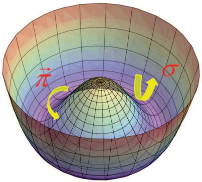
2. Nambu–Goldstone Bosons

- Sigma Model
- Goldstone Theorem
- Chiral Symmetry
- Effective Goldstone Theory

Sigma Model

$\mathbf{\Phi}^{\mathsf{T}} \equiv (\sigma, \vec{\pi})$



$$\mathcal{L}_\sigma = \frac{1}{2}\,\partial_\mu \mathbf{\Phi}^{\mathsf{T}} \partial^\mu \mathbf{\Phi} \;-\; \frac{\lambda}{4}\,\left(\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi} - v^2\right)^2$$

Global Symmetry: $\quad \mathbf{O(4)} \sim \mathbf{SU(2)} \otimes \mathbf{SU(2)}$

- $\mathbf{v^2 < 0}$**:** $\quad m_\Phi^2 = -\lambda\, v^2$
- $\mathbf{v^2 > 0}$**:** $\quad \langle 0|\sigma|0\rangle = v \quad , \quad \langle 0|\vec{\pi}|0\rangle = 0$

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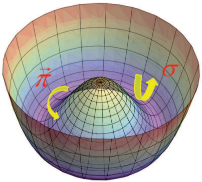

$$\mathcal{L}_\sigma = \frac{1}{2}\, \partial_\mu \mathbf{\Phi}^{\mathsf{T}} \partial^\mu \mathbf{\Phi} \;-\; \frac{\lambda}{4}\, \left(\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi} - v^2\right)^2$$

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SSB: $\quad \mathbf{O(4)} \to \mathbf{O(3)} \qquad \left[\frac{4\times 3}{2} - \frac{3\times 2}{2} = 3 \text{ broken generators}\right]$

$$\mathcal{L}_\sigma = \frac{1}{2}\left\{\partial_\mu\hat{\sigma}\,\partial^\mu\hat{\sigma} + \partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} - M^2\hat{\sigma}^2\right\} - \frac{M^2}{2v}\,\hat{\sigma}\left(\hat{\sigma}^2 + \vec{\pi}^2\right) - \frac{M^2}{8v^2}\left(\hat{\sigma}^2 + \vec{\pi}^2\right)^2$$

$$\hat{\sigma} \equiv \sigma - v \qquad ; \qquad M^2 = 2\,\lambda\, v^2$$

3 Massless Goldstone Bosons

1) $\mathbf{\Sigma}(x) \equiv \sigma(x)\, \mathbf{I}_2 + i\, \vec{\tau}\, \vec{\pi}(x)$; $\langle \mathbf{A} \rangle \equiv \mathrm{Tr}\,(\mathbf{A})$

$$\mathcal{L}_\sigma = \frac{1}{4}\, \langle \partial_\mu \mathbf{\Sigma}^\dagger\, \partial^\mu \mathbf{\Sigma} \rangle - \frac{\lambda}{16} \left(\langle \mathbf{\Sigma}^\dagger \mathbf{\Sigma} \rangle - 2\, v^2 \right)^2$$

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2) $\boldsymbol{\Sigma}(x) \equiv [v + S(x)]\;\mathbf{U}(x)$; $\mathbf{U} \equiv \exp\left\{\frac{i}{v}\,\vec{\tau}\,\vec{\phi}\right\} \to g_R\,\mathbf{U}\,g_L^\dagger$

$$\mathcal{L}_\sigma = \frac{v^2}{4}\,\left(1 + \frac{S}{v}\right)^2 \langle \partial_\mu \mathbf{U}^\dagger \partial^\mu \mathbf{U} \rangle + \frac{1}{2}\,\left(\partial_\mu S\,\partial^\mu S - M^2 S^2\right) - \frac{M^2}{2v}\,S^3 - \frac{M^2}{8v^2}\,S^4$$

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Derivative Golstone Couplings

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Derivative Golstone Couplings

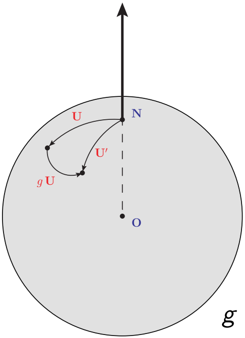
3) $E \ll M \sim v$:

$$\boxed{\mathcal{L}_\sigma \approx \frac{v^2}{4}\,\langle \partial_\mu \mathbf{U}^\dagger\,\partial^\mu \mathbf{U}\rangle}$$

O(N) Sigma Model: $\mathcal{L}_\sigma = \frac{1}{2}\,\partial_\mu \mathbf{\Phi}^\mathsf{T} \partial^\mu \mathbf{\Phi} \,-\, \frac{\lambda}{4}\,\left(\mathbf{\Phi}^\mathsf{T}\mathbf{\Phi} - v^2\right)^2$

$\mathbf{\Phi}^\mathsf{T} = (\phi_1, \cdots, \phi_N)$ Global $O(N)$ symmetry

- **Vacuum Manifold:** $|\mathbf{\Phi}|^2 = \sum_{i=1}^N \phi_i^2 = v^2$ Spherical surface S^{N-1}
- **Vacuum Choice:** $\mathbf{\Phi}_0^\mathsf{T} = (0, \cdots, 0, v)$ $O(N-1)$ symmetry



$\frac{1}{2}\, N(N-1) - \frac{1}{2}\,(N-1)(N-2) = N-1$ broken generators $\hat{T}_a$

Goldstones correspond to rotations of $\mathbf{\Phi}_0$ over S^{N-1}

$$\mathbf{\Phi} = \left(1 + \frac{S}{v}\right) U(x)\, \mathbf{\Phi}_0 \qquad , \qquad \underbrace{U(x) = \mathrm{e}^{i \sum_{a=1}^{N-1} \hat{T}_a \varphi_a(x)}}_{\text{Goldstone fields}}$$

$$\forall h \in O(N-1)\,, \qquad h\, \mathbf{\Phi}_0 = \mathbf{\Phi}_0$$

$$g \in O(N)\,, \quad U' \neq g\, U \quad \longrightarrow \quad U'(x) = g\; U(x)\; h^{-1}(g, U)$$

Symmetry Realizations

Symmetry **G** $\{T_a\}$ ⟷ **Conserved charges** $\mathcal{Q}_a$

Noether Theorem: $\partial_\mu j_a^\mu = 0 \quad ; \quad \mathcal{Q}_a = \int d^3x \, j_a^0(x) \quad ; \quad \frac{d}{dt}\,\mathcal{Q}_a = 0$

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Wigner–Weyl

$$\mathcal{Q}_a \,|\, 0 \,\rangle = 0$$

- Exact Symmetry
- Degenerate Multiplets
- Linear Representation

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Wigner–Weyl	**Nambu–Goldstone**
$\mathcal{Q}_a\,\vert 0\rangle = 0$	$\mathcal{Q}_a\,\vert 0\rangle \neq 0$
• Exact Symmetry • Degenerate Multiplets • Linear Representation	• Spontaneously Broken Symmetry • Massless Goldstone Bosons • Non-Linear Representation

Goldstone Theorem

$$\mathcal{Q} = \int d^3x \, j^0(x) \quad ; \quad \partial_\mu j^\mu = 0 \quad ; \quad \exists \, \mathcal{O} : \; v(t) \equiv \langle 0 | \, [\mathcal{Q}(t), \mathcal{O}] \, | 0 \rangle \neq 0$$

$$\Downarrow$$

$$\exists \, |n\rangle : \; \langle 0 | \mathcal{O} | n \rangle \, \langle n | j^0 | 0 \rangle \neq 0 \quad ; \quad E_n \, \delta^{(3)} \, (\vec{p}_n) = 0 \quad ; \quad M_n = 0$$

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Proof:

$$j^0(x) = \mathrm{e}^{iP\cdot x} \, j^0(0) \, \mathrm{e}^{-iP\cdot x} \quad ; \quad \sum_n |n\rangle\langle n| = 1$$

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$$=\, \sum_n \int d^3x\, \left\{ \mathrm{e}^{-ip_n\cdot x} \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle \,-\, \mathrm{e}^{ip_n\cdot x} \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\}$$

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$$= \sum_n \int d^3x\, \left\{ \mathrm{e}^{-ip_n\cdot x} \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \mathrm{e}^{ip_n\cdot x} \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\}$$

$$= (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n)\, \left\{ \mathrm{e}^{-iE_n t} \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \mathrm{e}^{iE_n t} \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\} \neq 0$$

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$$= \sum_n \int d^3x \, \left\{ \mathrm{e}^{-ip_n\cdot x}\, \langle 0|j^0(0)|n\rangle\langle n|\mathcal{O}|0\rangle \, - \, \mathrm{e}^{ip_n\cdot x}\, \langle 0|\mathcal{O}|n\rangle\langle n|j^0(0)|0\rangle \right\}$$

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$$\frac{d}{dt}\, v(t) \, = \, 0$$

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$$\begin{aligned}
v(t) &= \sum_n \int d^3x \, \left\{ \langle 0 | j^0(x) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(x) | 0 \rangle \right\} \\
&= \sum_n \int d^3x \, \left\{ \mathrm{e}^{-ip_n\cdot x} \, \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \mathrm{e}^{ip_n\cdot x} \, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\} \\
&= (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \, \left\{ \mathrm{e}^{-iE_n t} \, \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \mathrm{e}^{iE_n t} \, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\} \neq 0
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} v(t) = 0 = -i \, (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \, E_n \, \Big\{ & \mathrm{e}^{-iE_n t} \, \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle \\
& + \mathrm{e}^{iE_n t} \, \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \Big\}
\end{aligned}$$

□

Chiral Symmetry $\mathbf{m_q = 0}$ (Chiral Limit)

$$\mathcal{L}^0_{QCD} = -\frac{1}{4}\, G^{\mu\nu}_a G^a_{\mu\nu} + \bar{\mathbf{q}}_L\, i\gamma^\mu D_\mu\, \mathbf{q}_L + \bar{\mathbf{q}}_R\, i\gamma^\mu D_\mu\, \mathbf{q}_R$$

$\mathbf{q}^{\mathsf{T}} \equiv (u\,,\; d\,,\; s)$

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$\mathbf{q}^\mathbf{T} \equiv (u \, , \, d \, , \, s)$

$$q = \left(\frac{1-\gamma_5}{2}\right) q + \left(\frac{1+\gamma_5}{2}\right) q \equiv q_L + q_R$$

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$$\mathcal{L}_{QCD}^{0} = -\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} + \bar{\mathbf{q}}_L \, i \gamma^\mu D_\mu \mathbf{q}_L + \bar{\mathbf{q}}_R \, i \gamma^\mu D_\mu \mathbf{q}_R$$

$\mathbf{q}^{\mathbf{T}} \equiv (u \, , \, d \, , \, s)$

- $\mathcal{L}_{QCD}^{0}$ **invariant under** $\mathbf{G} \equiv \mathbf{SU(3)_L} \otimes \mathbf{SU(3)_R}$**:**

$$\bar{\mathbf{q}}_L \to g_L \, \bar{\mathbf{q}}_L \quad ; \quad \bar{\mathbf{q}}_R \to g_R \, \bar{\mathbf{q}}_R \quad ; \quad (g_L, g_R) \in \mathbf{G}$$

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- **Only** $\mathbf{SU(3)_V}$ **in the hadronic spectrum:** $(\pi, K, \eta)_{0^-}\, ;\, (\rho, K^*, \omega)_{1^-}\, ;\, \cdots$

$$M_{0^-} < M_{0^+} \quad ; \quad M_{1^-} < M_{1^+}$$

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- **The** $\mathbf{0^-}$ **octet is nearly massless:** $\mathbf{m_\pi \approx 0}$

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- **The** $\mathbf{0^-}$ **octet is nearly massless:** $\mathbf{m_\pi \approx 0}$

- **The vacuum is not invariant (SSB):** $\langle 0 | \, (\bar{\mathbf{q}}_L \mathbf{q}_R + \bar{\mathbf{q}}_R \mathbf{q}_L) \, | 0 \rangle \neq 0$

8 Massless $\mathbf{0^-}$ Goldstone Bosons

Noether QCD Currents: $\quad G \equiv SU(3)_L \otimes SU(3)_R$

$$J_X^{a\mu} = \bar{\mathbf{q}}_X \, \gamma^\mu \, \frac{\lambda^a}{2} \, \mathbf{q}_X \quad ; \quad \mathcal{Q}_X^a = \int d^3x \, J_X^{a0}(x) \qquad (a = 1, \cdots, 8 \; ; \; X = L, R)$$

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$$\mathbf{U} \longrightarrow g_R \; \mathbf{U} \; g_L^\dagger \qquad ; \qquad g_{L,R} \in SU(3)_{L,R}$$

Energy Scale	Fields	Effective Theory
M_W	W, Z, γ, g τ, μ, e, ν_i t, b, c, s, d, u	Standard Model
	$\downarrow$ OPE	
$\lesssim m_c$	$\gamma, g\ ;\ \mu, e, \nu_i$ s, d, u	$\mathcal{L}_{\rm QCD}^{(n_f=3)}, \mathcal{L}_{\rm eff}^{\Delta S=1,2}$
	$\downarrow$ $N_C \to \infty$	
M_K	$\gamma\ ;\ \mu, e, \nu_i$ π, K, η	χPT

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Derivative Coupling

Goldstones become free at zero momenta

$$\mathcal{L}_2 = \frac{f^2}{4}\langle \partial_\mu \mathbf{U}^\dagger \partial^\mu \mathbf{U} \rangle = \partial_\mu \pi^- \partial^\mu \pi^+ + \frac{1}{2}\partial_\mu \pi^0 \partial^\mu \pi^0 + \cdots$$

$$+ \frac{1}{6f^2}\left\{\left(\pi^+ \overleftrightarrow{\partial}_\mu \pi^-\right)\left(\pi^+ \overleftrightarrow{\partial}^\mu \pi^-\right) + 2\left(\pi^0 \overleftrightarrow{\partial}_\mu \pi^+\right)\left(\pi^- \overleftrightarrow{\partial}^\mu \pi^0\right) + \cdots\right\}$$

$$+ O\left(\pi^6/f^4\right)$$

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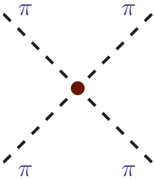
Chiral Symmetry Determines the Interaction:



$$T\left(\pi^+\pi^0 \to \pi^+\pi^0\right) = \frac{t}{f^2}$$

$t \equiv (p'_+ - p_+)^2$

Weinberg

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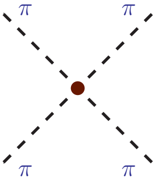

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Weinberg

Non-Linear Lagrangian: $2\pi \to 2\pi, 4\pi, \cdots$ related

Backup Slides

Symmetry: $U\,H\,U^\dagger = H \quad , \quad U\,|A\rangle = |B\rangle$

$$E_A = \langle A|\,H\,|A\rangle = \langle A|\,U^\dagger U\,H\,U^\dagger U\,|A\rangle = \langle B|\,H\,|B\rangle = E_B$$

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Spontaneously broken symmetry: $\mathcal{Q}_a\,|0\rangle$ not well defined

$$\langle 0|\mathcal{Q}_a\mathcal{Q}_b|0\rangle = \int d^3x\;\langle 0|J_a^0(x)\,\mathcal{Q}_b|0\rangle = \int d^3x\;\langle 0|J_a^0(0)\,\mathcal{Q}_b|0\rangle = \infty$$

$J_a^\mu(x) = \mathrm{e}^{iPx}\,J_a^\mu(0)\,\mathrm{e}^{-iPx}$

Goldstones and Coset-Space Coordinates: $G \xrightarrow{\text{SSB}} H$

Goldstone fields: $\vec{\phi} \equiv (\phi_1, \ldots, \phi_N) \quad \longrightarrow \quad \vec{\phi}\,' = \vec{\mathcal{F}}(g, \vec{\phi}) \quad , \quad g \in G$

$$N = \dim(G) - \dim(H) \quad , \quad \vec{\mathcal{F}}(e, \vec{\phi}) = \vec{\phi} \quad , \quad \vec{\mathcal{F}}(g_1 g_2, \vec{\phi}) = \vec{\mathcal{F}}\left(g_1, \vec{\mathcal{F}}(g_2, \vec{\phi})\right)$$

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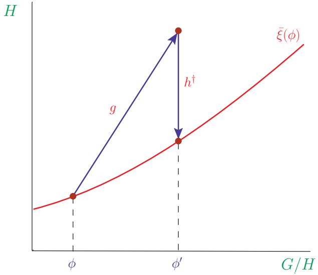
$\tilde{\mathcal{F}}$: **invertible mapping between Goldstone fields and G/H**



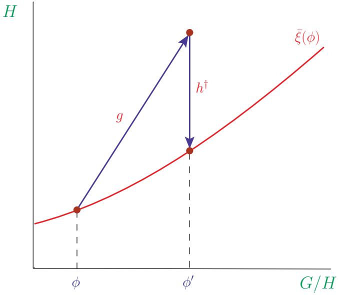
$$\vec{\mathcal{F}}(gh, \vec{0}) = \vec{\mathcal{F}}(g, \vec{0}) \qquad \forall g \in G\,,\ \forall h \in H$$

$$\vec{\mathcal{F}}(h, \vec{0}) = \vec{0} \quad , \quad h \in H \qquad \text{(vacuum invariant)}$$

$$\vec{\mathcal{F}}(g_i, \vec{0}) = \vec{\mathcal{F}}(g_j, \vec{0}) \quad \longrightarrow \quad g_i^{-1} g_j \in H$$

Coset representative: $\bar{\xi}(\phi) \in G$

Coset Space Coordinates: $G \equiv SU(3)_L \otimes SU(3)_R \xrightarrow{SCSB} H \equiv SU(3)_V$



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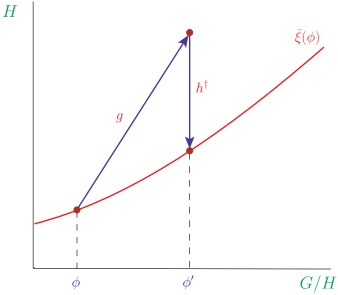

$$\bar{\xi}(\phi) \equiv (\xi_L(\phi), \xi_R(\phi)) \in G$$

$$\xi_L(\phi) \xrightarrow{G} g_L\, \xi_L(\phi)\, h^\dagger(\phi, g)$$

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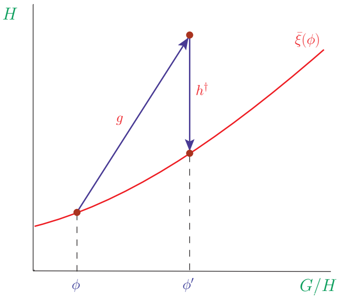

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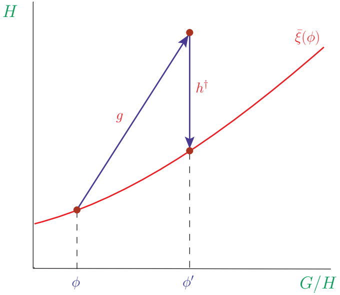

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Canonical choice:

$$\xi_R(\phi) \;=\; \xi_L(\phi)^\dagger \;\equiv\; \mathbf{u}(\phi) \quad \xrightarrow{G} \quad g_R\, \mathbf{u}(\phi)\, h^\dagger(\phi, g) \;=\; h(\phi, g)\, \mathbf{u}(\phi)\, g_L^\dagger$$

$$\mathbf{U}(\phi) \;=\; \mathbf{u}(\phi)^2 \;=\; \exp\left\{ i\, \frac{\sqrt{2}}{f}\, \mathbf{\Phi} \right\}$$