi_c) = d\sigma_{hf}^{(3)} = \frac{1}{2!} d\varphi_{n+2} \left(\frac{1}{\xi_1} \right)_c \left(\frac{1}{\xi_2} \right)_c \left(\xi_1^2 \xi_2^2 \mathcal{M}_{n+2}^{(1)f} \right)$$

$$d\sigma_{n+3}^{(3)}(\xi_c) = d\sigma_{hhh}^{(3)} = \frac{1}{3!} d\varphi_{n+3} \left(\frac{1}{\xi_1} \right)_c \left(\frac{1}{\xi_2} \right)_c \left(\frac{1}{\xi_3} \right)_c \left(\xi_1^2 \xi_2^2 \xi_3^2 \mathcal{M}_{n+3}^{(0)f} \right)$$

with $\mathcal{M}_m^{(1)f}$ as above and $\mathcal{M}_m^{(2)f} = \mathcal{M}_m^{(2)} + \frac{\hat{\mathcal{E}}^1}{1!} \mathcal{M}_m^{(1)} + \frac{\hat{\mathcal{E}}^2}{2!} \mathcal{M}_m^{(2)}$

- there seems to be a pattern here
- ⇒ **postulate** behaviour at N^ℓLO

$$d\sigma_{n+j}^{(\ell)} = d\varphi_{n+j}^{d=4} \frac{1}{j!} \left(\frac{1}{\xi_1}\right)_c \cdots \left(\frac{1}{\xi_j}\right)_c \xi_1^2 \cdots \xi_j^2 \mathcal{M}_{n+j}^{(\ell-j)f}(\xi_c)$$

$$\mathcal{M}_m^{(\ell)f} = \sum_{j=0}^{\ell} \frac{\hat{\mathcal{E}}^j}{j!} \mathcal{M}_m^{(\ell-j)}$$

The FKS² scheme: future

mass hierarchy $m \ll M, \sqrt{s}, .. \rightarrow \beta \sim 1$

- integrand $\propto \frac{1}{1-y\beta}$ has a pseudo-singularity at $y = 1$
- integrable but annoying. so far: increase statistics
- find a suitable counter-term to remedy this (cf. [Dittmaier 1999])
- **but** don't destroy parton-level picture !
- this is different from QCD where collinear emission is always unresolved

- f -matrix elements (or eikonal-subtracted matrix elements) are central

$$\mathcal{M}_n^{(1)f} = \mathcal{M}_n^{(1)} + \hat{\mathcal{E}}(\xi_c) \mathcal{M}_n^{(0)} = \text{finite}$$

- can we calculate these objects directly in $d = 4$?
- simplest example with $p^2 = q^2 = m^2$

$$\begin{aligned} \mathcal{M}_n^{(1)f}(\xi_c) = & \left(\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \frac{1}{k^2 - 2k \cdot p} \frac{1}{k^2 - 2k \cdot q} \right) \\ & + \left(\int \frac{d^d k}{(2\pi)^d} \delta(k^2) \delta(k^0) \frac{\xi_c^{-2\epsilon}}{2\epsilon} \frac{(k^0)^2}{(k^2 - 2k \cdot p)(k^2 - 2k \cdot q)} \right) \end{aligned}$$

$$\mathcal{M}_n^{(1)f(1)} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - 2k \cdot p} \frac{1}{k^2 - 2k \cdot q} \left(\frac{1}{k^2 + i0^+} + \delta(k^2) \delta(k^0) \frac{(k^0)^2}{2\epsilon} \right)$$

- idea: use LTD on $\frac{1}{k^2 + i0^+} = \frac{1}{k^2 - i0^+ k^0} - \Theta(k^0) \delta(k^2)$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - 2k \cdot p} \frac{1}{k^2 - 2k \cdot q} \frac{1}{k^2 - i0^+ k^0} + \int d\phi \frac{1}{-2k \cdot p} \frac{1}{-2k \cdot q} \left(\frac{(k^0)^2}{2\epsilon} \delta(k^0) - 1 \right)$$

- does any of this make sense?
- how do I calculate these things numerically?