Cohomological methods for quantum gravity

Alex Mitchell

SHEP group
University of Southampton
Southampton student conference

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Outline

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- Based on ArXiv:1802.04281 and ArXiv:1806.02206 by Tim Morris
- Gravity is difficult to quantize
- From the renormalization group perspective the coupling $\kappa \propto G^{\frac{1}{2}}$ is irrelevent, i.e. $[\kappa] = -1$
- As a result no naive UV complete theory of quantum gravity is possible, despite tricks at low loop level calculations in free gravity

- Expanding the metric as

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa H_{\mu\nu} \quad \text{with} \quad H_{\mu\nu} = h_{\mu\nu} + \tfrac{1}{2}\delta_{\mu\nu}\phi$$

- we find

$$\mathcal{L} = \tfrac{1}{2}(\partial_\lambda h_{\mu\nu})^2 - \tfrac{1}{2}(\partial_\lambda \phi)^2$$

- This significantly restricts what eigenoperators can be constructed around the UV Gaussian fixed point, in particular the sign change defines our Sturm-Liouville measure

$$exp(\frac{-(h_{\mu\nu})^2}{2\Omega_\Lambda}) \qquad exp(\frac{\phi^2}{2\Omega_\Lambda})$$

- This leads to 2 quantization condition equations, where we demand square integrability

$$\int_{-\infty}^{\infty} dh_{\mu\nu} \quad e^{\frac{-(h_{\mu\nu})^2}{2\Omega_\Lambda}} \; \mathcal{O}_n(h_{\mu\nu}) \; \mathcal{O}_m(h_{\mu\nu}) = K\delta_{nm}$$

$$\int_{-\infty}^{\infty} d\phi \quad e^{\frac{\phi^2}{2\Omega_\Lambda}} \; \mathcal{O}_n(\phi) \; \mathcal{O}_m(\phi) = K\delta_{nm}$$

- In the second we are no longer permitted polynomials, instead we find our super-relevant tower operators

- These tower operators are

$$\delta_\Lambda^{(n)}(\phi) := \frac{\partial^n}{\partial\phi^n}\delta_\Lambda^{(0)} \quad , \qquad \delta_\Lambda^{(0)} := \frac{1}{\sqrt{2\pi\Omega_\Lambda}} e^{\left(\frac{-\phi^2}{2\Omega_\Lambda}\right)}$$

- These are perturbative in their couplings, non-perturbative in $\hbar$ and effervescent, an infinite tower of these can be associated to an operator and so these are summed into a 'coupling function'

$$f_\Lambda^\sigma(\phi)\ \sigma(\partial, \partial\phi, h, c, \bar{c}, b, \Phi^*)$$

- with

$$f_\Lambda^\sigma(\phi) = \sum_{n=n_\sigma}^\infty g_n^\sigma \delta_\Lambda^{(n)}(\phi)$$

- Unfortunately now we can renormalize every possible interaction we could ever want!
- This isn't predictive and we also have an infinite number of couplings
- However this hasn't been gravity, we still need to incorporate diffeomorphism invariance to make this a theory of quantum gravity

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- Before we begin to incorporate diffeomorphism we remind ourselves of Faddeev-Popov ghosts introduced in Yang-Mills theories to gauge fix gauge fields and define their propagator

$$\mathcal{L}_{ghost} = \partial_\lambda c^a \partial^\lambda \bar{c}^a + g f^{abc} (\partial^\lambda \bar{c}^a) A^b_\lambda c^c$$

- Famously odd; scalar fields with fermionic statistics and non-asymptotic states, regarded as a useful tool

- The implementation and expansion of ghosts was expanded upon by Becchi, Rouet, Stora, Tyutin (BRST) who regarded the ghost as a useful field for their global symmetry
- They introduce a nilpotent operator; an action invariant under the action of this operator is invariant under the gauge symmetry associated to that operator
- The auxiliary field b_μ is also introduced to implement the BRST symmetry off-shell, it has no kinetic term and doesn't propagate by itself, we also split our fields into 'families'

ONE TAKE AWAY

Diffeomorphism invariance is implemented at the quantum level by requiring a global BRST symmetry

(this can be found order by order using the anti-field formalism)

- We have our BRST operator Q such that

$$QS = 0$$

- It is 'nilpotent' such that $Q^2\mathcal{O} = 0$ or more concisely $Q^2 = 0$
- We say that $\mathcal{O}$ is 'closed' under Q
- e.g. for gravitons and ghosts at the free level

$$Q_0 H_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu \qquad Q_0 c_\mu = 0$$

$$Q_0^2 H_{\mu\nu} = Q_0(\partial_\mu c_\nu + \partial_\nu c_\mu) = 0$$

- An operator is exact if it can be defined as

$$\mathcal{O} = QK \text{ , such that } Q\mathcal{O} = Q^2 K = 0$$

- Only interested in operators that are closed but not exact. These are in the 'cohomology' of the BRST operator

- We're free to add exact operators to those in the cohomology, this does not affect the physics and will have important implications for diffeomorphism invariance

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- We can further extend this treatment past Yang-Mills theories to gravity using 'Batalin-Vilkovisky anti-field formalism' (or BV or anti-field formalism for short)
- This is necessary for gravity as here we have much more freedom for field redefinitions; BV accounts for this as well as implementation of BRST symmetry and also now renormalization group flow
- 3 major parts; the introduction of 'anti-fields', the anti-bracket and a measure term

- For each field Φ^A we associate an anti-field Φ^*_A with opposite Grassman grading
- Anti-fields ϕ^*_A are introduced, at 1st order, as a source for our BRST transformations

$$S \to S + (Q\Phi^A)\Phi^*_A$$

- Objects conjugate to the ghost anti-field c^*_μ are the commutator of gauge transformations
- Free to switch between gauge invariant and gauge fixed actions, results are equivalent. In the former we may use the minimal basis where we can exclude terms involving $\bar{c}_\mu$ and b_μ past the free level

- We also have our quantum master equation, those actions which satisfy this are invariant under the BRST transformation and the gauge symmetry associated to it

$$\frac{1}{2}(S,S) - \Delta S = 0$$

$$(X,Y) = \frac{\partial_r X}{\partial \Phi^A}\frac{\partial_l Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A}\frac{\partial_l Y}{\partial \Phi^A} \qquad \Delta X (-)^A \frac{\partial_l}{\partial \Phi^A}\frac{\partial_l}{\partial \Phi^*_A} X$$

- where the anti-bracket corresponds to the classical part and the measure the quantum part

- With our new anti-fields and measure terms we must expand our old BRST operator into our new BRST operator

$$s = Q + Q^{-} - \Delta^{-} - \Delta^{=}$$

- Δ^{-} and $\Delta^{=}$ generate tadpoles so we focus on Q and Q^{-}. The full quantum BRST operator is

$$s\mathcal{O} = (S, \mathcal{O}) - \Delta\mathcal{O}$$

- such that

$$Q\Phi^{A} = (S, \Phi^{A}) \quad \text{and} \quad Q^{-}\Phi^{*}_{A} = (S, \Phi^{*}_{A})$$

- We can now expand our action, the BRST charge and the Koszul-Tate differential in κ and find relations amongst these

$$S = S_0 + \kappa S_1 + \frac{1}{2}\kappa^2 S_2 + \ldots$$

$$Q = Q_0 + \kappa Q_1 + \ldots \quad \text{and} \quad Q^- = Q_0^- + \kappa Q_1^- + \ldots$$

- This leads to important relationships between our BRST transformations and the action at given orders

$$s_0 S_1 = 0 \ , \quad s_0 S_2 = -\frac{1}{2}(S_1, S_1) \ , \quad s_0 S_3 = -(S_1, S_2) \ \ldots$$

- Importantly, knowing our BRST transformations is equivalent to knowing our actions and vice versa!

- We can use this to constrain our action; we introduce an 'anti-ghost number' (agh) grading to further constrain our actions
- We find agh 0 are the top terms, agh 1 correspond to BRST transformations of the graviton and agh 2 are conjugate to the transformations of the ghost

$$S_i = \Sigma_n S_i^n$$

- We find Q maintains the agh and Q^- lowers it by 1, hence we have a set of 'anti-field cascade equations'

$$Q_0 S_1^2 = 0 \;, \quad Q_0 S_1^1 + Q_0^- S_1^2 = 0, \quad Q_0 S_1^0 + Q_0^- S_1^1 = 0$$

- Given the restrictions of dimensionality, field number, total ghost number 0, Lorentz invariance our agh 2 operators are heavily constrained, up to the addition of exact pieces

$$S_1^2 = \int d^4x \ \ c_\alpha \partial_\beta c_\alpha c^*_\beta$$

- We can then act on this with Q_0^- and identify the agh 1 part

$$S_1^1 = \int d^4x \ \ 2c_\alpha \Gamma^{(1)\alpha}{}_{\beta\gamma} H^*_{\beta\gamma}$$

- and repeating this process we find our top terms in the S_1^0 part, 13 terms one would find from the classical expansion of $\mathcal{L}_{EH}$

- We then take our continuum limits where we then restrict our Hilbert space to find the classical result plus linearised cosmological constant
- We are now perturbatively renormalizable in κ and diffeomorphism invariant
- Our coupling fucntions and the infinite couplings are trading in for a constant κ

$$f^{\sigma}_{\Lambda}(\phi) \rightarrow \kappa$$

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- We can now extend this treatment to 2nd order, again from the restrictions our agh 2 and agh 1 terms are heavily constrained

$$\partial_\alpha H_{\mu\nu} c_\alpha c_\nu c^*_\mu \ , \quad \partial_\alpha H_{\mu\nu} H_{\alpha\beta} c_\beta H^*_{\mu\nu}$$

- we are free to add exact pieces and these too are highly constrained

$$QH_{\mu\nu} H_{\nu\alpha} c_\alpha c^*_\mu, \quad QH_{\nu\alpha} H_{\alpha\beta} H_{\beta\mu} H^*_{\mu\nu}$$

- The addition of exact pieces at 1st and 2nd order are alternative expressions for diffeomorphism invariance, for example the Lie derivative can be used to find alternative expressions for $Q_1 H_{\mu\nu}$
- Our anti-field cascade equations are also expanded once we consider our BRST operators to higher order

$$Q_0 S_2^2 + Q_1 S_1^2 = 0$$

$$Q_0 S_2^1 + Q_0^- S_2^2 + Q_1 S_1^1 + Q_1^- S_1^2 = 0$$

$$Q_0 S_2^0 + Q_0^- S_2^1 + Q_1 S_1^0 + Q_1^- S_1^1 = 0$$

- We also find at higher orders that the anti-fields acting as sources becomes less simple, with terms schematically of the form

$$\Phi^A \Phi^B \Phi^*_A \Phi^*_B$$

- This means we may have BRST transformations that mix our fields and anti-fields, or includes anti-fields in the BRST transformations of the regular fields

- We restrict our action here significantly, from any action possible to those that are diffeomorphism invariant, in a vigorous manner
- However this still allows all actions that are diffeomorphism invariant i.e. f(R) theories
- However through the consequence of a poorly posed Cauchy initial value problem for the 2 RG flows of our theory and expressing the metric as tetrads there is some evidence that these f(R) theories are not independent and can be eliminated

- To conclude we have been working on a minimal route to quantizing gravity
- We resolve issues of irrelevancy using our tower operator and have begun gauging the theory under the anti-field formalism
- The next step is to produce the 2nd order expansion of EH and use that plus the 1st order expansion under the action of BRST charges to investigate the structure of the theory
- In addition to this we will also be able to begin investigating the running of couplings and the nature of marginal operators

Thanks for listening!