Cluster Adjacency of Scattering Amplitudes

Jack Foster

University of Southampton

5th April 2019

Based on work with James Drummond, Omer Gurodgan, and Georgios Papathanasiou: 1710.10953, 1810.08149, & 1812.04640

Outline

Introduction

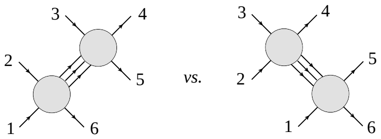
- Scattering amplitudes are important for matching theory with experiment
- Standard methods quickly become cumbersome, e.g. Feynman diagrams, but solutions can be surprisingly simple - Parke-Taylor

$$A_{\text{tree}}^{\text{MHV}}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, (n-1)^+, n^+)$$
$$= \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1n\rangle\langle n1\rangle}$$
$$p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}, \quad \langle ij\rangle = \lambda_{i\alpha}\lambda_j^{\alpha}$$

- Amplitudes are rich in analytic structure - poles, branch cuts etc.
- Understanding structure could circumvent calculations!
- Mathematically interesting

Analytics of Amplitudes

- The analytic properties of amplitudes have been studied since the '60's
- The Steinmann conditions are one such property which prohibit certain consecutive discontinuities



The s_{345} and s_{234} channels of $3 \to 3$ scattering [Caron-Huot et al, '16]

- A discontinuity in s_{345} followed by one in s_{234} is forbidden
- Mirrors the factorisation of amplitudes into products of trees (on poles)

Britto-Cachazo-Feng-Witten Recursion Relation

- The BCFW recursion relation is an efficient method of calculating higher-point amplitudes from lower-point ones by using their analytic properties

$$A_n = \sum_{i=2}^{n-1} \sum_{s} A_L^s(z_{P_i}) \frac{1}{P_i^2} A_R^{-s}(z_{P_i})$$

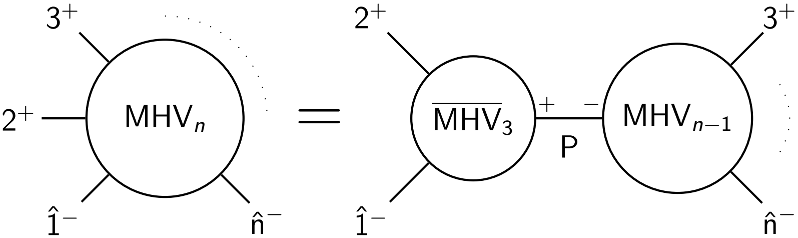
- We use three-point amplitudes as the building blocks of higher point amplitudes

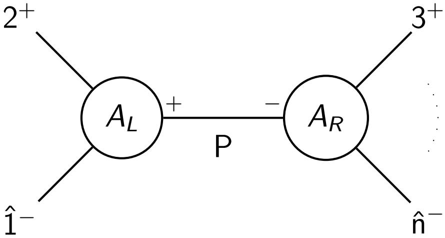
$$A_3^{MHV}(i^-, j^-) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \quad A_3^{\overline{MHV}}(i^+, j^+) = \frac{[ij]^4}{[12][23][31]}$$

- We introduce a complex shift in two of the spinors

$$\lambda_1 \to \lambda_1 - z\lambda_n, \quad \tilde{\lambda}_n \to \tilde{\lambda}_n + z\tilde{\lambda}_1, \qquad z \in \mathbb{C}$$

- Tree-amplitudes factorise on the pole $z_{P_i} = \frac{\langle 12 \rangle}{\langle n2 \rangle}$





- We assume that A_R has the form of A_3^{MHV}

$$A_L = A_3^{\overline{MHV}}(\hat{1}^-, 2^+, -\hat{P}^+) = -\frac{[2(-\hat{P})]^3}{[12][(-\hat{P})1]},$$

$$A_R = A_{n-1}^{MHV}(\hat{P}^-, 3^+, \ldots, \hat{n}^-) = \frac{\langle \hat{n}\hat{P}\rangle^3}{\langle \hat{P}3\rangle\langle 34\rangle \cdots \langle (n-1)n\rangle}$$

- Thus the BCFW recursion relation gives us (after some spinor manipulation)

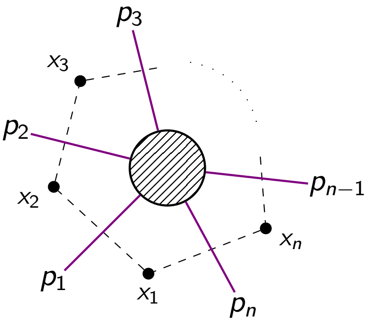
$$A_n^{MHV}(1^-, n^-) = A_L \frac{1}{(p_1 + p_2)^2} A_R = \frac{\langle n1 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n-1 n \rangle \langle n1 \rangle}$$

- Much simpler calculation than Feynman diagrams - only **one** diagram to compute
- Hints at further understanding of amplitudes from the study of their poles

$\mathcal{N} = 4$ Super Yang-Mills (SYM)

- Most symmetric QFT (without gravity) - conformal and supersymmetric
- SU(N) gauge group, "QCD-like" - provides good mathematical playground for perturbative QCD and QFT in general
- One side of the AdS/CFT correspondence - dual to IIB string theory
- Amplitudes in planar limit ($N \to \infty$) have revealed many interesting structures
- Tree-level amplitudes are equivalent to non-SUSY Yang-Mills gluon tree-amplitudes

Dual Conformal Symmetry

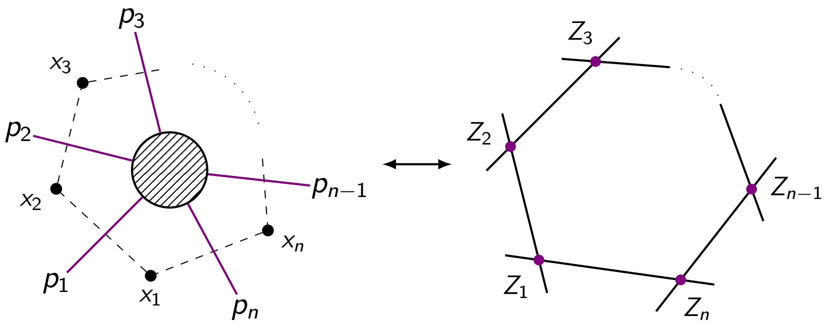


Duality between an n-point amplitude and a light-like Wilson loop

- Can identify external null momenta with dual coodinates

$$p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}$$

Dual Conformal Symmetry and Twistors



Equivalence between amplitude in momentum space and closed, light-like polygon in twistor space

- Define momentum twistors $Z_i^a = (\lambda_i^\alpha, x_i^{\alpha\dot{\alpha}}\lambda_{i\alpha}), \quad Z_i^a \in \mathbb{P}^3$
- Lines in $\tilde{M}_4 \longleftrightarrow$ points in $\mathbb{P}^3$ (and vice versa)

Twistors

- Natural objects to consider are SL(4) invariant brackets

$$\langle ijkl \rangle = \epsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D$$
$$s_{i\ldots j} = (x_i - x_{j+1})^2 \sim \langle i-1ijj+1 \rangle$$

- Brackets invariant up to a scaling ($Z \sim tZ$) so must appear in amplitudes as homogeneous combinations

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{\langle 1236 \rangle \langle 3456 \rangle}{\langle 1346 \rangle \langle 2356 \rangle}$$

Polylogarithms and Symbols

- Polylogarithms are common pure functions to appear in amplitudes

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(x) \equiv -\log(1-x)$$

- For instance, the one-loop, six-point amplitude in SYM is

$$Y_6 = -\sum_{i=1}^{3} \mathrm{Li}_2 \left(1 - \frac{1}{u_i}\right)$$

- The *symbol* is a map which encodes the discontinuities and derivatives of pure functions

$$\mathcal{S}[\mathrm{Li}_n(x)] = -(1-x) \otimes \underbrace{x \otimes \cdots \otimes x}_{n-1 \text{ times}}$$

Symbols and Twistors

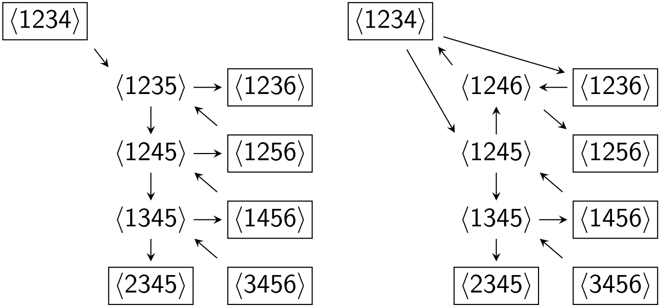
- Taking the symbol of Y_6

$$\mathcal{S}[Y_6] = \sum_{i=1}^{3} u_i \otimes (1 - u_i) - u_i \otimes u_i$$

- Further expansion would result in individual twistor brackets in each slot of the symbol
- Which twistor brackets sit next to each other in the symbol is controlled by an algebraic structure call a **cluster algebra**

Cluster Algebras

- First introduced by Fomin and Zelevinsky in 2001 - connected to amplitudes by Golden et al in 2013
- Move from one cluster to another via *mutation*



Cluster Adjacency

- For $n < 8$ the cluster algebra is finite - for $n = 6$ there are 14 different clusters
- By examining all clusters we can construct *neighbour sets* for each node - the set of all nodes that can be found in a cluster with a given node
- If we choose one node, we find its disallowed neighbours are

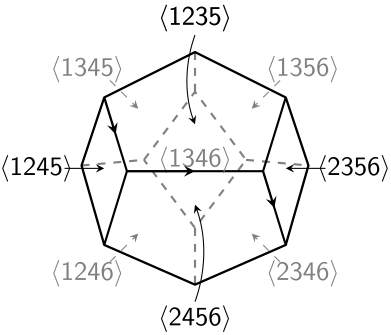
$$\langle 1245\rangle \;\Big|\; \underbrace{\langle 2356\rangle \quad \langle 1346\rangle}_{\text{Steinmann}} \quad \underbrace{\langle 1356\rangle \quad \langle 2346\rangle}_{\text{Cluster}}$$

- We define *cluster adjacency* as:

Consecutive discontinuities can only be taken around branch points who appear together in a cluster.

The Stasheff Polytope

- To a cluster algebra we can associate a polyhedron - an *associahedron* - which encodes all the information about the algebra **including** cluster adjacency



- All Mandelstams have positive values inside the polytope

Tree Amplitudes

- Cluster adjacency originally defined in terms of loop integrals and amplitudes [Drummond, JF, Gurdogan '17]
- Discontinuities of loops are non-abelian whereas poles of trees are abelian
- We normalise the maximally-helicity-violating (MHV) tree amplitude to one
- Hence the first non-trivial amplitudes to study are NMHV amplitudes which in $\mathcal{N}=4$ SYM are made up of R-invariants

$$[ijklm] \sim \frac{1}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}$$

- For example the six-point NMHV amplitude is given by

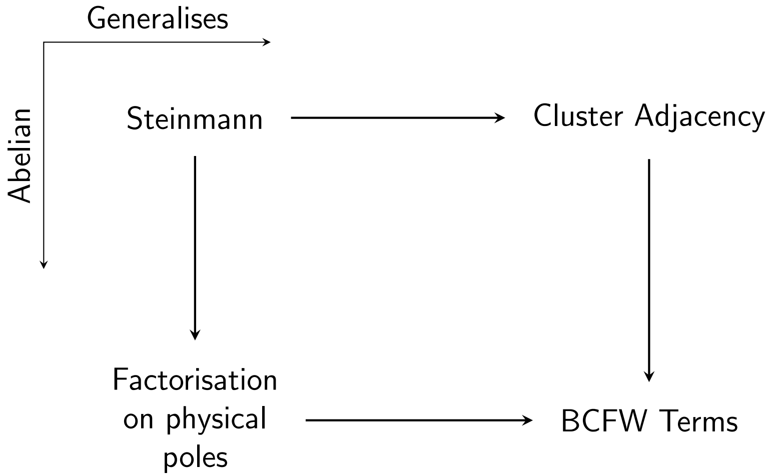
$$\mathcal{A}_6^{NMHV} = [12345] + [13456] + [12356]$$

Tree Amplitudes

- If we take the first term

$$[12345] \sim \frac{1}{\langle 1234\rangle\langle 2345\rangle\langle 1345\rangle\langle 1245\rangle\langle 1235\rangle}$$

- One can find a cluster which contains all of these poles
- Using knowledge of how to rotate the labels of the nodes via mutation we prove that this is true for *all* R-invariants
- We also conjecture that all individual terms in N^kMHV tree amplitudes are also cluster adjacent


Generalises
Abelian
Steinmann
Cluster Adjacency
Factorisation on physical poles
BCFW Terms

Seven-Point, Four-Loop, NMHV Amplitude

- NMHV amplitudes have symbol final entries of the form

$$R \log a_{ij}$$

- Derivatives of symbols act from the end $\rightarrow$ poles and branch points talk to each other in NMHV amplitudes
- Cluster adjacency imposes that the poles of the R-invariant be compatible with the twistor brackets in a_{ij}
- This allowed us to calculate a previously unknown amplitude - the seven-point, four-loop, NMHV amplitude in SYM

Summary

- Amplitudes are rich in analytic structure
- Simpler techniques than Feynman diagrams
- Cluster adjacency controls the analytic properties of scattering amplitudes
- Geometric interpretation of Steinmann conditions along with other conditions
- Cuts of loops have an abelianised version in the poles of trees
- Cuts and poles talk to each other through cluster adjacency

Outlook

- Tree amplitudes in SYM are the same as in pure Yang-Mills - could one apply cluster adjacency to non-supersymmetric gauge theories?
- Interesting mathematics - new class of functions; cluster adjacent polylogs
- Geometric interpretations of amplitudes - are they linked and could they provide a way of writing down amplitudes without calculation?

Thank you!