

3. Chiral Perturbation Theory

- Chiral Symmetry Breakings & External Sources
- Lowest-Order χ PT
- Weinberg's Power Counting
- Loops
- χ PT at $\mathcal{O}(p^4)$ and Beyond

Explicit Symmetry Breaking

$$\begin{aligned}\mathcal{L}_{QCD} &\equiv \mathcal{L}_{QCD}^0 + \bar{\mathbf{q}}(\not{\mathbf{v}} + \not{\mathbf{a}}\gamma_5)\mathbf{q} - \bar{\mathbf{q}}(\mathbf{s} - i\gamma_5\mathbf{p})\mathbf{q} \\ &= \mathcal{L}_{QCD}^0 + \bar{\mathbf{q}}_L\not{\mathbf{q}}_L + \bar{\mathbf{q}}_R\not{\mathbf{q}}_R - \bar{\mathbf{q}}_R(\mathbf{s} + i\mathbf{p})\mathbf{q}_L - \bar{\mathbf{q}}_L(\mathbf{s} - i\mathbf{p})\mathbf{q}_R\end{aligned}$$

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$$\mathbf{l}_\mu \equiv \mathbf{v}_\mu - \mathbf{a}_\mu = e\mathcal{Q}A_\mu + \dots$$

$$\mathcal{Q} \equiv \frac{1}{3} \text{diag}(2, -1, -1)$$

$$\mathbf{r}_\mu \equiv \mathbf{v}_\mu + \mathbf{a}_\mu = e\mathcal{Q}A_\mu + \dots$$

$$\mathbf{s} = \mathcal{M} + \dots$$

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$$\mathcal{M} \equiv \text{diag}(m_u, m_d, m_s)$$

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Local $SU(3)_L \otimes SU(3)_R$ Symmetry:

$$\mathbf{q}_L \rightarrow \mathbf{g}_L \mathbf{q}_L$$

$$\mathbf{q}_R \rightarrow \mathbf{g}_R \mathbf{q}_R$$

$$\mathbf{l}_\mu \rightarrow \mathbf{g}_L \mathbf{l}_\mu \mathbf{g}_L^\dagger + i \mathbf{g}_L \partial_\mu \mathbf{g}_L^\dagger$$

$$\mathbf{r}_\mu \rightarrow \mathbf{g}_R \mathbf{r}_\mu \mathbf{g}_R^\dagger + i \mathbf{g}_R \partial_\mu \mathbf{g}_R^\dagger$$

$$(\mathbf{s} + i\mathbf{p}) \rightarrow \mathbf{g}_R (\mathbf{s} + i\mathbf{p}) \mathbf{g}_L^\dagger$$

Lowest-Order Effective Lagrangian:

$$\mathcal{L} = \frac{f^2}{4} \langle D_\mu \mathbf{U} D^\mu \mathbf{U}^\dagger + \chi \mathbf{U}^\dagger + \mathbf{U} \chi^\dagger \rangle$$

$$D_\mu \mathbf{U} = \partial_\mu \mathbf{U} - i \mathbf{r}_\mu \mathbf{U} + i \mathbf{U} \mathbf{l}_\mu$$

$$\chi \equiv 2 B_0 (\mathbf{s} + i \mathbf{p})$$

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Currents:

$$\mathbf{J}_L^\mu = \frac{\partial}{\partial \mathbf{l}_\mu} \mathcal{L}_2 = \frac{i}{2} f^2 D^\mu \mathbf{U}^\dagger \mathbf{U} = \frac{f}{\sqrt{2}} D^\mu \Phi + \dots$$

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$$\langle 0 | (J_A^\mu)_{12} | \pi^+(p) \rangle = i \sqrt{2} f p^\mu$$



$$f = f_\pi \approx 92.2 \text{ MeV}$$

$$(\pi^+ \rightarrow \mu^+ \nu_\mu)$$

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$$\bar{\mathbf{q}}_L^j \mathbf{q}_R^i = -\frac{\partial \mathcal{L}_2}{\partial (\mathbf{s} - i \mathbf{p})^{ji}} = -\frac{f^2}{2} B_0 \mathbf{U}^{ij}$$



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$$\langle 0 | \bar{\mathbf{q}}^j \mathbf{q}^i | 0 \rangle = -f^2 B_0 \delta_{ij}$$

Quark Masses:

$$\frac{f^2}{4} \langle \chi \mathbf{U}^\dagger + \mathbf{U} \chi^\dagger \rangle \rightarrow \mathcal{L}_m = -B_0 \langle \mathcal{M} \Phi^2 \rangle$$

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$$\begin{aligned} \mathcal{L}_m = -B_0 & \left\{ (m_u + m_d) \left[\pi^+ \pi^- + \frac{1}{2} \pi^0 \pi^0 \right] + (m_u + m_s) K^+ K^- \right. \\ & \left. + (m_d + m_s) K^0 \bar{K}^0 + \frac{1}{6} (m_u + m_d + 4 m_s) \eta^2 + \frac{1}{\sqrt{3}} (m_u - m_d) \pi^0 \eta \right\} \end{aligned}$$

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Isospin limit: $m_u = m_d = \hat{m}$

$$\frac{M_\pi^2}{2 \hat{m}} = \frac{M_K^2}{\hat{m} + m_s} = \frac{3 M_\eta^2}{2 \hat{m} + 4 m_s} = B_0$$

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- Gell-Mann–Okubo: $4 M_K^2 = M_\pi^2 + 3 M_\eta^2$

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- Gell-Mann–Oakes–Renner: $f^2 M_\pi^2 = -\hat{m} \langle 0 | \bar{u} u + \bar{d} d | 0 \rangle$

Quark Mass Ratios:

Dashen
Theorem

$$(M_{K^0}^2 - M_{K^\pm}^2)_{\text{em}} = (M_{\pi^0}^2 - M_{\pi^\pm}^2)_{\text{em}} + \mathcal{O}(e^2 p^2)$$

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Proof: $e^2 \langle \mathcal{Q}_R U \mathcal{Q}_L U^\dagger \rangle = -\frac{2e^2}{f^2} (\pi^+ \pi^- + K^+ K^-) + \mathcal{O}(\phi^4)$; $\mathcal{Q}_X \rightarrow g_X \mathcal{Q}_X g_X^\dagger$ \square

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$$\frac{m_d - m_u}{m_d + m_u} = \frac{(M_{K^0}^2 - M_{K^\pm}^2) - (M_{\pi^0}^2 - M_{\pi^\pm}^2)}{M_{\pi^0}^2} \approx 0.29$$

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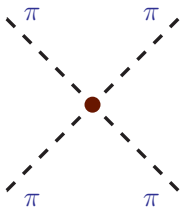


$$m_u : m_d : m_s = 0.55 : 1 : 20.3$$

Weinberg

$$\frac{f^2}{4} \langle \chi \mathbf{U}^\dagger + \mathbf{U} \chi^\dagger \rangle = -B_0 \langle \mathcal{M} \Phi^2 \rangle + \frac{B_0}{6 f^2} \langle \mathcal{M} \Phi^4 \rangle + \dots$$

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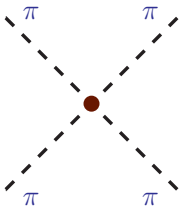


$$T(\pi^+ \pi^0 \rightarrow \pi^+ \pi^0) = \frac{t - M_\pi^2}{f_\pi^2}$$

$$t \equiv (p'_+ - p_+)^2$$

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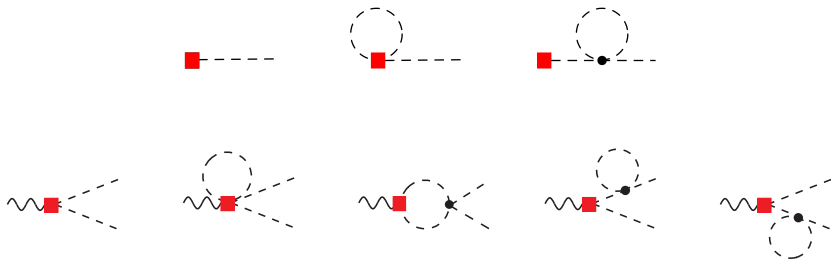
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Weinberg

$\mathcal{L}_2 \iff$ Current Algebra 60's

Chiral Loops:



- Vertices $\rightarrow p^2$
- Propagators $\rightarrow p^{-2}$
- Loops $\rightarrow p^4$



**1-loop
diagrams
contribute
at $\mathcal{O}(p^4)$**

Scaling with Momenta:

A general connected diagram with N_d vertices of $\mathcal{O}(p^d)$, B_I propagators and L loops, scales as $\mathcal{O}(p^D)$ with

$$D = 4L - 2B_I + \sum_d N_d d$$

$$\# \text{ vertices} \equiv V = \sum_d N_d, \quad L = B_I - V + 1$$



$$D = 2L + 2 + \sum_d N_d (d - 2)$$

Chiral Power Counting

$$\mathbf{U} \quad \mathcal{O}(p^0)$$

$$D_\mu \mathbf{U}, \mathbf{l}_\mu, \mathbf{r}_\mu \quad \mathcal{O}(p^1)$$

$$\chi, \mathbf{F}_{L,R}^{\mu\nu} \quad \mathcal{O}(p^2)$$

$$\mathbf{F}_L^{\mu\nu} \equiv \partial^\mu \mathbf{l}^\nu - \partial^\nu \mathbf{l}^\mu - i [\mathbf{l}^\mu, \mathbf{l}^\nu]$$

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General connected diagram with N_d vertices of $\mathcal{O}(p^d)$ and L loops:

$$D = 2L + 2 + \sum_d N_d (d - 2) \quad \text{Weinberg}$$

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- $D = 2$: $L = 0$, $d = 2$
- $D = 4$: $L = 0$, $d = 4$, $N_4 = 1$
 $L = 1$, $d = 2$

$\mathcal{O}(p^4)$ χ PT

i) \mathcal{L}_4 at tree level (Gasser–Leutwyler)

$$\begin{aligned}\mathcal{L}_4 = & L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle \\ & + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle \\ & + L_5 \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 \\ & + L_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi \rangle \\ & - i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle\end{aligned}$$

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ii) \mathcal{L}_2 at one loop (unitarity): $T_4 \sim p^4 \{a \log(p^2/\mu^2) + b(\mu)\}$

- Chiral Logarithms unambiguously predicted

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- Chiral Logarithms unambiguously predicted
- L_i 's fixed by QCD dynamics. 1-loop divergences $\rightarrow L_i^r(\mu)$

$\mathcal{O}(p^4)$ χ PT

i) \mathcal{L}_4 at tree level (Gasser–Leutwyler)

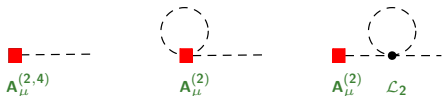
$$\begin{aligned}\mathcal{L}_4 = & L_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U \rangle \langle D^\mu U^\dagger D^\nu U \rangle \\ & + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U \rangle + L_4 \langle D_\mu U^\dagger D^\mu U \rangle \langle U^\dagger \chi + \chi^\dagger U \rangle \\ & + L_5 \langle D_\mu U^\dagger D^\mu U (U^\dagger \chi + \chi^\dagger U) \rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U \rangle^2 \\ & + L_7 \langle U^\dagger \chi - \chi^\dagger U \rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi \rangle \\ & - i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U \rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu} \rangle\end{aligned}$$

ii) \mathcal{L}_2 at one loop (unitarity): $T_4 \sim p^4 \{a \log(p^2/\mu^2) + b(\mu)\}$

- Chiral Logarithms unambiguously predicted
- L_i 's fixed by QCD dynamics. 1-loop divergences $\rightarrow L_i^r(\mu)$

iii) Wess–Zumino–Witten term (chiral anomaly): $\pi^0, \eta \rightarrow \gamma\gamma$

Meson Decay Constants:



$$\mu_P \equiv \frac{M_P^2}{32\pi^2 f^2} \log \left(\frac{M_P^2}{\mu^2} \right)$$

$$f_\pi = f \left\{ 1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu) \right\}$$

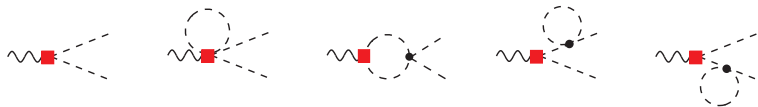
$$f_K = f \left\{ 1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_{\eta_8} + \frac{4M_K^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu) \right\}$$

$$f_{\eta_8} = f \left\{ 1 - 3\mu_K + \frac{4M_{\eta_8}^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu) \right\}$$

$$\frac{f_K}{f_\pi} = 1.193 \pm 0.003 \quad \Rightarrow \quad L_5^r(M_\rho) = (1.2 \pm 0.1) \cdot 10^{-3} \quad \Rightarrow \quad \frac{f_{\eta_8}}{f_\pi} = 1.31 \pm 0.02$$

Vector Form Factor:

$$\langle \pi^+ \pi^- | J_{\text{em}}^\mu | 0 \rangle = (p_+ - p_-)^\mu F_\pi^V(s)$$

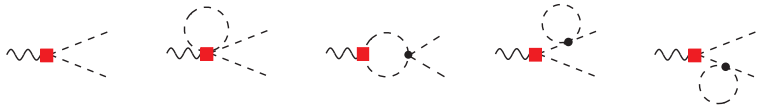


$$\begin{aligned} F_\pi^V(s) &= 1 + \frac{2L_9^r(\mu)}{f^2} s - \frac{s}{96\pi^2 f^2} \left[A\left(\frac{m_\pi^2}{s}, \frac{m_\pi^2}{\mu^2}\right) + \frac{1}{2} A\left(\frac{m_K^2}{s}, \frac{m_K^2}{\mu^2}\right) \right] \\ &= 1 + \frac{1}{6} \langle r^2 \rangle_\pi^V s + \dots \end{aligned}$$

$$A\left(\frac{m_P^2}{s}, \frac{m_P^2}{\mu^2}\right) = \log\left(\frac{m_P^2}{\mu^2}\right) + \frac{8m_P^2}{s} - \frac{5}{3} + \sigma_P^3 \log\left(\frac{\sigma_P+1}{\sigma_P-1}\right) \quad , \quad \sigma_P \equiv \sqrt{1 - \frac{4m_P^2}{s}}$$

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$$= 1 + \frac{1}{6} \langle r^2 \rangle_\pi^V s + \dots$$

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$$\langle r^2 \rangle_\pi^V = \frac{12 L_9^r(\mu)}{f^2} - \frac{1}{32\pi^2 f^2} \left\{ 2 \log\left(\frac{M_\pi^2}{\mu^2}\right) + \log\left(\frac{M_K^2}{\mu^2}\right) + 3 \right\}$$

$$\langle r^2 \rangle_\pi^V = (0.439 \pm 0.008) \text{ fm}^2 \quad \longrightarrow \quad L_9^r(M_\rho) = (6.9 \pm 0.7) \cdot 10^{-3}$$

$\mathcal{O}(p^4)$ χ PT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$	Source	Γ_i
1	1.0 ± 0.1	$K_{e4}, \pi\pi \rightarrow \pi\pi$	3/32
2	1.6 ± 0.2	$K_{e4}, \pi\pi \rightarrow \pi\pi$	3/16
3	-3.8 ± 0.3	$K_{e4}, \pi\pi \rightarrow \pi\pi$	0
4	0.0 ± 0.3	Zweig rule	1/8
5	1.2 ± 0.1	F_K/F_π	3/8
6	0.0 ± 0.4	Zweig rule	11/144
7	-0.3 ± 0.2	GMO, $L_{5,8}$	0
8	0.5 ± 0.2	$M_{K^0} - M_{K^+}, L_5, (m_s - \hat{m})/(m_d - m_u)$	5/48
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$	1/4
10	-5.2 ± 0.17	$\tau \rightarrow \nu_\tau + \text{hadrons}, \pi \rightarrow e\nu\gamma$	-1/4

- $$L_i = L_i^r(\mu) + \Gamma_i \frac{\mu^{D-4}}{32\pi^2} \left\{ \frac{2}{D-4} + \gamma_E - \log(4\pi) - 1 \right\}$$

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- $\Lambda_\chi \sim 1 \text{ GeV} \quad \rightarrow \quad L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim 2 \times 10^{-3}$

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- $\Lambda_\chi \sim 1 \text{ GeV} \quad \rightarrow \quad L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim 2 \times 10^{-3}$
- χ PT Loops $\sim 1/(4\pi f_\pi)^2$

Chiral Anomaly:

$$\delta Z[v, a, s, p] = -\frac{N_C}{16\pi^2} \int d^4x \langle \delta\beta(x) \Omega(x) \rangle$$

$$g_{L,R} \approx 1 + i\delta\alpha \mp i\delta\beta$$

$$\Omega(x) = \varepsilon^{\mu\nu\sigma\rho} \left[v_{\mu\nu} v_{\sigma\rho} + \frac{4}{3} \nabla_\mu a_\nu \nabla_\sigma a_\rho + \frac{2}{3} i \{v_{\mu\nu}, a_\sigma a_\rho\} + \frac{8}{3} i a_\sigma v_{\mu\nu} a_\rho + \frac{4}{3} a_\mu a_\nu a_\sigma a_\rho \right]$$
$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i [v_\mu, v_\nu] \quad , \quad \nabla_\mu a_\nu = \partial_\mu a_\nu - i [v_\mu, a_\nu] \quad , \quad \varepsilon_{0123} = 1$$

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$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu] \quad , \quad \nabla_\mu a_\nu = \partial_\mu a_\nu - i[v_\mu, a_\nu] \quad , \quad \varepsilon_{0123} = 1$$

$$S[U, \ell, r]_{wzw} = -\frac{iN_C}{240\pi^2} \int d\sigma^{ijklm} \langle \Sigma_i^L \Sigma_j^L \Sigma_k^L \Sigma_l^L \Sigma_m^L \rangle$$

$$-\frac{iN_C}{48\pi^2} \int d^4x \varepsilon_{\mu\nu\alpha\beta} (W(U, \ell, r)^{\mu\nu\alpha\beta} - W(\mathbf{1}, \ell, r)^{\mu\nu\alpha\beta})$$

Wess-Zumino-Witten

$$W(U, \ell, r)_{\mu\nu\alpha\beta} = \langle U l_\mu l_\nu l_\alpha U^\dagger r_\beta + \frac{1}{4} U l_\mu U^\dagger r_\nu U l_\alpha U^\dagger r_\beta + i U \partial_\mu l_\nu l_\alpha U^\dagger r_\beta$$

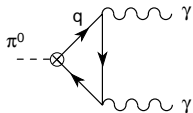
$$+ i \partial_\mu r_\nu U l_\alpha U^\dagger r_\beta - i \Sigma_\mu^L l_\nu U^\dagger r_\alpha U l_\beta + \Sigma_\mu^L U^\dagger \partial_\nu r_\alpha U l_\beta - \Sigma_\mu^L \Sigma_\nu^L U^\dagger r_\alpha U l_\beta$$

$$+ \Sigma_\mu^L l_\nu \partial_\alpha l_\beta + \Sigma_\mu^L \partial_\nu l_\alpha l_\beta - i \Sigma_\mu^L l_\nu l_\alpha l_\beta + \frac{1}{2} \Sigma_\mu^L l_\nu \Sigma_\alpha^L l_\beta - i \Sigma_\mu^L \Sigma_\nu^L \Sigma_\alpha^L l_\beta \rangle$$

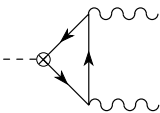
$$- (L \leftrightarrow R)$$

$$\Sigma_\mu^L = U^\dagger \partial_\mu U \quad , \quad \Sigma_\mu^R = U \partial_\mu U^\dagger$$

$\pi^0 \rightarrow \gamma\gamma$:



+



$$A_3^\mu \equiv \bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d$$

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = \left(\frac{N_c}{3}\right)^2 \frac{\alpha^2 M_\pi^3}{64 \pi^3 f_\pi^2} = 7.7 \text{ eV}$$

Exp: $(7.63 \pm 0.16) \text{ eV}$

There are no QCD corrections

The chiral anomaly contributes to: $\pi^0 \rightarrow \gamma\gamma$, $\eta \rightarrow \gamma\gamma$
 $\gamma 3\pi$, $\gamma \pi^+ \pi^- \eta$, $K\bar{K}3\pi$, ...

$\mathcal{O}(p^6)$ χ PT

i) $\mathcal{L}_6 = \sum_i c_i \mathcal{O}_i^{p^6}$ at tree level

Bijnens-Colangelo-Ecker, Fearing-Scherer

90 + 4 [53 + 4] terms in SU(3) [SU(2)] χ PT (even-intrinsic parity only)

ii) \mathcal{L}_4 at one loop, \mathcal{L}_2 at two loops

Bijnens-Colangelo-Ecker

Double chiral logarithms

Many Calculations: $M_\phi, f_\phi, \gamma\gamma \rightarrow \pi\pi, \pi\pi \rightarrow \pi\pi, \pi K \rightarrow \pi K, K_{l4},$
 $\pi \rightarrow e \bar{\nu}_e \gamma, F_V(s), F_S(s), \Pi_{V,A}(s), \dots$

Amoros-Bijnens-Dhonte-Talavera, Ananthanarayan-Colangelo-Gasser-Leutwyler, Bellucci-Gasser-Sainio, Brgui, Bijnens et al, Descotes-Genon et al, Golowich-Kambor, Post-Schilcher...

Theoretical Challenge: QCD calculation of the χ PT couplings

$$K^+ \rightarrow \pi^0 \ell^+ \nu_\ell, \quad K^0 \rightarrow \pi^- \ell^+ \nu_\ell:$$

$$c_{K^+\pi^0} = \frac{1}{\sqrt{2}}, \quad c_{K^0\pi^-} = 1$$

$$\langle \pi | \bar{s} \gamma^\mu u | K \rangle = c_{K\pi} [(P_K + P_\pi)^\mu f_+^{K\pi}(t) + (P_K - P_\pi)^\mu f_-^{K\pi}(t)]$$

- **Lowest order** [$\mathcal{O}(p^2)$]: $f_+^{K\pi}(t) = 1$, $f_-^{K\pi}(t) = 0$

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- **π^0 - η mixing:** $f_+^{K^+\pi^0}(0) = 1 + \frac{3}{4} \frac{m_d - m_u}{m_s - \hat{m}} = 1.017$

$$K^+ \rightarrow \pi^0 \ell^+ \nu_\ell, \quad K^0 \rightarrow \pi^- \ell^+ \nu_\ell:$$

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- **$\mathcal{O}(p^4)$:** $f_+^{K^0\pi^-}(0) = 0.977$, $\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = 1.022$

Gasser-Leutwyler '85

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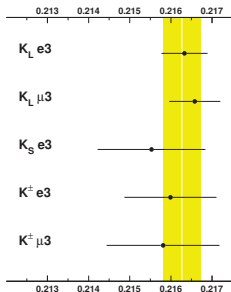
Needed to determine V_{us}



$$|\mathbf{V}_{us} f_+(0)| = 0.21654 \pm 0.00041$$

Flavianet Kaon WG, arXiv:1005.2323; Moulson arXiv:1411.5252

$$\langle \pi^- | \bar{s} \gamma_\mu u | K^0 \rangle = (p_\pi + p_K)_\mu f_+(t) + (p_K - p_\pi)_\mu f_-(t)$$

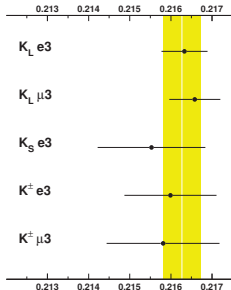


$$K \rightarrow \pi \ell \nu_\ell$$

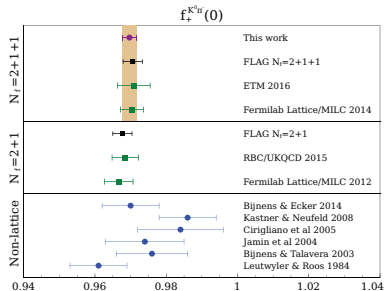
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Fermilab-MILC 1809.02827

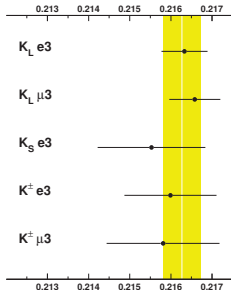


$$K \rightarrow \pi \ell \nu_\ell$$

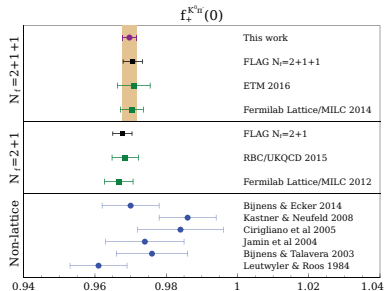
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Fermilab-MILC 1809.02827



$$f_+(0) = 0.9696(19)$$



$$|V_{us}| = 0.22333(60)$$

$$f_+(0) = 1 + f_2 + f_4 + \dots$$

Large $\mathcal{O}(p^6)$ χ PT correction

χ PT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$				
	$\mathcal{O}(p^4)$	$\mathcal{O}(p^6)$	Lattice	$R\chi T$	$R\chi T_{SD}$
1	1.0 ± 0.1	0.53 ± 0.06		0.6	0.9
2	1.6 ± 0.2	0.81 ± 0.04		1.2	1.8
3	-3.8 ± 0.3	-3.07 ± 0.20		-2.8	-4.8
4	0.0 ± 0.3	0.3 (fixed)	0.09 ± 0.34	0.0	0.0
5	1.2 ± 0.1	1.01 ± 0.06	1.19 ± 0.25	1.2^\dagger	1.1
6	0.0 ± 0.4	0.14 ± 0.05	0.16 ± 0.20	0.0	0.0
7	-0.3 ± 0.2	-0.34 ± 0.09		-0.3	-0.3
8	0.5 ± 0.2	0.47 ± 0.10	0.55 ± 0.15	0.5^\dagger	0.4
9	6.9 ± 0.7	5.9 ± 0.4		6.9^\dagger	7.1
10	-5.2 ± 0.1	-4.1 ± 0.4		-5.8	-5.3

\dagger = input

χ PT: Bijnens-Ecker, ARNPS 64 (2014) 149

Lattice: Dowdall et al, PR D88 (2013) 074504

$R\chi T$: Ecker et al, NP B321 (1989) 311, PL B223 (1989) 425; A.P., hep-ph/0205030